Math 241, Fall 2014, Final Exam

Name and section number:

Solutions
by
D. Yuen

Question	Points	Score
1	16	
2	8	
3	16	
4	11	
5	10	
6	10	
7	14	
8	10	
9	10	
10	9	
11	16	
12	10	
13	10	
Total:	150	

- You may not use notes or electronic devices on the test.

- Please ask if anything seems confusing or ambiguous.

- Show all your work.

- Good luck!

1. Compute the following limits. **Do not** use L'hospital's rule. If the limit is either positive or negative infinity say which. Simplify your answers.

(a) (4 points) $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$ $\quad$ Limit type $\frac{0}{0}$. Do More Work

$= \lim_{x \to 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)}$

$= \lim_{x \to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{2+2} = \frac{1}{4}$

(b) (4 points) $\lim_{x \to \infty} \frac{4-x^3-x}{7(x+1)^3} \cdot \frac{1/x^3}{1/x^3}$ $\quad$ Limit type $\frac{-\infty}{\infty}$. Do more work.

$= \lim_{x \to \infty} \frac{4/x^3 - 1 - \frac{1}{x^2}}{7(1+\frac{1}{x})^3} = \frac{0-1-0}{7(1+0)^3} = \frac{-1}{7}$

(c) (4 points) $\lim_{x \to -1^+} \frac{x^3+1}{x^2+2x+1}$. $\quad$ Limit type $\frac{0}{0}$. Do more work

$= \lim_{x \to -1^+} \frac{(x+1)(x^2-x+1)}{(x+1)^2} = \lim_{x \to -1^+} \frac{x^2-x+1}{x-1}$

$\quad$ Limit type $\frac{1}{0^+} \to \infty$

$= \boxed{\infty}$

(d) (4 points) $\lim_{x \to 4^-} \frac{x^2-5x+1}{x+17}$. $\quad$ Limit type $\frac{16-20+1}{21}$

$\quad$ No issues

$= \frac{-3}{21} = \frac{-1}{7}$

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2. (8 points) **Using the definition of the derivative, compute** $f'(x)$ **if** $f(x) = x^2 - x$.

(Warning: you get no credit if you use the rules of differentiation).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x - h - x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 - h}{h}$$

$$= \lim_{h \to 0} (2x + h - 1)$$

$$= 2x + 0 - 1$$

$$= 2x - 1.$$

3. Differentiate the following functions. **Do not** simplify your answers.

(a) (4 points) $f(x) = \sqrt[3]{x}\sec(x)$.

$$f'(x) = \tfrac{1}{3}x^{-2/3}\sec(x) + \sqrt[3]{x}\,\sec(x)\tan(x)$$

(b) (4 points) $s(t) = \dfrac{t^3 - 3t}{7t + 4}$.

$$s'(t) = \frac{(3t^3-3)(7t+4)-(t^3-3t)\,7}{(7t+4)^2}$$

(c) (4 points) $g(\theta) = \cos(\theta + \theta^{-1})$.

$$g'(\theta) = -\sin\!\left(\theta + \tfrac{1}{\theta}\right) \cdot \left(1 - \tfrac{1}{\theta^2}\right)$$

(d) (4 points) $V(x) = \displaystyle\int_0^{x^3} \sqrt{1+t^2}\,dt$.

$$V'(x) = \sqrt{1+(x^3)^2} \cdot 3x^2$$

4. A (male) tortoise is walking along a straight line with his position at time t given by

$$s(t) = t^2 - 2t + 1, \quad 0 \le t \le 3$$

(distance is measured in meters, and time is measured in minutes).

(a) (3 points) At what time(s) between zero and three minutes is he stopped?
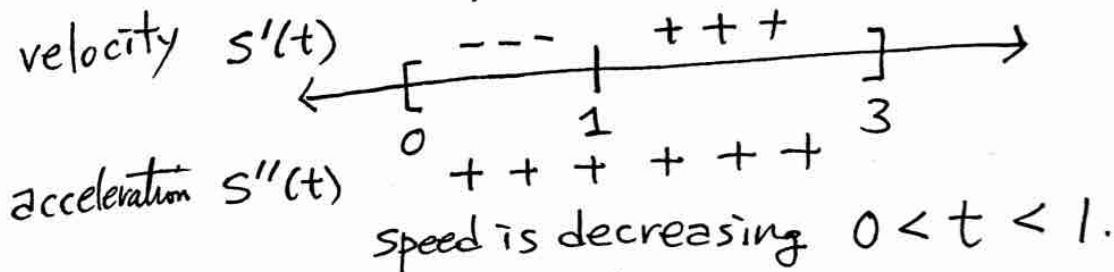
$$s'(t) = 2t - 2 \overset{set}{=\!=} 0$$

$$t = 1$$

Answer: stopped at time $t = 1$ minute

(b) (4 points) At what time(s) between zero and three minutes is his speed decreasing?

