Math 241, Fall 2016, Final Exam

Name and section number:

Solutions
by
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Question	Points	Score
1	6	
2	6	
3	8	
4	12	
5	6	
6	8	
7	8	
8	18	
9	10	
10	9	
11	15	
12	6	
13	8	
Total:	120	

- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- Show all your work (unless for a multiple choice question).
- You do **not** need to simplify your answers.
- Good luck!

1. Choose the option that best describes the limit in each case.

(a) (2 points) $\lim_{x\to 2} \frac{x^3-2}{x^2+x+1}$.

Limit type $\frac{\to 8-2}{\to 7} = \frac{6}{7}$

$(a)\ 4/5, \quad (b)\ 5/6, \quad \boxed{(c)\ 6/7,} \quad (d)$ does not exist.

(b) (2 points) $\lim_{x\to\infty} \frac{4-7x^2}{(x+5)^2}$. $\frac{\cdot 1/x^2}{\cdot 1/x^2}$ Limit type $\frac{-\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{4}{x^2}-7}{\left(1+\frac{5}{x}\right)^2} = \frac{-7}{1^2} = -7$$

$(a)\ 4, \quad \boxed{(b)\ -7,} \quad (c)\ 5, \quad (d)$ does not exist.

(c) (2 points) $\lim_{x\to 3^+} \frac{x^2-2x+1}{x-3}$.

Limit type $\frac{\to 9-6+1}{\to 0} \quad \frac{\to 4}{\to 0^+} \to \infty$

Key is $x-3 \to 0^+$

$(a)\ 0, \quad (b)\ 1, \quad \boxed{(c)\ \infty,} \quad (d)\ -\infty.$

2. (6 points) Explain why the function $f(x) = x^4 - 4x^3 + 1$ has a zero in the interval $[0, 1]$. Be clear about which theorem, or theorems, you use in your explanation.

Intermediate Value Theorem,

f is continuous every where.

$f(0) = 1$

$f(1) = -2.$

Because $f(0) > 0 > f(1)$,

there for some $0 < c < 1$,

$f(c) = 0.$

3. (8 points) **Using the definition of the derivative as a limit**, compute $f'(4)$ if $f(x) = \sqrt{x}$.

(Warning: you will get no credit if you use the rules of differentiation).

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h} \cdot \frac{\sqrt{4+h}+\sqrt{4}}{\sqrt{4+h}+\sqrt{4}}$$

$$= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+\sqrt{4})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+\sqrt{4}}$$

$$= \frac{1}{\sqrt{4+0}+\sqrt{4}}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4}$$

4. Differentiate the following functions. You do not need to simplify your answers.

(a) (4 points) $f(x) = \frac{1+x^{1/3}}{1+x}$.

$$f'(x) = \frac{\frac{1}{3}x^{-2/3}(1+x) - (1+x^{1/3})\,1}{(1+x)^2}$$

(b) (4 points) $f(x) = (x^3+1)^3$.

$$f'(x) = 3(x^3+1)^2\, 3x^2$$

(c) (4 points) $f(x) = x\tan(x)$.

$$f'(x) = 1\tan(x) + x\sec^2(x)$$

5. (6 points) Use either linearization / linear approximation, or a differential together with the fact that $\sqrt{9} = 3$ to find an approximation to $\sqrt{10}$.

Let $f(x) = \sqrt{x}$.

