Math 242 Chain Rule Review

Chain rule is used when you are taking the derivative of a function that can be written as a composition of functions. The rule is as follows:

$$\text{If } f(x) = (v \circ u)(x) = v(u(x)) \text{ then } f'(x) = v(u(x))u'(x)$$

Examples:

1. Differentiate $f(x) = \ln(x^2 + 1)$

   Method 1:
   Notice that $f(x) = v(u(x))$ where $v(x) = \ln(x)$ and $u(x) = x^2 + 1$. Then by the chain rule, $f'(x) = v'(u(x))u'(x)$. Evaluating $v'(x)$ and $u'(x)$ we get $v'(x) = \frac{1}{x}$ and $u'(x) = 2x$ so $f'(x) = \frac{1}{u(x)}(2x) = \frac{2x}{u(x)} = \frac{2x}{x^2+1}$

   Method 2:
   Let u = the inside function i.e. let $u = x^2 + 1$ then $u' = 2x$ and $f(x) = \ln(u)$ Now pretend that u is x and take the derivative of f as you normally would BUT by chain rule you MUST multiple by u' afterward so $f'(x) = \frac{1}{u}u' = \frac{1}{2x+1}(2x) = \frac{2x}{x^2+1}$

   Method 3:
   Recognize that the chain rule says to keep the inside the same and take the derivative of the outside then multiple by the derivative of the inside. So applying that to $\ln(x^2+1)$, keeping the inside the same and taking the derivative of the outside gives $\frac{1}{x^2+1}$ then multiplying by the derivative of the inside gives $f'(x) = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1}$

   (Note: All these methods are very similar because they come from the same rule. There are only small differences.)

2. Differentiate $f(x) = (4x + 5)^6$

   Method 1:
   $v(x) = x^6$ and $u(x) = 4x + 5$ so $v'(x) = 6x^5$ and $u'(x) = 4$ thus $f'(x) = 6(4x + 5)^5(4) = 24(4x + 5)^5$

   Method 2:
   $u(x) = 4x + 5$, $u'(x) = 4$ so $f'(x) = 6(u(x))^5 u'(x) = 6(4x + 5)^5(4) = 24(4x + 5)^5$.

   Method 3:
   Outside derivative is $6(4x+5)^5$ and inside derivative is 4 so $f'(x) = 6(4x+5)^5(4) = 24(4x+5)^5$

3. Differentiate e^{2x^2+4x}

   Method 1:
   $v(x) = e^x$, $u(x) = 2x^2 + 4x$ so $v'(x) = e^x$ and $u'(x) = 4x + 4$. So $f'(x) = e^{2x^2+4x}(4x+4)$

   Method 2:
   $u(x) = 2x^2 + 4x$ and $u'(x) = 4x + 4$ so $f'(x) = e^u u' = e^{2x^2+4x}(4x+4)$

   Method 3
   Outside derivative: e^{2x^2+4x}. Inside derivative: $4x+4$. So $f'(x) = e^{2x^2+4x}(4x+4)$

4. Differentiate $f(x) = \dfrac{\sin(4x) + 7}{75x^2}$

   Recall the Quotient Rule: If $f(x) = \dfrac{u(x)}{v(x)}$ then $f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$
   So $f'(x) = \dfrac{(\sin(4x)+7)'(75x^2) - (\sin(4x)+7)(75x^2)'}{(75x^2)^2}$
   To evaluate $(\sin(4x)+7)'$ we need to use chain rule. Using which method you like best, you should get $(\sin(4x)+7)' = 4\cos(4x)$. Now we can finish the problem:
   $f'(x) = \dfrac{4\cos(4x)(75x^2) - (\sin(4x)+7)(150x)}{5625x^4}$

