Math 241

Fall 2015

Final Exam

Name: ________________

Section Number: ________________

Instructor: ________________

Solutions
by D. Yuen

Question	Points	Score
1	9	
2	9	
3	6	
4	24	
5	10	
6	8	
7	12	
8	8	
9	6	
10	16	
11	9	
12	20	
13	4	
14	9	
Total:	150	

Read all of the following information before starting the exam.

- Electronic devices (calculators, cell phones, computers), books, and notes are not allowed.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- This test has 14 pages total including this cover sheet and is worth 150 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Compute the following limits. If the limit does not exist, state so. If the limit is either positive or negative infinity, say which.

(a) (3 points)

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 2x}$$

Type $\frac{4-6+2}{4-4} \to \frac{0}{0}$

Try factoring

$$= \lim_{x \to 2} \frac{(x-2)(x-1)}{x(x-2)} = \lim_{x \to 2} \frac{x-1}{x}$$

$$= \frac{1}{2}$$

(b) (3 points)

$$\lim_{x \to \infty} \frac{7 - \sqrt{x}}{7 + \sqrt{x}}$$

Type $\frac{\infty}{\infty}$

Try dividing by highest term from denominator

$$= \lim_{x \to \infty} \frac{7 - \sqrt{x}}{7 + \sqrt{x}} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} = \lim_{x \to \infty} \frac{\frac{7}{\sqrt{x}} - 1}{\frac{7}{\sqrt{x}} + 1}$$

$$= \frac{0 - 1}{0 + 1} = -1$$

(c) (3 points)

$$\lim_{x \to 0} x \sin \frac{1}{x}$$

Type $0 \cdot$ oscillating

Try sandwich theorem

Note $-1 \leq \sin \frac{1}{x} \leq 1$

Then $-|x| \leq x \sin \frac{1}{x} \leq |x|$

Since $\lim_{x \to 0} (-|x|) = 0$ and $\lim_{x \to 0} |x| = 0$,

then by sandwich theorem, $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

2. Consider the limit

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$$

Type $\frac{0}{0}$.
Here try multiplying by the conjugate

(a) (5 points) Find the limit.

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h\to 0} \frac{(4+h)-2^2}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

(b) (4 points) This limit represents the derivative of some function $f(x)$ at some number a. Find $f(x)$ and a

$f(x) = \sqrt{x}$

$a = 4$ Note $f(4) = 2$.

because $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

3. (6 points) Find all numbers a such that the function

$$f(x) = \begin{cases} 2x^3 & \text{if } x < a \\ x^4 + x^2 & \text{if } x \geq a \end{cases}$$

is continuous. *Please circle your answer.*

The above $f(x)$ is continuous for all x except <u>possibly</u> at the boundary between the two subdomains of $(x < a)$ and $(x \geq a)$, namely at $x = a$. To be continuous at a, we would need $\lim_{x \to a} f(x) = f(a)$.

We have

$$f(a) = a^4 + a^2.$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} 2x^3 = 2a^3$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (x^4 + x^2) = a^4 + a^2.$$

Need all three equal.

Thus we need $a^4 + a^2 = 2a^3$.

$$a^4 - 2a^3 + a^2 = 0$$

$$a^2(a^2 - 2a + 1) = 0$$

$$a^2(a-1)^2 = 0$$

$$\boxed{a = 0, \ a = 1}$$

4. Differentiate the following functions. Do not simplify your answers.

(a) (6 points)

$$F(x) = \sqrt{x}\,(x^5 + 3)$$

product rule

$$F'(x) = \frac{1}{2}x^{-1/2}(x^5+3) + \sqrt{x}\,(5x^4)$$

(b) (6 points)

$$g(x) = 7(\sin(x^5))^2$$

$$g'(x) = 7 \cdot 2(\sin(x^5)) \cdot \cos(x^5) \cdot 5x^4$$

(c) (6 points)

$$f(x) = \frac{\tan(\pi^3 x)}{x^2+1}$$

Quotient rule

$$f'(x) = \frac{\sec^2(\pi^3 x) \cdot \pi^3 (x^2+1) - \tan(\pi^3 x)\, 2x}{(x^2+1)^2}$$

(d) (6 points)

$$g(x) = \int_x^{x^2} \frac{1}{t^3+1}\,dt = \int_0^{x^2} \frac{1}{t^3+1}\,dt + \int_x^{0} \frac{1}{t^3+1}\,dt$$

OR any other constant

$$= \int_0^{x^2} \frac{1}{t^3+1}\,dt - \int_0^{x} \frac{1}{t^3+1}\,dt$$

By FTC and chain rule

$$g'(x) = \frac{1}{(x^2)^3+1} \cdot 2x - \frac{1}{x^3+1}$$

5. (10 points) Find an equation of the tangent line to the graph of the function $f(x) = x\cos x$ at $x = \pi/2$.