$$s''(t) = 2.$$

positive acceleration always

velocity $s'(t)$

acceleration $s''(t)$

speed is decreasing $0 < t < 1$.

(c) (4 points) What is the distance between the tortoise's position at time zero, and his position after three minutes?
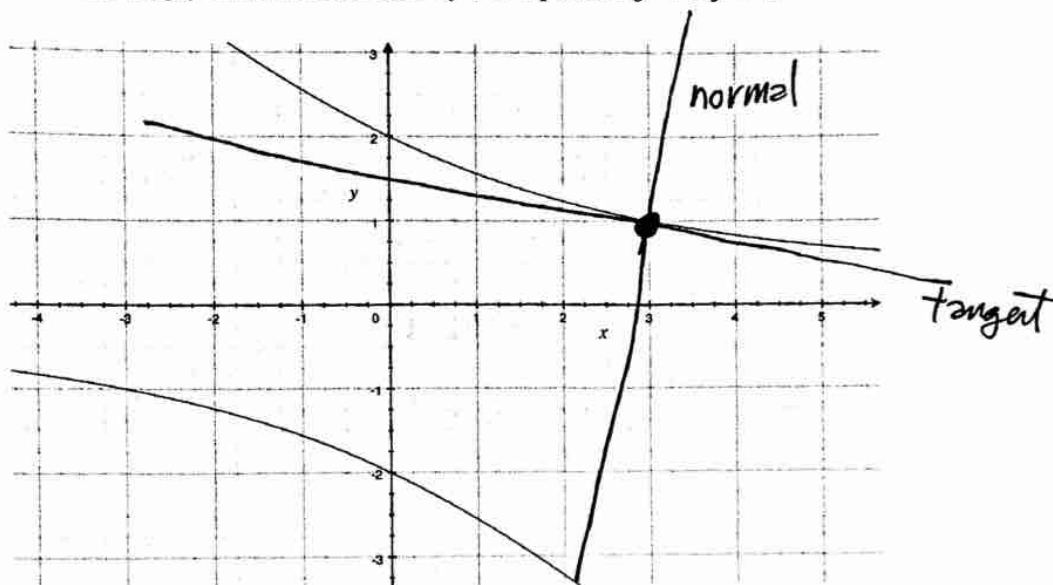
Distance between positions is

$$s(3) - s(0) = 4 - 1 = 3.$$

⌜As opposed to distance travelled, which would be
$$|s(3) - s(1)| + |s(1) - s(0)| \quad \text{because of turning}$$
$$= |4 - 0| + |0 - 1| = 5 \quad \text{around at time } t = 1⌟$$

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5. The following picture is described by the equation $y^2 + xy = 4$.

(a) (6 points) Find the equation for the tangent line to this graph at the point $(3, 1)$.

Implicit differentiation.

$$2y\,y' + 1y + xy' = 0$$

At $x = 3, y = 1$, we have

$$2y' + 1 + 3y' = 0$$
$$5y' + 1 = 0 \implies y' = -1/5.$$

Tangent line: $y - 1 = -1/5\,(x - 3)$.

