$= (2)(2) = 4.$ ▶ Now Try Exerc... 3.

INCORPORATING TECHNOLOGY

Using Tables to Find Limits Consider the function

$$y = \frac{x^2 - 9}{x - 3}$$

from Example 4(a). This function is undefined at $x = 3$, but if you examine its values near $x = 3$, as in Fig. 4, you will be convinced that 4.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

To generate the tables in Fig. 4, first press [Y=] and assign the function $\frac{x^2-9}{x-3}$ to... Press [2nd] [TBLSET] and set **Indpnt** to **Ask**, leaving the other settings at their default values. Finally, press [2nd] [TABLE] and enter the values shown for X.

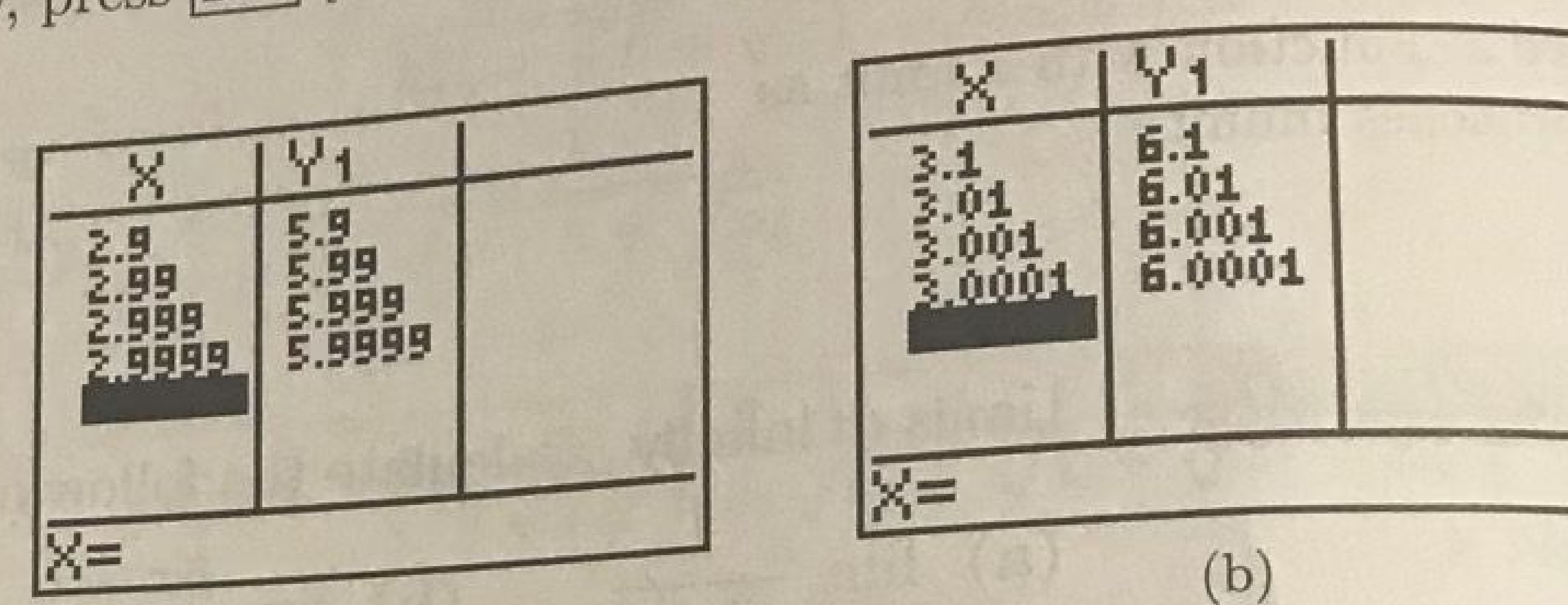


Figure 4

Check Your Understanding 1.4

Determine which of the following limits exist. Compute the limits that exist.