Linearization at base point $a = 9$ yields:

$$f(x) = \sqrt{x} \Rightarrow f(9) = 3$$

$$f'(x) = \frac{1}{2} x^{-1/2} \Rightarrow f'(9) = \frac{1}{2} 9^{-1/2} = \frac{1}{6}$$

$$L(x) = 3 + \frac{1}{6}(x - 9).$$

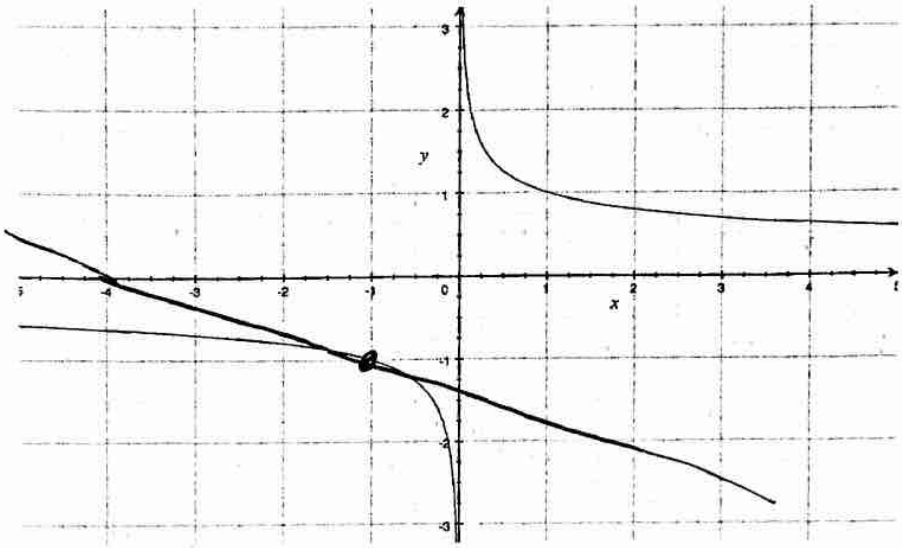
Then

$$\sqrt{10} = f(10) \approx L(10) = 3 + \frac{1}{6}(10-9)$$

$$= 3 + \frac{1}{6}$$

$$= \boxed{\frac{19}{6}}$$

6. The graph below is of the equation $y^3x = 1$.



(a) (6 points) Find an equation for the tangent line to this graph at the point $(-1, -1)$.

Implicit differentiation

$$3y^2y'x + y^3 = 0$$

$$3y^2y'x = -y^3$$

$$y' = \frac{-y^3}{3y^2x}$$

At $x=-1$, $y=-1$,

$$y' = \frac{-(-1)^3}{3(-1)^2(-1)} = -\frac{1}{3}$$

Tangent line is $y-(-1) = -\frac{1}{3}(x--1)$

Alternatively, solve for y

$$y^3 = x^{-1}$$

$$y = x^{-1/3}$$

$$y' = -\frac{1}{3}x^{-4/3}$$

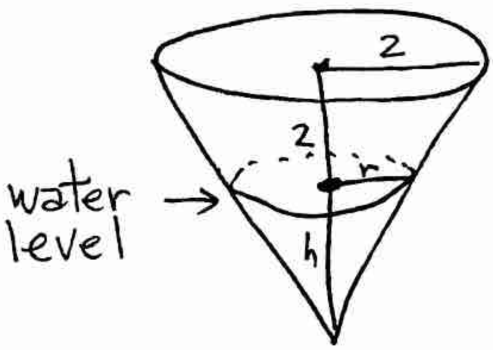
At $x=-1$

$$y' = -\frac{1}{3}$$

(b) (2 points) Sketch the tangent line computed above on the graph.

7. (8 points) A container is in the shape of a downward pointing cone. The container is 2 meters high, and has a radius at the top end of 2 meters. Water drips out at a rate of 1 cubic meter every hour. How fast is the height of water in the container changing at the point when the height is 1 meter?