$f'(x) = 1\cos x + x(-\sin x)$ $\qquad \cos\frac{\pi}{2} = 0$

$f'(\frac{\pi}{2}) = 1 \cdot 0 + \frac{\pi}{2}(-1)$ $\qquad \sin\frac{\pi}{2} = 1$

$= -\frac{\pi}{2}$

Along with $f(\frac{\pi}{2}) = \frac{\pi}{2}\cos\frac{\pi}{2} = 0$,

The tangent line is $y = 0 + \frac{-\pi}{2}(x - \frac{\pi}{2})$.

6. (8 points) Consider the curve

$$x^3 + \sin y = x^3 y.$$

Find the x-coordinate of a point where the curve has a horizontal tangent line.

Implicit differentiation first.

$$3x^2 + \cos y \cdot y' = 3x^2 y + x^3 y'$$

$$\cos y\, y' - x^3 y' = 3x^2 y - 3x^2$$

$$y'(\cos y - x^3) = 3x^2 y - 3x^2$$

$$y' = \frac{3x^2 y - 3x^2}{\cos y - x^3}.$$

Want to find $y' = 0$. Set $3x^2 y - 3x^2 = 0$

$$3x^2(y-1) = 0$$

So $x = 0$ OR $y = 1$.

So $\boxed{x=0}$ works

7. (12 points) Find maximum and minimum values of the function

$$f(x) = \frac{x}{(2x+1)^2}$$

on the interval $0 \le x \le 2$.

$$f'(x) = \frac{1(2x+1)^2 - x\,2(2x+1)\,2}{(2x+1)^4}$$

$$= \frac{(2x+1)\left[(2x+1) - 4x\right]}{(2x+1)^4}$$

$$= \frac{1-2x}{(2x+1)^3} \Longrightarrow \text{set } 1-2x=0 \text{ for critical point}$$

$1 = 2x$

$\frac{1}{2} = x$

denominator $= 0$ at $x = -\frac{1}{2}$ but that is not in domain

By Extreme Value theorem, we just need to compare f at critical points and endpoints

$x = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{2^2} = \frac{1}{8}$

Endpoints

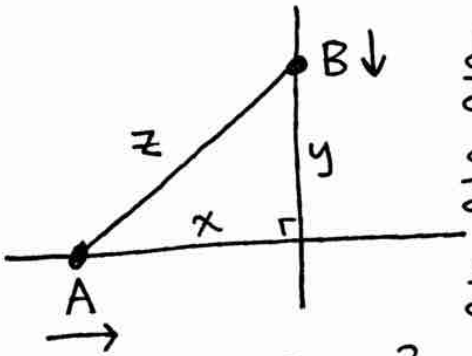
$x = 0 \Rightarrow f(0) = 0$

$x = 2 \Rightarrow f(2) = \frac{2}{5^2} = \frac{2}{25}$ $\qquad \frac{2}{25} < \frac{1}{8}$

Max value of $\frac{1}{8}$ at $x = \frac{1}{2}$

Min value of 0 at $x = 0$.

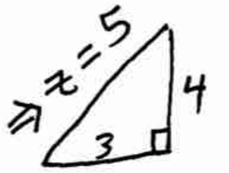
8. (8 points) Two straight streets meet at right angles. Person A is walking on one of the streets towards the East at 2 miles per hour, and person B is walking on the other street towards the South at 1 mile per hour; both are walking towards the intersection. When A is 3 miles from the intersection and B is 4 miles from the intersection, at what speed does the distance between A and B shrink?



$\frac{dx}{dt} = -2$ mi/h

$\frac{dy}{dt} = -1$ mi/h

$\frac{dz}{dt} = ?$ when $x=3,\ y=4$



$$z^2 = x^2 + y^2$$

$$\frac{d}{dt} \quad 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2(5)\frac{dz}{dt} = 2(3)(-2) + 2(4)(-1)$$

$$10\frac{dz}{dt} = -20$$

$$\boxed{\frac{dz}{dt} = -2 \text{ mph}}$$

9. (6 points) Use linear approximation to estimate the number $(.95)^{10}$.