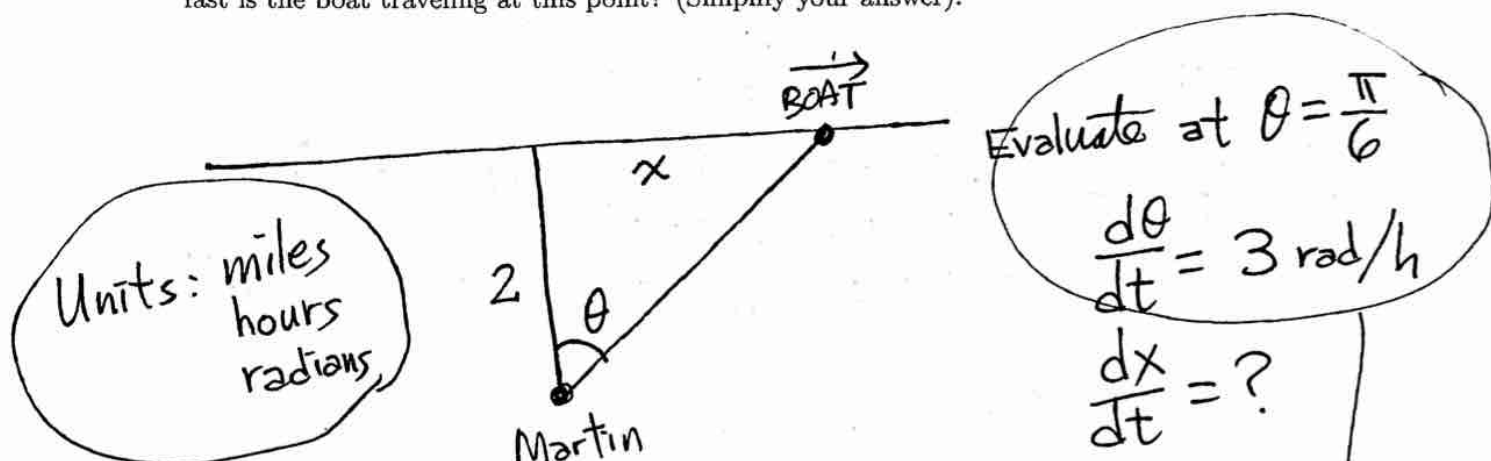(b) (2 points) Find the equation for the normal line to this graph at $(3, 1)$.

slope of normal line is 5. (negative reciprocal)

Line is $y - 1 = 5\,(x - 3)$

(c) (2 points) On the graph above sketch the normal and tangent lines at the point $(3, 1)$ (computed above).

6. (10 points) Martin stands still on the coast. There is a boat two miles straight north of his position. The boat starts moving east, and Martin measures how the angle between the following lines changes: the line pointing straight north from where he stands; and the line from where he stands to the boat.

When the angle Martin is measuring is $\pi/6$, it is changing at a rate of 3 radians / hour. How fast is the boat traveling at this point? (Simplify your answer).

$\overrightarrow{BOAT}$

x

Units: miles hours radians

2

θ

Martin

Evaluate at $\theta = \dfrac{\pi}{6}$

$\dfrac{d\theta}{dt} = 3 \text{ rad/h}$

$\dfrac{dx}{dt} = ?$

Relation is $\dfrac{x}{2} = \tan\theta$

$x = 2\tan\theta$

$\dfrac{dx}{dt} = 2\sec^2\theta \dfrac{d\theta}{dt}$

$\dfrac{dx}{dt} = 2\left(\sec\dfrac{\pi}{6}\right)^2 (3)$

$= 2\left(\dfrac{2}{\sqrt{3}}\right)^2 3$

$= 8 \text{ mi/h.}$

$\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\sec\dfrac{\pi}{6} = \dfrac{2}{\sqrt{3}}$

7. Consider the function $f(x) = 3x^2 - x^3$.

   (a) (6 points) Find the absolute minimum and maximum of f on the interval $[0, 4]$, and where they occur.

Use E.V.T. $f'(x) = 6x - 3x^2 \overset{set}{=} 0$

$$3x(2-x) = 0$$
$$x = 0, 2$$

Compare
critical points
and endpoints
$$x = 0 \Rightarrow f(0) = 0$$
$$x = 2 \Rightarrow f(2) = 12 - 8 = 4 \quad \leftarrow MAX.$$
$$x = 4 \Rightarrow f(4) = 48 - 64 = -16 \quad \leftarrow MIN$$

   (b) (4 points) Find all inflection points of the graph of f (just give the x value(s)).

$$f''(x) = 6 - 6x \overset{set}{=} 0$$
$$x = 1$$

$f''(x)$ ++ o - - - →   IP at $x = 1$.

f $\smile$ 1 $\frown$

   (c) (4 points) Find all critical points of f (just give the x value(s)), and say whether they are local minima, local maxima or neither.

From above, $x = 0, 2$

$f'(x)$ - - - o + + o - - →

x 0   2

Conclusions
on f →   ↗ LOCAL MAX at $x = 2$ ↘

LOCAL MIN at $x = 0$

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8. (10 points) On the coordinate system below, sketch the graph of a function f with the following properties. Make sure all the features listed are clear from your graph.