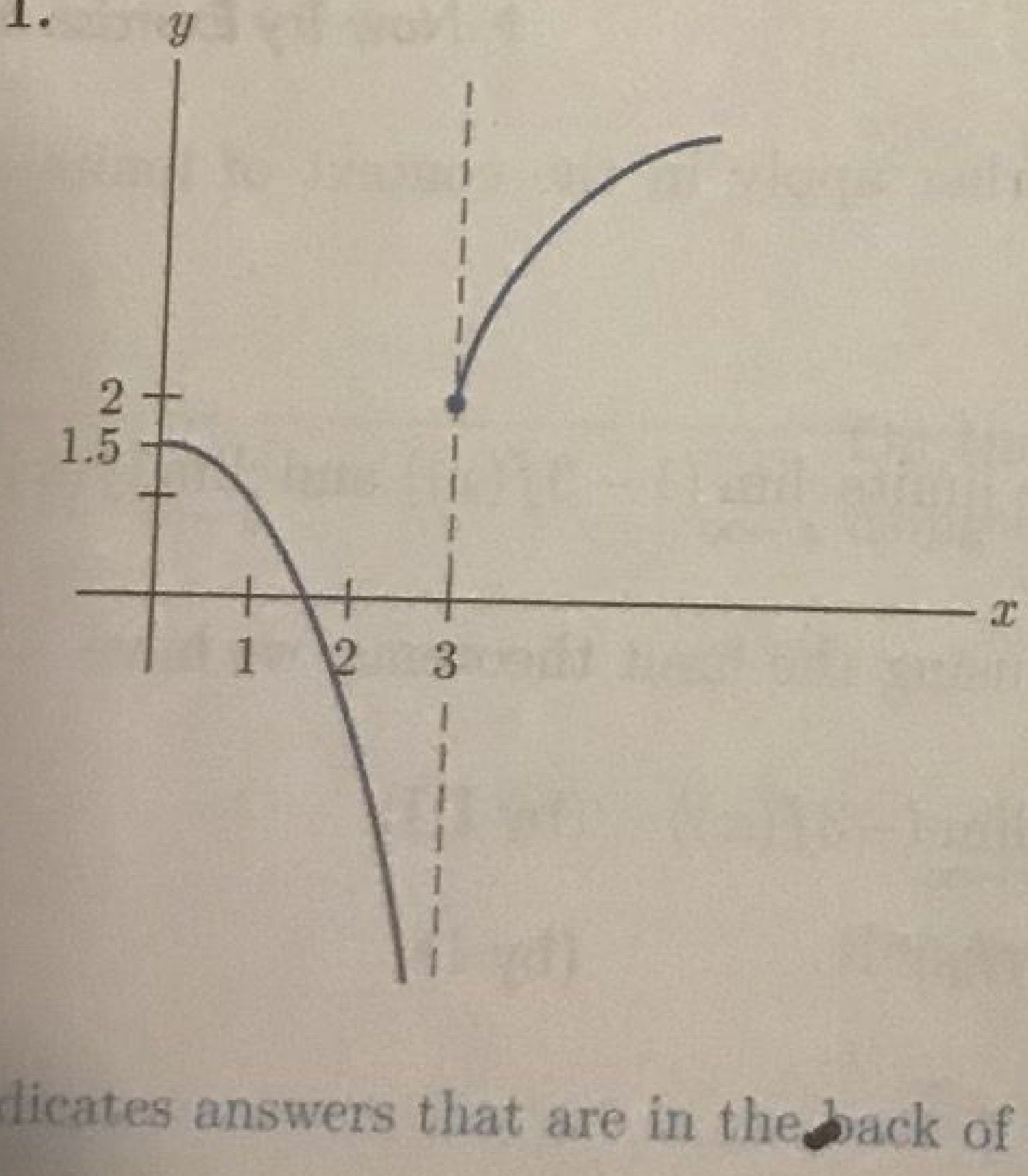
1. $\lim_{x \to 6} \frac{x^2 - 4x - 12}{x - 6}$