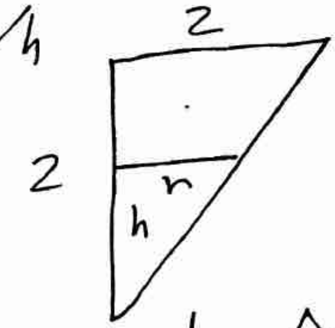
Hint: the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$*, where* r *is the radius, and* h *is the height.*



Given $\frac{dV}{dt} = -1 m^3/h$

volume of water is

$$V = \frac{1}{3}\pi r^2 h$$



By similar $\triangle$s,

$$\frac{2}{2} = \frac{r}{h}$$

$$h = r$$

$$V = \frac{1}{3}\pi r^2 (r)$$

$$V = \frac{1}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{1}{3}\pi 3r^2 \frac{dr}{dt}$$

At $h=1$, $\frac{dV}{dt} = -1$,

$$-1 = \pi (1)^2 \frac{dr}{dt}$$

$$-\frac{1}{\pi} = \frac{dr}{dt}$$

Water falling at $\frac{1}{\pi}$ m/h.

8. Consider the function $f(x) = x^3 - 2x^2$.

(a) (4 points) Find all critical points of f (just give the x value(s)), and classify them as local minima, local maxima, or neither. Justify your answer for full credit.

$$f'(x) = 3x^2 - 4x \overset{\text{SET}}{=} 0$$

$$x(3x-4) = 0$$

Critical points: $x=0 \quad x = 4/3$

$f'(x)$: $++$ | 0 at $x = 0$ | $--$ | 0 at $x = 4/3$ | $++$

f Conclusions: $\nearrow$ LOCAL MAX $\searrow$ LOCAL MIN $\nearrow$

(b) (4 points) Find all inflection points of f (just give the x value(s)). Justify your answer for full credit.

$$f''(x) = 6x - 4 \overset{set}{=} 0$$

$$x = 2/3$$

$f''(x)$: $--$ | 0 at $x = 2/3$ | $++$

f Conclusions: $\frown$ $\smile$ IP at $x = 2/3$

(c) (4 points) On what intervals is f concave up, and on which is it concave down?

Concave ~~up~~ down on $(-\infty, 2/3)$.

Concave ~~down~~ up on $(\frac{2}{3}, \infty)$

(d) (4 points) Find the absolute minimum and maximum of f on the interval $[0, 2]$, and where they occur.

Use E.V.T. Compare critical points and endpoints

$x = 0 \quad f(0) = 0$

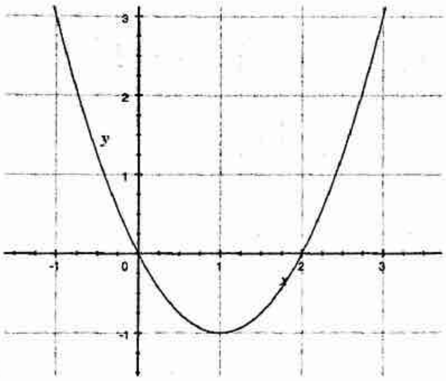
$x = 4/3 \quad f(\frac{4}{3}) = (\frac{4}{3})^3 - 2(\frac{4}{3})^2 = \frac{64}{27} - \frac{96}{27} = -\frac{32}{27}$