- f has x intercepts at -3 and 1;
- f has a vertical asymptote at $x = 2$ (and no other vertical asymptotes);
- $\lim_{x \to -\infty} f(x) = 1$, $\lim_{x \to +\infty} f(x) = 1$;
- f is continuous on the intervals $(-\infty, 2)$ and $(2, \infty)$ (and not on any bigger intervals);
- f is differentiable on the intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$ (and not on any bigger intervals);
- f is decreasing on $(-\infty, -2)$ and $(2, \infty)$, and increasing on $(-2, 2)$;
- f is concave down on $(-\infty, -2)$ and $(-2, 1)$, and concave up on $(1, 2)$ and $(2, \infty)$.
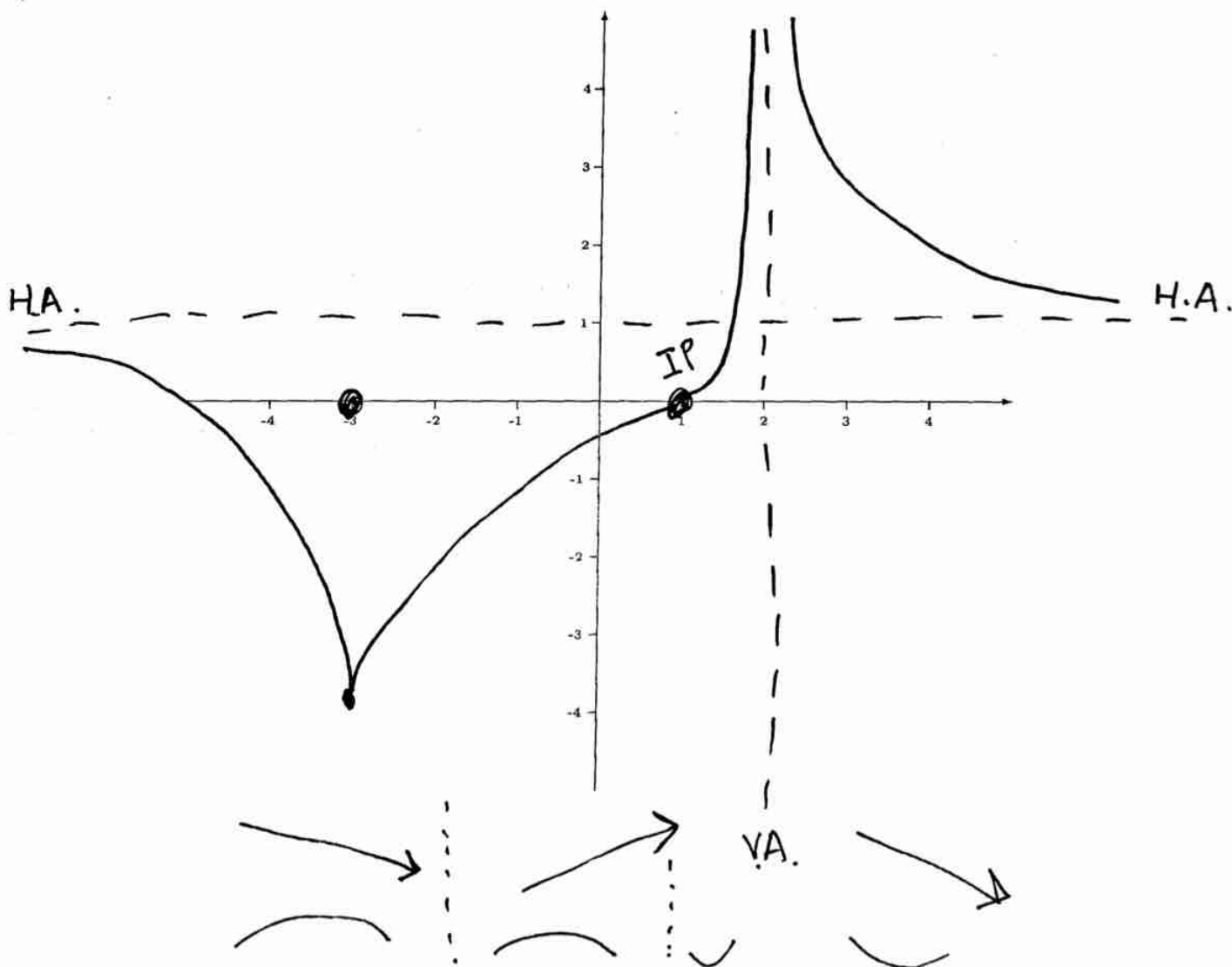
9. (10 points) Emma wants to enclose a rectangular field with total area $200m^2$. Along one side of the field, she will use a pre-existing straight wall, but on the other three sides, she needs to buy fence.