2. $\lim_{x \to 6} \frac{4x + 12}{x - 6}$

EXERCISES 1.4

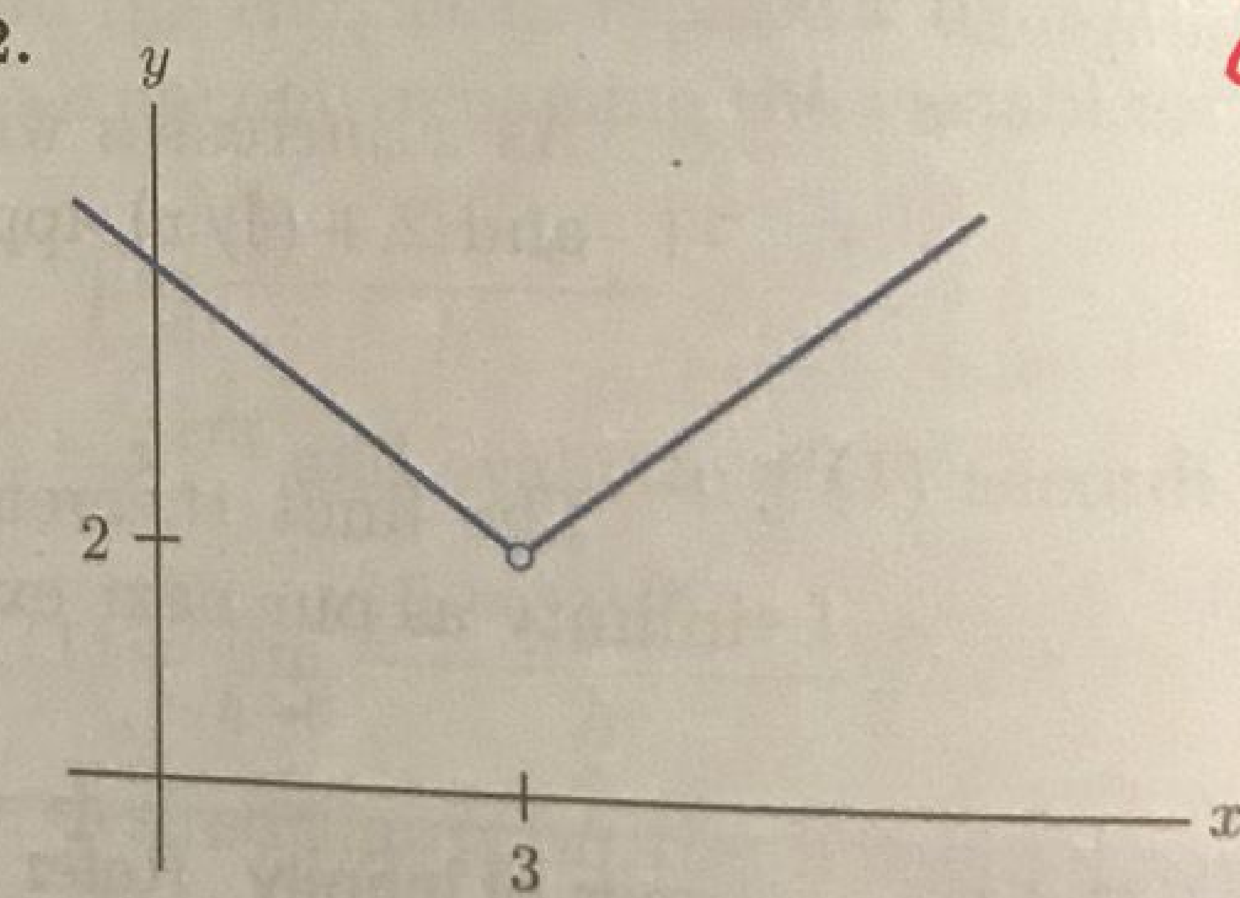
For each of the following functions $g(x)$, determine whether or not $\lim_{x \to 3} g(x)$ exists. If so, give the limit.

1.