$x = 2 \quad f(2) = 8 - 8 = 0.$

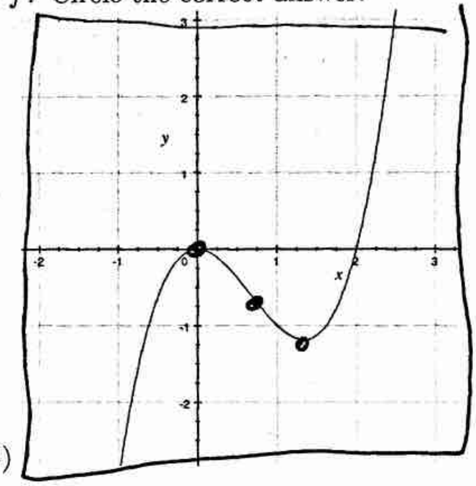
Max of 0 at $x = 0, 2$

Min of $-32/27$ at $x = 4/3$

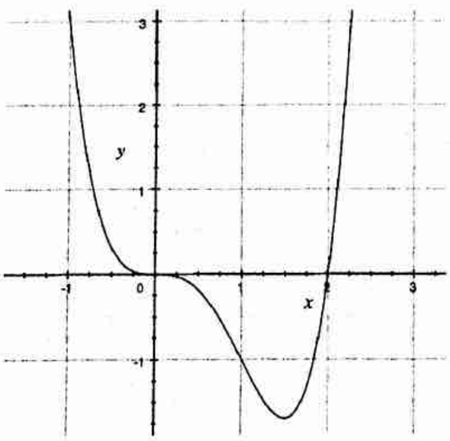
(e) (2 points) Which of the following is the graph of f? Circle the correct answer.



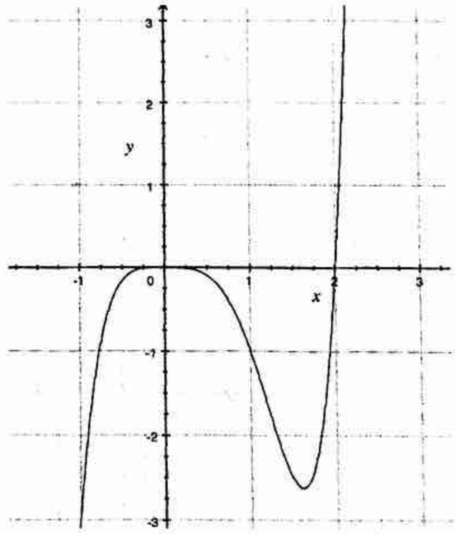
(A)



(B)

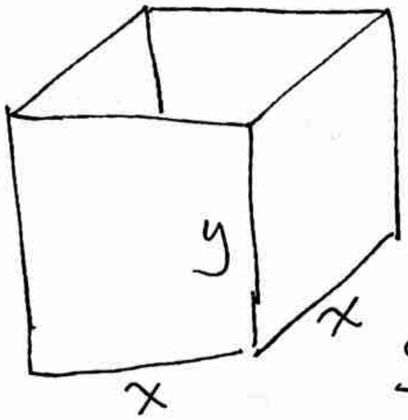


(C)



(D)

9. (10 points) A rectangular box with a base and sides, but no top, is to be made with a volume of 1 cubic foot. The base of the box is to be square. Find the dimensions of the box which give rise to the minimum surface area.



Constraint (volume) $x^2 y = 1$

$\Rightarrow y = 1/x^2$

Surface area is

$$S = \overset{\text{BOTTOM}}{x^2} + \overset{\text{4 sides}}{4xy}$$

$$S = x^2 + 4x\left(\frac{1}{x^2}\right)$$

$$S = x^2 + \frac{4}{x}. \quad \text{Domain } x > 0$$

$$\frac{dS}{dx} = 2x - \frac{4}{x^2} = \frac{2x^3 - 4}{x^3} \leftarrow \text{set } 2x^3 - 4 = 0$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

$\frac{dS}{dx}$: $- - -$ 0 $+ + +$

x: $($ 0 ... $\sqrt[3]{2}$ ...

Conclusions on S: $\rightarrow$ ABS MIN

Min at $x = \sqrt[3]{2}$ ft,

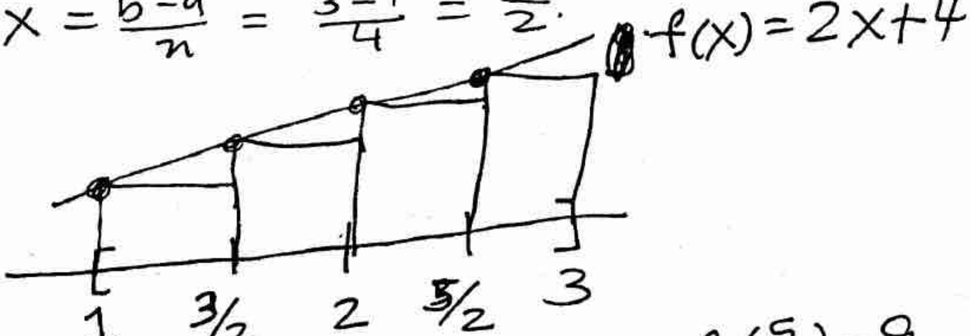
$$y = \frac{1}{(\sqrt[3]{2})^2} \text{ ft.}$$