If it costs \$2 for each meter of fence, what is the least amount she can spend to enclose her field? (Simplify your answer.)

pre-existing wall

y y

x
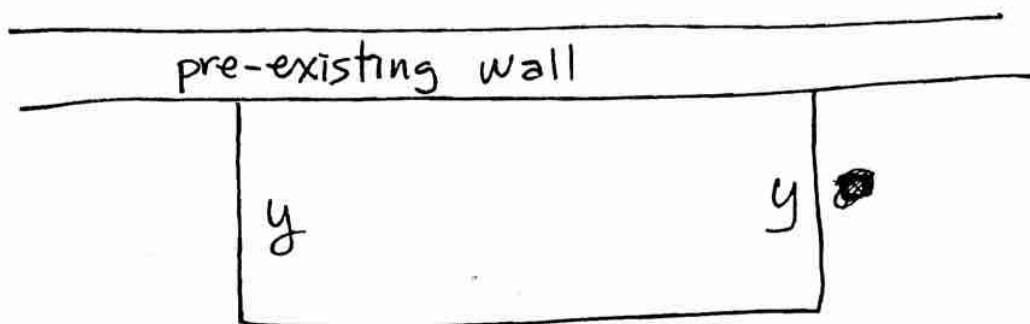
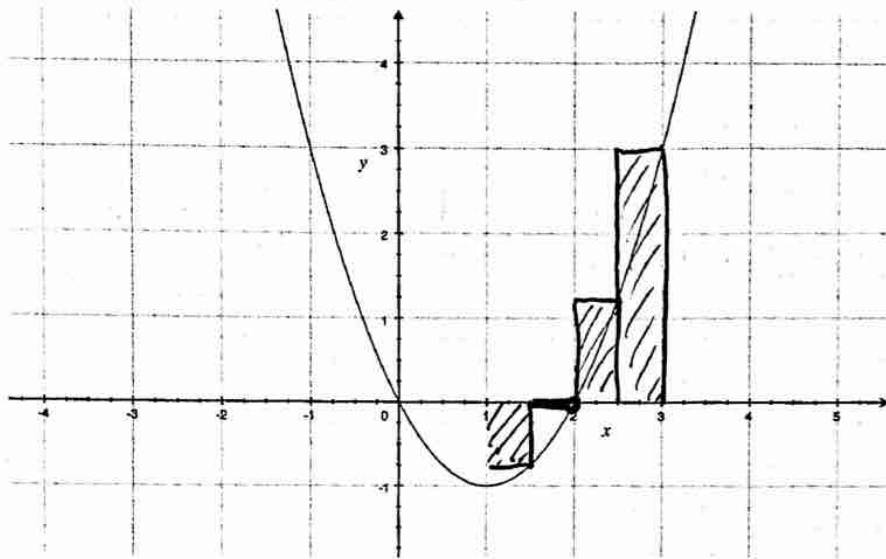Constraint area is $200 = xy \Rightarrow y = \dfrac{200}{x}$

Objective is cost

$$C = 2(x + 2y) = 2x + 4y$$
$$= 2x + 4\left(\dfrac{200}{x}\right) = 2x + \dfrac{800}{x}$$

Domain is $x > 0$.

10. Consider the function $f(x) = x^2 - 2x$ as pictured below.

(a) (6 points) Compute a Riemann sum for this function that approximates the integral $\int_1^3 f(x)\,dx$. Use four equal-width intervals for your Riemann sum, and use the right endpoint of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

$\Delta x = \frac{1}{2}$

The interval is:

$$1 \qquad \frac{3}{2} \qquad 2 \qquad \frac{5}{2} \qquad 3$$

$$f\left(\tfrac{3}{2}\right) = -\tfrac{3}{4}, \quad f(2) = 0, \quad f\left(\tfrac{5}{2}\right) = \tfrac{5}{4}, \quad f(3) = 3$$

Riemann sum is

$$\left(-\tfrac{3}{4} + 0 + \tfrac{5}{4} + 3\right)\tfrac{1}{2}$$

(b) (2 points) Sketch the rectangles that correspond to part (a) on the graph above.