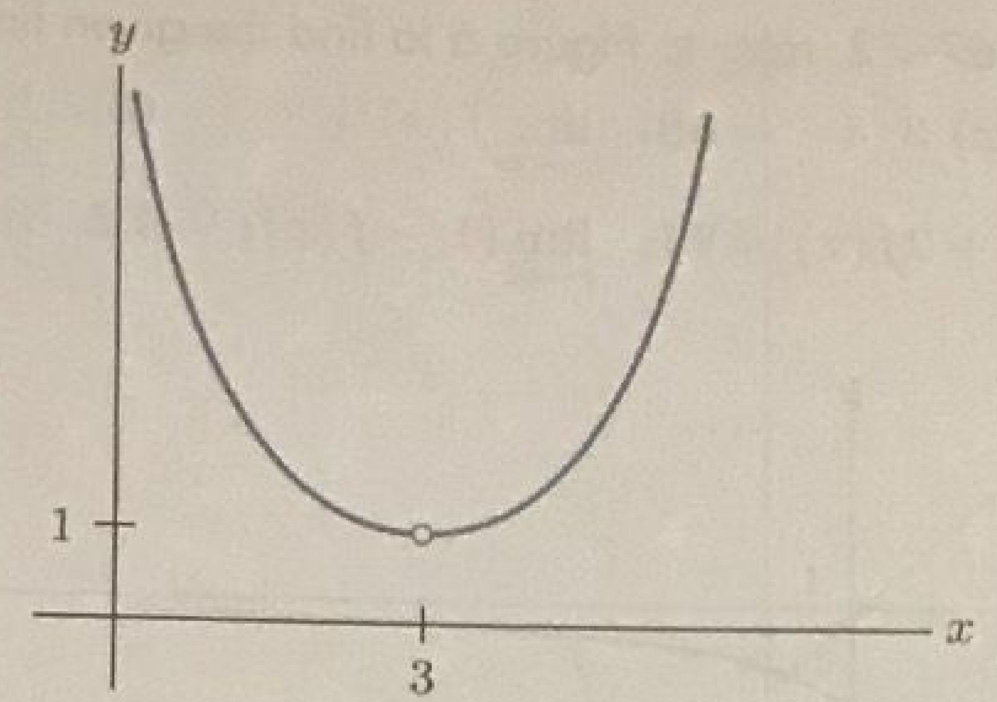


No limit

2.



* indicates answers that are in the back of the book.

3.


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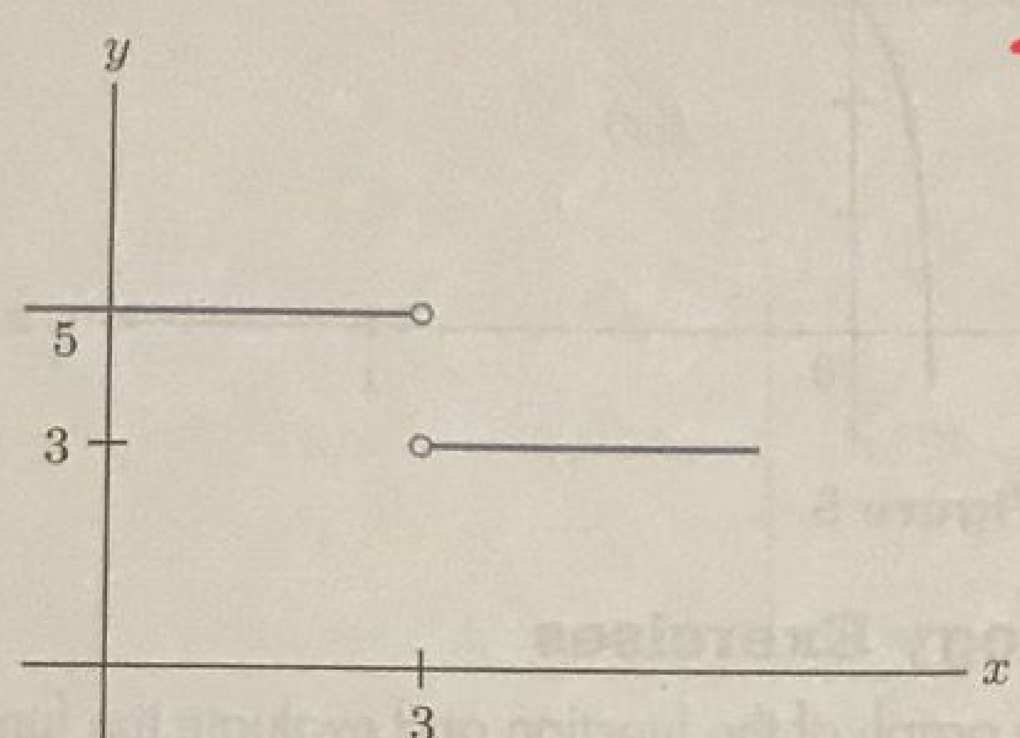
4.