10. Let $f(x) = 2x + 4$.

(a) (6 points) Compute a Riemann sum for this function to estimate the integral $\int_1^3 f(x)dx$. Use four equal length intervals, and use the left endpoint of each interval to give the height of the rectangles that contribute to the Riemann sum.

$a=1,\ b=3,\ n=4.$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}.$



$f(1)=6 \quad f(\frac{3}{2})=7 \quad f(2)=8 \quad f(\frac{5}{2})=9.$

Riemann sum $= (6+7+8+9)\frac{1}{2} = \boxed{15}$

(b) (3 points) What is the difference between your estimate from part (a) and the actual value of $\int_1^3 f(x)dx$?

$\int_1^3 f(x)dx = \int_1^3 (2x+4)dx$ (OR use geometry)

$= x^2 + 4x \Big|_1^3$

$= (9+12) - (1+4) = 21 - 5 = 16.$

The difference is $|15-16| = 1.$

11. Compute each of the following integrals.

(a) (5 points) $\int_0^3 (1+x^2)dx$.

$$x + \frac{1}{3}x^3 \Big|_0^3$$

$$= (3+9) - (0+0) = \boxed{12}$$

(b) (5 points) $\int_0^1 x(1+x^2)^3 dx$.

Sub $u = 1 + x^2$ $\quad$ $x=0 \Rightarrow u=1$

$du = 2x\,dx$ $\quad$ $x=1 \Rightarrow u=2$

$\frac{1}{2}du = x\,dx$

$$\int_1^2 u^3 \tfrac{1}{2} du$$

$$= \frac{1}{2}\frac{1}{4}u^4\Big|_1^2 = \frac{1}{8}(16-1) = \boxed{\frac{15}{8}}$$

[OR $= \frac{1}{8}(1+x^2)^4 \Big|_0^1$ using original limits of integration]

(c) (5 points) $\int (\sqrt{x^3} - \cos(2x))dx$.

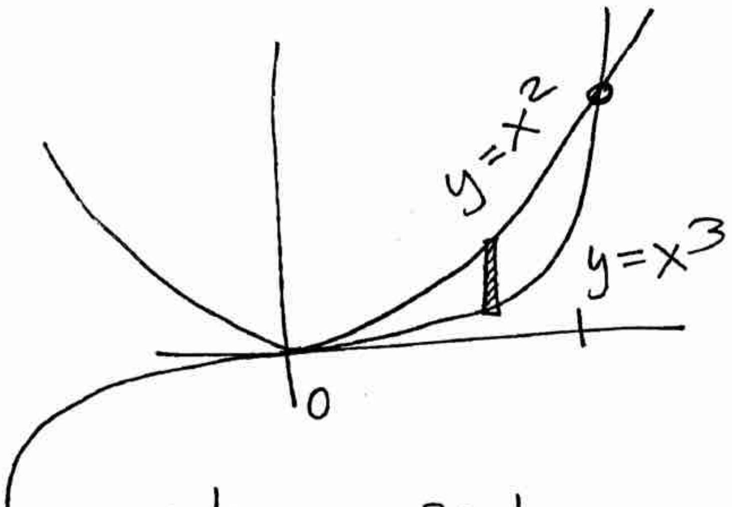
$$\int x^{3/2}dx - \int \cos(2x)\,dx$$

$$= \frac{2}{5}x^{5/2} - \frac{1}{2}\sin(2x) + C$$

by formula,

OR can sub $u = 2x$.