1^{10} is a nice value.

Use $f(x) = x^{10}$ at basepoint $a=1$. $\Rightarrow f(1) = 1$

$f'(x) = 10x^9 \Rightarrow f'(1) = 10$.

The linear approximation is

$$L(x) = 1 + 10(x-1)$$

At $x = .95$

$$L(.95) = 1 + 10(-.05)$$

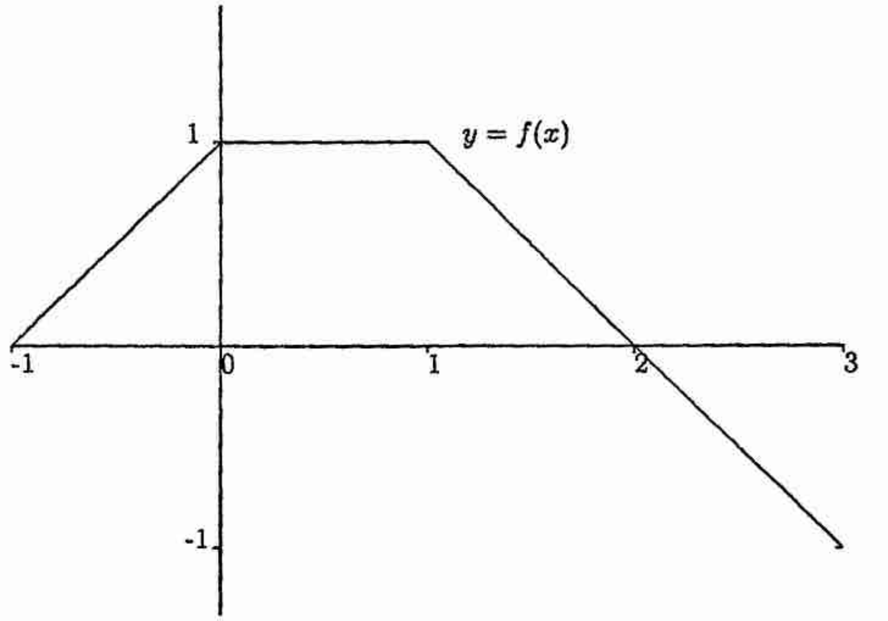
$$= 1 - .5 = .5 \Leftarrow \text{approximation to } (.95)^{10}.$$

10. The graph of a function f is given below.

Set

$$g(x) = \int_0^x f(t)\, dt.$$

Note that the graph of the function f is given directly below. All the questions below are about the function g, NOT f.



(a) (2 points) Find $g(0)$. $= \int_0^0 f(t)\,dt = 0$

$\int_2^3 f(t)\,dt$

(b) (2 points) Find $g(3)$. $= \int_0^3 f(t)\,dt = 1 + \frac{1}{2} - \frac{1}{2} = 1$

$\int_0^1 f(t)\,dt \quad \int_1^2 f(t)\,dt$

(c) (2 points) Is the function g differentiable on $(-1,3)$?
Please choose and circle the correct option:

yes no

$g'(x) = f(x)$, because f is continuous

Note $-1 < x < 3$ only

(d) (2 points) Indicate all critical numbers for the function g on the interval $(-1, 3)$.

$g'(x) = f(x)$ so $g'(x) = 0$ when $x = 2$.

(e) (2 points) For which value of x does the function $g(x)$ take its maximum value on the interval $[-1, 3]$?

Look at 1ST derivative diagram

$g'(x)$: $($ $+++$ 0 $--$ $)$

x: -1 2 3

Max $g(x)$ at $x = 2$

(f) (2 points) What is the maximum value of the function $g(x)$ on the interval $[-1, 3]$?

$$g(2) = \int_0^2 f(t)\,dt = 1 + \frac{1}{2} = \frac{3}{2}$$

(g) (2 points) What is the minimum value of the function $g(x)$ on the interval $[-1, 3]$?