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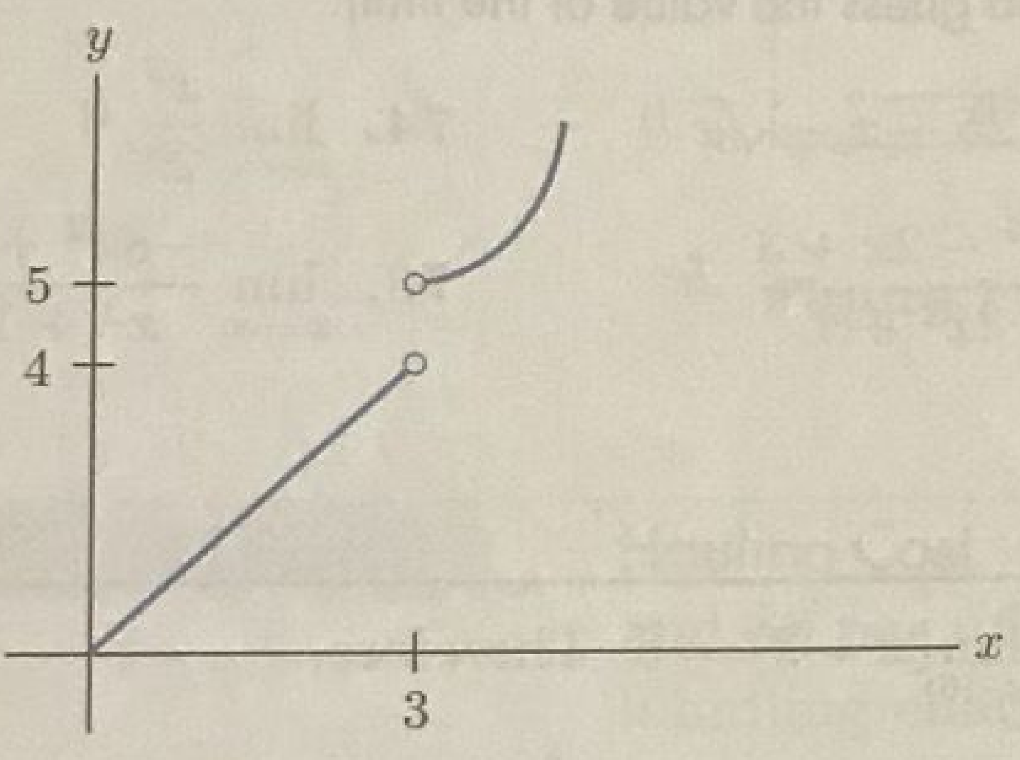
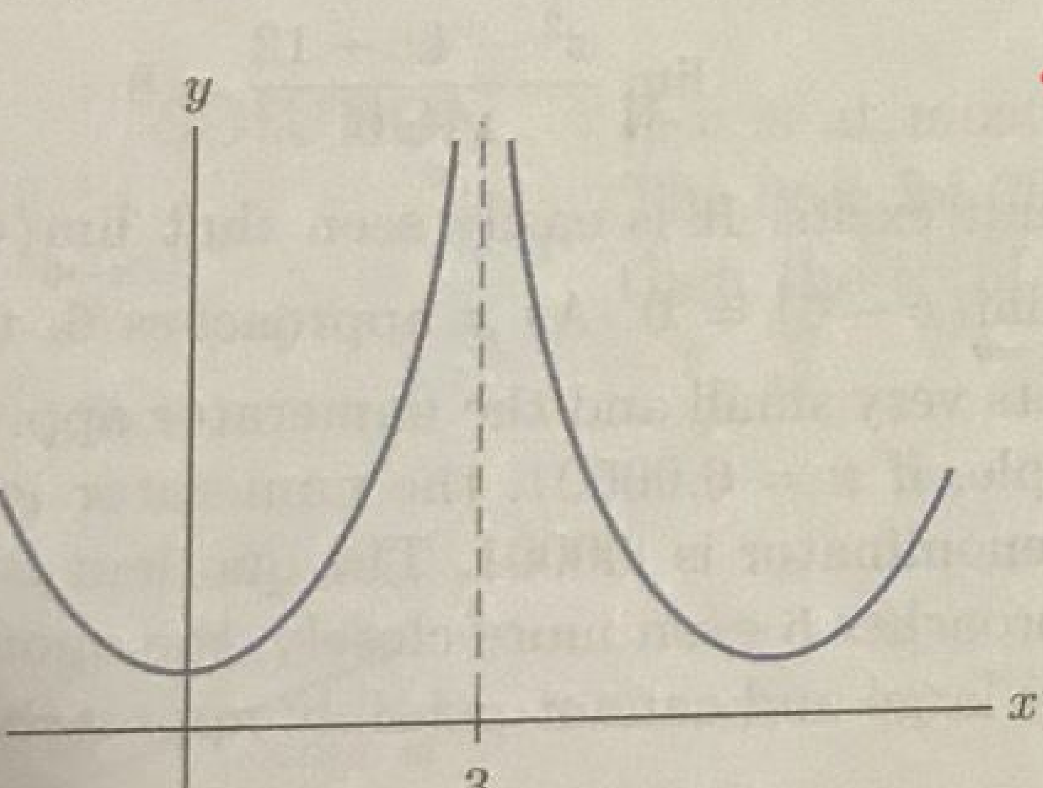
5.