12. (6 points) Find the area of the bounded region between the graphs $y = x^3$ and $y = x^2$.



They intersect where $x^3 = x^2$

$$x^3 - x^2 = 0$$

$$x^2(x-1) = 0$$

$$x = 0, \quad x = 1.$$

For $0 < x < 1$, we have $x^3 < x^2$ (just test $x = \frac{1}{2}$ for example)

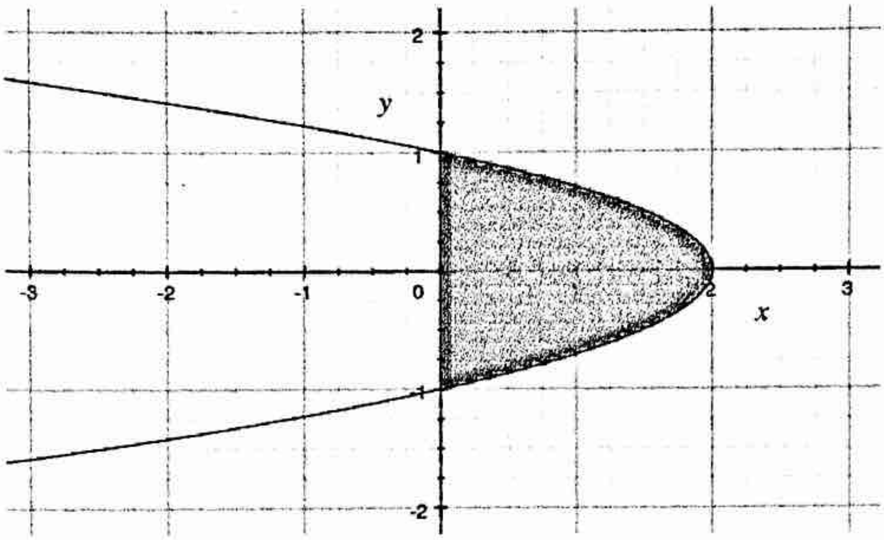
$$A = \int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - (0-0)$$

$$= \boxed{\frac{1}{12}}$$

13. (8 points) The following picture shows the region between the graph of $x = 2 - 2y^2$ and the y axis.



Find the volume of the shape obtained by rotating this region about the y axis. Use any method you like.

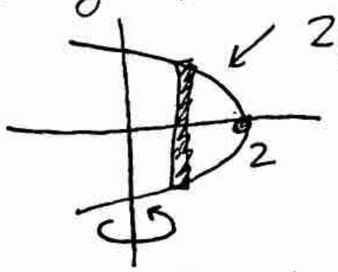
Using dx,

$x = 2 - 2y^2$

$2y^2 = 2 - x$

$y^2 = \frac{2-x}{2}$

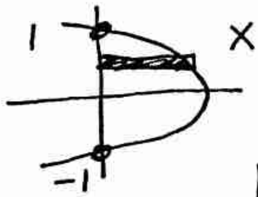
$y = \pm\sqrt{\frac{2-x}{2}}$



Get shell method.

$$V = 2\pi \int_0^2 \underbrace{(x - 0)}_{\text{radius}} \left(\sqrt{\frac{2-x}{2}} - \left(-\sqrt{\frac{2-x}{2}}\right)\right) dx$$

Using dy,



$x = 2 - 2y^2$

Get washers

$$V = \pi \int_{-1}^{1} (2 - 2y^2)^2 \, dy$$

Easier to evaluate

$$V = \pi \int_{-1}^{1} (4 - 8y^2 + 4y^4) \, dy$$

$$= \pi \left(4y - \frac{8}{3}y^3 + \frac{4}{5}y^5\right)\Big|_{-1}^{1} = \pi\left(\left(4 - \frac{8}{3} + \frac{4}{5}\right) - \left(-4 + \frac{8}{3} - \frac{4}{5}\right)\right)$$

$$= \pi\left(\frac{64}{15}\right).$$