By Extreme value theorem, it would be at $x = -1, 2,$ or 3.

$$g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\frac{1}{2} \Leftarrow \text{MIN}$$

$g(2) = 3/2$

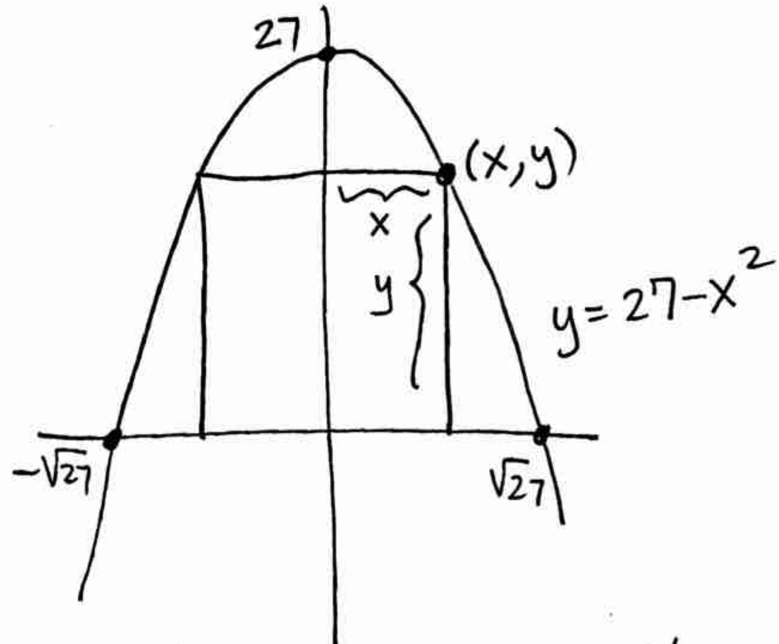
$g(3) = 1$.

(h) (2 points) Indicate the intervals where the function g is concave up.

$$g''(x) = f'(x) = \begin{cases} 1 & \text{for } -1 < x < 0 \\ 0 & \text{for } 0 < x < 1 \\ -1 & \text{for } 1 < x < 2 \end{cases}$$

So $g''(x) > 0$ for $-1 < x < 0$.

11. (9 points) Find the maximum area of a rectangle which has two vertices on the x-axis, and another two vertices on the parabola $y = 27 - x^2$ above x-axis.



Area is $A = 2xy$ with $0 \leq x \leq \sqrt{27}$

$$A = 2x(27 - x^2)$$
$$A = 54x - 2x^3$$
$$\frac{dA}{dx} = 54 - 6x^2 \overset{\text{set}}{=} 0$$
$$54 = 6x^2$$
$$9 = x^2$$
$$\pm 3 = x$$

$x = 3$ is only critical point in domain

1^{ST} derivative diagram

$\frac{dA}{dx}$: [+ + + 0 − − −]

x: 0 3 $\sqrt{27}$

A ↗ MAX ↘

so $x = 3$ is abs. max with area

$$A = 2 \cdot 3(27 - 9) = 108$$

12. Evaluate the integrals

(a) (6 points)

$$\int_0^{2\pi} \frac{5+\cos x}{3}\,dx$$

$$= \frac{1}{3}(5x+\sin x)\Big|_0^{2\pi}$$

$$= \frac{1}{3}\big((10\pi+0)-(0+0)\big) = \frac{10\pi}{3}$$

(b) (6 points)

$$\int \frac{x-5}{(x^2-10x)^2}\,dx$$

Try substitution

$u = x^2 - 10x$

$du = (2x-10)\,dx$

$= 2(x-5)\,dx$

$$= \int \frac{1}{2} \cdot \frac{2(x-5)\,dx}{(x^2-10x)^2}$$

$$= \int \frac{1}{2}\,\frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{1}{2}\,\frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{2}(x^2-10x)^{-1} + C$$