No limit

6.


[redacted]

Determine which of the following limits exist. Compute the limits that exist.

7. $\lim_{x\to 1}(1-6x)$ −5

8. $\lim_{x\to 2}\frac{x}{x-2}$ No limit

9. $\lim_{x\to 3}\sqrt{x^2+16}$ 5

10. $\lim_{x\to 4}(x^3-7)$ 57

11. $\lim_{x\to 5}\frac{x^2+1}{5+x}$ 13/5

12. $\lim_{x\to 6}\left(\sqrt{6x}+3x-\frac{1}{x}\right)(x^2-4)$ 2288/3

13. $\lim_{x\to 7}(x+\sqrt{x-6})(x^2-2x+1)$ 288

14. $\lim_{x\to 8}\frac{\sqrt{5x-4}-1}{3x^2+2}$ 5/194

15. $\lim_{x\to -5}\frac{\sqrt{x^2-5x-36}}{8-3x}$ $\sqrt{14}/23$

16. $\lim_{x\to 10}(2x^2-15x-50)^{20}$ 0

17. $\lim_{x\to 0}\frac{x^2+3x}{x}$ 3

18. $\lim_{x\to 1}\frac{x^2-1}{x-1}$ 2

19. $\lim_{x\to 2}\frac{-2x^2+4x}{x-2}$ −4

20. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$ 5

21. $\lim_{x\to 4}\frac{x^2-16}{4-x}$ −8

22. $\lim_{x\to 5}\frac{2x-10}{x^2-25}$ 1/5

23. $\lim_{x\to 6}\frac{x^2-6x}{x^2-5x-6}$ 6/7

24. $\lim_{x\to 7}\frac{x^3-2x^2+3x}{x^2}$ 38/7

25. $\lim_{x\to 8}\frac{x^2+64}{x-8}$ No limit

26. $\lim_{x\to 9}\frac{1}{(x-9)^2}$ No limit

27. Compute the limits that exist, given that

$$\lim_{x\to 0}f(x)=-\frac{1}{2} \quad\text{and}\quad \lim_{x\to 0}g(x)=\frac{1}{2}.$$

(a) $\lim_{x\to 0}(f(x)+g(x))$ 0

(b) $\lim_{x\to 0}(f(x)-2g(x))$ −3/2

(c) $\lim_{x\to 0}f(x)\cdot g(x)$ −1/4

(d) $\lim_{x\to 0}\frac{f(x)}{g(x)}$ −1

28. Use the limit definition of the derivative to show that if $f(x)=mx+b$, then $f'(x)=m$.

Use limits to compute the following derivatives.

29. $f'(3)$, where $f(x)=x^2+1$ 6

30. $f'(2)$, where $f(x)=x^3$ 12

31. $f'(0)$, where $f(x)=x^3+3x+1$ 3

32. $f'(0)$, where $f(x)=x^2+2x+2$ 2

In Exercises 33–36, apply the three-step method to compute $f'(x)$ for the given function. Follow the steps that we used in Example 6. Make sure to simplify the difference quotient as much as possible before taking limits.

33. $f(x)=x^2+1$

34. $f(x)=-x^2+2$

35. $f(x)=x^3-1$

36. $f(x)=-3x^2+1$

In Exercises 37–48, use limits to compute $f'(x)$. [Hint: In Exercises 45–48, use the rationalization trick of Example 8.]

37. $f(x)=3x+1$

38. $f(x)=-x+11$ $f'(x)=-1$

39. $f(x)=x+\frac{1}{x}$

40. $f(x)=\frac{1}{x^2}$ $f'(x)=-2/x^3$

41. $f(x)=\frac{x}{x+1}$

42. $f(x)=-1+\frac{2}{x-2}$

43. $f(x)=\frac{1}{x^2+1}$

44. $f(x)=\frac{x}{x+2}$ $f'(x)=2/(x+2)^2$

45. $f(x)=\sqrt{x+2}$

46. $f(x)=\sqrt{x^2+1}$ $f'(x)=x/\sqrt{x^2+}$

47. $f(x)=\frac{1}{\sqrt{x}}$

48. $f(x)=x\sqrt{x}$ $f'(x)=3/2\sqrt{x}$

Each limit in Exercises 49–54 is a definition of $f'(a)$. Determine the function $f(x)$ and the value of a.