(c) (1 point) Does your solution to (a) overestimate or underestimate $\int_1^3 f(x)\,dx$?

Overestimate, the rectangles' tops are above the graph.

11. Compute the following integrals. Simplify your answers.

(a) (4 points) $\int_0^{\pi/2} \sin(2\theta)d\theta.$

$$= -\tfrac{1}{2}\cos(2\theta)\Big|_0^{\pi/2}$$

$$= -\tfrac{1}{2}(\cos\pi - \cos 0) = -\tfrac{1}{2}(-1-1) = \boxed{1}$$

(b) (4 points) $\int \sqrt[3]{x^5} - 7x^{-3}dx.$

$$\int (x^{5/3} - 7x^{-3})\,dx$$

$$= \tfrac{3}{8}x^{8/3} - \tfrac{7}{-2}x^{-2} + C = \tfrac{3}{8}x^{8/3} + \tfrac{7}{2}x^{-2} + C$$

(c) (4 points) $\int_{-1}^2 x\sqrt{1+x^2}dx.$

Sub $u = \sqrt{1+x^2}$ $x = -1 \Rightarrow u = 2$

$du = 2x\,dx$ $x = 2 \Rightarrow u = 5$

$$= \int_{-1}^2 \sqrt{1+x^2}\cdot x\,dx$$

$$= \int_2^5 \sqrt{u}\cdot \tfrac{1}{2}du = \tfrac{1}{2}\tfrac{2}{3}u^{3/2}\Big|_2^5$$

$$= \tfrac{1}{3}(5^{3/2} - 2^{3/2}) = \tfrac{1}{3}(5\sqrt{5} - 2\sqrt{2})$$
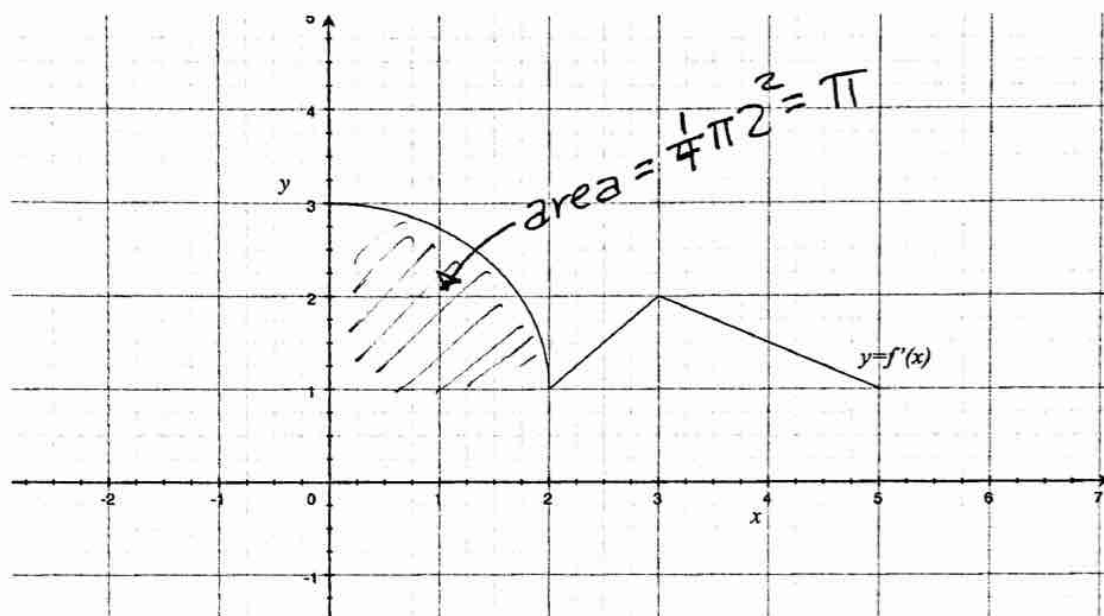
(d) (4 points) $\int \cos^7(\theta)\sin(\theta)d\theta.$

Sub $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= \int u^7(-du)$$

$$= -\tfrac{u^8}{8} + C$$

$$= -\tfrac{1}{8}\cos^8(\theta) + C$$

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12. A function f of a single variable x is defined on the interval $[0, 5]$ and differentiable on $(0, 5)$. The following picture shows the graph of the **derivative** f' of f.