5. Differentiate $f(x) = (x^2+5)^3 \tan(4x)$

   Recall the Product Rule: If $f(x) = u(x)v(x)$ then $f'(x) = u'(x)v(x) + u(x)v'(x)$. Thus $f'(x) = [(x^2+5)^3]' \tan(4x) + (x^2+5)^3 \tan(4x)'$. To find $[(x^2+5)^3]'$, we use the chain rule with inner function x^2+5 to get $[(x^2+5)^3]' = 6x(x^2+5)^2$ and to find $\tan(4x)'$ we use chain rule with inner function $4x$ to get $\tan(4x)' = 4\sec^2(4x)$. Putting this all together we get that $f'(x) = 6x(x^2+5)^2\tan(4x) + (x^2+5)^3 4\sec^2(4x)$

As you can see from the previous two exercises, you can use chain rule twice in one problem. In fact, you can use chain rule 5 billion times in one problem if it's intense enough. The next example will require you to use chain rule twice in one expression. This rule can be written as follows:

$$\text{If } f(x) = v(u(w(x))) \text{ then } f'(x) = v'(u(w(x)))u'(w(x))w'(x)$$

This equation comes from applying the chain rule twice to f. Note that you could you it more than twice in the future with the pattern of the equation continuing.

Example:

6. Differentiate $f(x) = e^{\sec(4x^2)}$.

   Method 1:
   Let $w(x)$ be the inner most function so $w(x) = 4x^2$. Let $u(x)$ be the second inner function so $u(x) = \sec x$ and then let $v(x)$ be the outermost function so $v(x) = e^x$ then we have $f(x) = v(u(w(x)))$. Finding the derivative of each we get, $w'(x) = 8x$, $u'(x) = \sec x \tan x$, and $v'(x) = e^x$.
   Putting this together with the chain rule we have
   $f'(x) = v'(u(w(x)))u'(w(x))w'(x) = e^{\sec(4x^2)} \sec(4x^2)\tan(4x^2)(8x)$
   $= 8x \sec(4x^2)\tan(4x^2)e^{\sec(4x^2)}$

   Method 2:
   Let $u(x)$ be the inner most function so $u(x) = 4x^2$ so the expression is now $e^{\sec u}$. Let $v(x)$ be the second inner function so $v(x) = \sec u$. Using chain rule where we pretend v is an x and take the derivative then multiply by $v'(x)$ (if you're confused look back at Example 1), we have that $f'(x) = e^v v'(x)$ and similarly pretending that u is an x and using the chain rule we have $v'(x) = \sec u(\tan u)u'(x)$. Evaluate $u'(x)$ to get $u'(x) = (4x^2)' = 8x$ so $v'(x) = \sec(4x^2)\tan(4x^2)(8x)$. Plugging this into the equation for $f'(x)$ we get $f'(x) = e^{\sec u} 8x \sec(4x^2)\tan(4x^2) = 8x\sec(4x^2)\tan(4x^2)e^{\sec(4x^2)}$

   Method 3:
   Recall that Method 3 says to keep the inner function the same, find the derivative of the outer function and multiply that by the derivative of the inner function. Keeping the inner function $\sec(4x^2)$ the same, the derivative of the outer function is $e^{\sec(4x^2)}$ so $f'(x) = e^{\sec(4x^2)}(\sec(4x^2))'$ by the chain rule. So we have to find $(\sec(4x^2))'$ but this function also has an inner function so we'll call $\sec(4x^2)$ the new outer function and $4x^2$ the new inner function. Keeping the new inner function $4x^2$ the same, the derivative of the new outer function is $\sec(4x^2)\tan(4x^2)$ and the derivative of the new inner function is just $(4x^2)' = 8x$. Therefore we know $(\sec(4x^2))' = \sec(4x^2)\tan(4x^2)(8x)$ by the chain rule. Thus,
   $f'(x) = e^{\sec(4x^2)}(\sec(4x^2))' = e^{\sec(4x^2)}\sec(4x^2)\tan(4x^2)(8x)$
   $= 8x\sec(4x^2)\tan(4x^2)e^{\sec(4x^2)}$