$f(x)=x^2; a=1$ 49. $\lim_{h\to 0}\frac{(1+h)^2-1}{h}$

50. $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$ $f(x)=x^3,\ a=$

$f(x)=1/x;\ a=10$ 51. $\lim_{h\to 0}\frac{\frac{1}{10+h}-.1}{h}$

52. $\lim_{h\to 0}\frac{(64+h)^{1/3}-4}{h}$

$f(x)=\sqrt{x};\ a=9$ 53. $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$

54. $\lim_{h\to 0}\frac{(1+h)^{-1/2}-1}{h}$

28. $f(x+h)=mx+mh+b$, $f(x+h)-f(x)=mh$, $f(x+h)-f(x)/h=m$, $\lim_{h\to 0}m=m$ 42. $f'(x)=-2/(x-2)^2$. 52. $f(x)=\sqrt[3]{x}$, $a=64$ 54. $f(x)=1/\sqrt{x}$, $a=1$

In Exercises 55–60, match the given limit with a derivative and then find the limit by computing the derivative.

55. $\lim_{h\to 0} \frac{(h+2)^2-4}{h}$

56. $\lim_{h\to 0} \frac{(h-1)^3+1}{h}$ $\frac{d}{dx}x^3\Big|_{x=-1} = 3$

57. $\lim_{h\to 0} \frac{\sqrt{h+2}-\sqrt{2}}{h}$

58. $\lim_{h\to 0} \frac{\sqrt{h+4}-2}{h}$ $\frac{d}{dx}\sqrt{x}\Big|_{x=4} = \frac{1}{4}$

59. $\lim_{h\to 0} \frac{(8+h)^{1/3}-2}{h}$

60. $\lim_{h\to 0} \frac{1}{h}\left[\frac{1}{h+1}-1\right]$ $\frac{d}{dx}\frac{1}{x}\Big|_{x=1} = -1$

Compute the following limits.

61. $\lim_{x\to\infty} \frac{1}{x^2}$ 0

62. $\lim_{x\to -\infty} \frac{1}{x^2}$ 0

63. $\lim_{x\to\infty} \frac{5x+3}{3x-2}$ 5/3

64. $\lim_{x\to\infty} \frac{1}{x-8}$ 0

65. $\lim_{x\to\infty} \frac{10x+100}{x^2-30}$ 0

66. $\lim_{x\to\infty} \frac{x^2+x}{x^2-1}$ 1

55. Take $f(x)=x^2$; then, the given limit is $f'(2)=4$.
57. Take $f(x)=\sqrt{x}$; then, the given limit is $f'(2)=1/2\sqrt{2}$.
59. Take $f(x)=x^{1/3}$; then, the given limit is $f'(8)=1/12$.

In Exercises 67–72, refer to Figure 5 to find the given limit.