In the picture, the portion of the graph on the interval $(0, 2)$ identifies with a quarter-circle of radius 2 and center $(0, 1)$; the portions of the graph on the intervals $[2, 3]$ and $[3, 5]$ are line segments.

(a) (2 points) What is $f''(2.5)$? $f''(2,5) = 1$

(b) (4 points) If $f(0) = 0$, what is $f(3)$?

$$f(3) - f(0) = \int_0^3 f'(x)dx = \pi + 3 + \frac{1}{2} \quad (\text{add up areas})$$
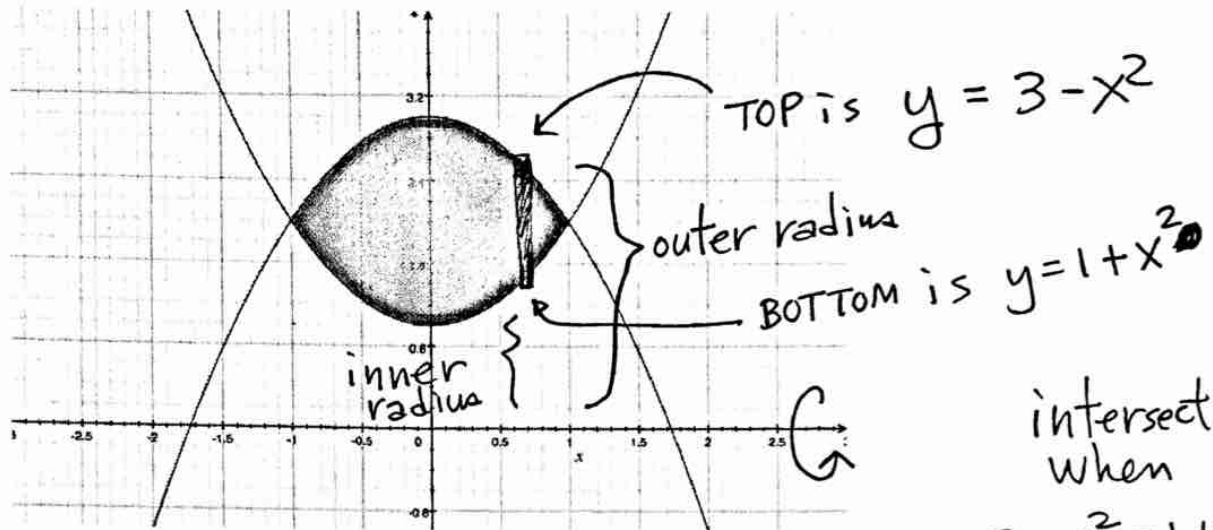
$$\text{Thus } f(3) = f(0) + \pi + \frac{7}{2} = 0 + \pi + \frac{7}{2} = \pi + \frac{7}{2}.$$

(c) (4 points) On which interval(s) is the graph of f concave up, and on which concave down?

f is concave up when $(f')' > 0$ which by the above is when $2 < x < 3$.

f is concave down when $(f')' < 0$ which is $0 < x < 2$ and $3 < x < 5$.

13. (10 points) The following picture shows the region between the graphs of $y = 3 - x^2$ and $y = 1 + x^2$.

TOP is $y = 3 - x^2$

outer radius

BOTTOM is $y = 1 + x^2$

inner radius

intersect when

$3 - x^2 = 1 + x^2$

$2 = 2x^2$

$1 = x^2$

$\pm 1 = x$

Find the volume of the shape obtained by revolving this region about the x axis.

Use vertical rectangles (dx-problem)

Get washers.

$$V = \pi \int_{-1}^{1} \left((3 - x^2)^2 - (1 + x^2)^2 \right) dx$$

$$= \pi \int_{-1}^{1} \left(9 - 6x^2 + x^4 - (1 + 2x^2 + x^4) \right) dx$$

$$= \pi \int_{-1}^{1} (8 - 8x^2) \, dx$$

$$= \pi \left(8x - \tfrac{8}{3} x^3 \right) \Big|_{-1}^{1}$$

$$= \pi \left(\left(8 - \tfrac{8}{3} \right) - \left(-8 + \tfrac{8}{3} \right) \right) = \pi \, \frac{32}{3}$$