(c) (8 points)

$$\int_0^{\pi/3} 160(\cos x)^4 \sin x \, dx$$

Try sub
$u = \cos x$
$du = -\sin x \, dx$
$x = 0 \Rightarrow u = 1$
$x = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$

$$\int_0^{\pi/3} -160 (\cos x)^4 (-\sin x) dx$$

$$= \int_1^{1/2} -160 \, u^4 du$$

$$= -160 \frac{u^5}{5} \Big|_1^{1/2} = -32\left(\left(\tfrac{1}{2}\right)^5 - 1^5 \right)$$

$$= -32\left(\tfrac{1}{32} - 1 \right) = -1 + 32 = 31$$

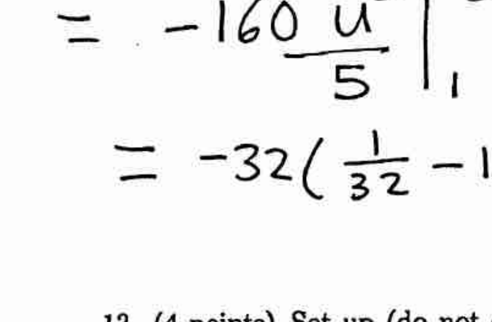
13. (4 points) Set up (do not compute) an integral for the area of the region bounded by the curves

$$y = \sin x, \quad y = \sin x \cos x$$

between $x = 0$ and $x = \pi$.

which is bigger?
Where do they intersect?

Set $\sin x = \sin x \cos x$
$0 = \sin x \cos x - \sin x$
$0 = \sin x (\cos x - 1)$
$\sin x = 0$ OR $\cos x = 1$
$x = 0, \pi$ $\quad$ $x = 0$.



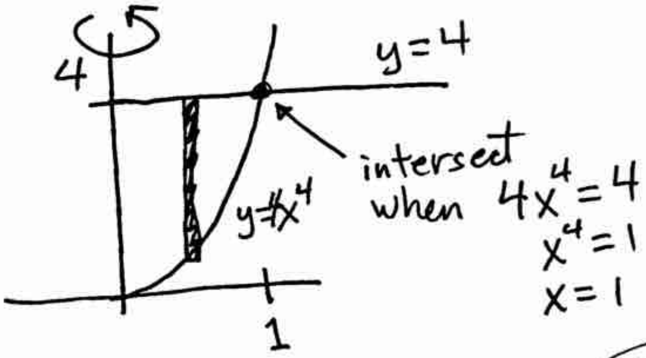
Intersect only at $x = 0, \pi$

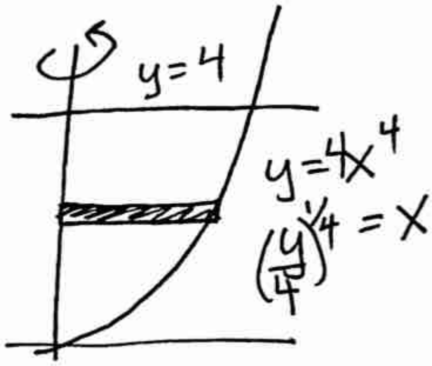
$$\text{Area} = \int_0^{\pi} (\sin x - \sin x \cos x) dx$$

14. (9 points) The region inside the first quadrant ($x \geq 0,\ y \geq 0$) bounded by

$$y = 4x^4,\ x = 0,\ y = 4$$

is rotated around the y-axis. Find the volume.





As x-problem

Get shell method.

$$V = 2\pi\int_0^1 \underbrace{(x-0)}_{\text{radius}}\underbrace{(4-4x^4)}_{\text{height}}\,dx$$

$$= 2\pi\int_0^1 (4x-4x^5)\,dx$$

$$= 2\pi\left(2x^2-\frac{4}{6}x^6\right)\Big|_0^1$$

$$= 2\pi\left(\left(2-\frac{2}{3}\right)-(0-0)\right)$$

$$= 2\pi\cdot\frac{4}{3} = \frac{8\pi}{3}$$

OR

As y-problem

$$V = \pi\int_0^4 \Big(\underbrace{\left(\left(\tfrac{y}{4}\right)^{\frac{1}{4}}-0\right)^2}_{\text{outer radius}} - \underbrace{(0-0)^2}_{\text{inner radius}}\Big)\,dy$$

$$= \pi\int_0^4 \left(\frac{y}{4}\right)^{\frac{1}{2}}dy = \pi\int_0^4 \frac{1}{2}y^{\frac{1}{2}}dy$$

$$= \pi\frac{1}{2}\frac{2}{3}y^{\frac{3}{2}}\Big|_0^4$$

$$= \pi\left(\frac{1}{2}\frac{2}{3}4^{3/2}-0\right)$$

$$= \pi\frac{1}{2}\frac{2}{3}\cdot 8$$

$$= \frac{1}{2}\frac{16\pi}{3} = \frac{8\pi}{3}$$