67. $\lim_{x\to 0} f(x)$ 3/4

68. $\lim_{x\to\infty} f(x)$

69. $\lim_{x\to 0} xf(x)$ 0

70. $\lim_{x\to\infty} (1+2f(x))$

71. $\lim_{x\to\infty} (1-f(x))$ 0

72. $\lim_{x\to 0} [f(x)]^2$

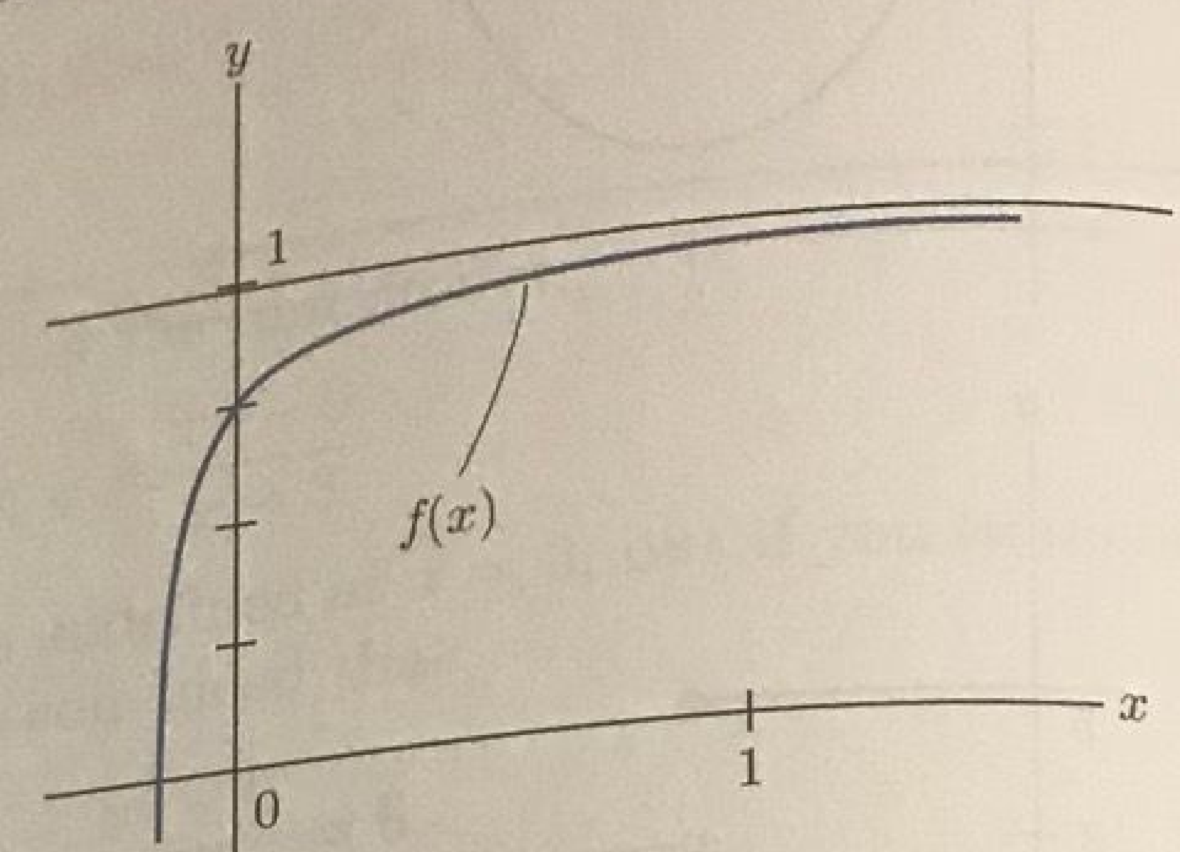


Figure 5

Technology Exercises

Examine the graph of the function and evaluate the function at large values of x to guess the value of the limit.

73. $\lim_{x\to\infty} \sqrt{25+x}-\sqrt{x}$ 0

74. $\lim_{x\to\infty} \frac{x^2}{2^x}$ 0

75. $\lim_{x\to\infty} \frac{x^2-2x+3}{2x^2+1}$ .5

76. $\lim_{x\to\infty} \frac{-8x^2+1}{x^2+1}$ −8

Solutions to Check Your Understanding 1.4

1. The function under consideration is a rational function. Since the denominator has value 0 at $x=6$, we cannot immediately determine the limit by just evaluating the function at $x=6$. Also,

$$\lim_{x\to 6}(x-6)=0.$$

Since the function in the denominator has the limit zero, we cannot apply Limit Theorem VI. However, since the definition of limit considers only values of x different from 6, the quotient can be simplified by factoring and canceling:

$$\frac{x^2-4x-12}{x-6} = \frac{(x+2)(x-6)}{(x-6)} = x+2 \quad \text{for } x \neq 6.$$

Now, $\lim_{x\to 6}(x+2)=8$. Therefore,

$$\lim_{x\to 6}\frac{x^2-4x-12}{x-6} = 8.$$

2. No limit exists. It is easily seen that $\lim_{x\to 6}(4x+12)=36$ and $\lim_{x\to 6}(x-6)=0$. As x approaches 6, the denominator gets very small and the numerator approaches 36. For example, if $x=6.00001$, the numerator is 36.00004 and the denominator is .00001. The quotient is 3,600,004. As x approaches 6 even more closely, the quotient gets arbitrarily large and cannot possibly approach a limit.

1.5 Differentiability and Continuity

In the preceding section, we defined differentiability of $f(x)$ at $x=a$ in terms of a limit. If this limit does not exist, then we say that $f(x)$ is *nondifferentiable* at $x=a$. Geometrically, the nondifferentiability of $f(x)$ at $x=a$ can manifest itself in several different ways. First, the graph of $f(x)$ could have no tangent line at $x=a$. Second, the graph could have a vertical tangent line at $x=a$. (Recall that slope is not defined for vertical lines.) Some of the various geometric possibilities are illustrated in Fig. 1.

The following example illustrates how nondifferentiable functions can arise in practice.