Math 241/251A Final Exam

Your name: ______________________

Select your instructor and section time:

- ☐ Luca Candelori (Thursday 1:30pm)
- ☐ Luca Candelori (Friday 10:30am)
- ☐ Erik Guentner (Wednesday 8:30am)
- ☐ Asaf Hadari (Thursday 10:30am)
- ☐ Piper Harron (Thursday 12:00pm)
- ☐ Piper Harron (Friday 9:30am)
- ☐ Mushfeq Khan (Wednesday 10:30)
- ☐ Mushfeq Khan (Wednesday 1:30pm)
- ☐ Daisuke Takagi (Thursday 1:30pm)
- ☐ Daisuke Takagi (Friday 11:30am)
- ☐ David Webb (Friday 8:30)
- ☐ David Yuen (Thursday 8:30am)
- ☐ David Yuen (Thursday 10:30am)

1 (16)	
2 (4)	
3 (10)	
4 (15)	
5 (3)	
6 (10)	
7 (12)	
8 (10)	
9 (10)	
10 (10)	
11 (18)	
12 (6)	
13 (6)	
14 (10)	
TOTAL (140)	

Justify all your work. Answers without suitable justification will receive no credit.

Problem 1. (16 points) Evaluate the following limits. If the limit is infinite, indicate whether it is ∞ or $-\infty$. (Do not use l'Hôspital's rule.)

a. $\displaystyle\lim_{x\to\infty} \frac{x^3+x}{3x^3-1}$

type: ∞/∞

$$\lim_{x\to\infty} \frac{x^3+x}{3x^3-1} = \lim_{x\to\infty} \frac{x^3/x^3 + x/x^3}{3x^3/x^3 - 1/x^3} = \lim_{x\to\infty} \frac{1+1/x^2}{3-1/x^3} = \frac{1+0}{3-0} = \boxed{\frac{1}{3}}$$

b. $\displaystyle\lim_{x\to 2^+} \frac{4-2x}{|2x-4|}$

Type: $0/0$

$2^+ \Rightarrow x>2 \Rightarrow 2x-4>0 \Rightarrow |2x-4| = 2x-4$

$$\lim_{x\to 2^+} \frac{4-2x}{|2x-4|} = \lim_{x\to 2^+} \frac{4-2x}{2x-4} = \lim_{x\to 2^+} \frac{-\cancel{(2x-4)}}{\cancel{2x-4}} = \lim_{x\to 2^+} -1 = \boxed{-1}$$

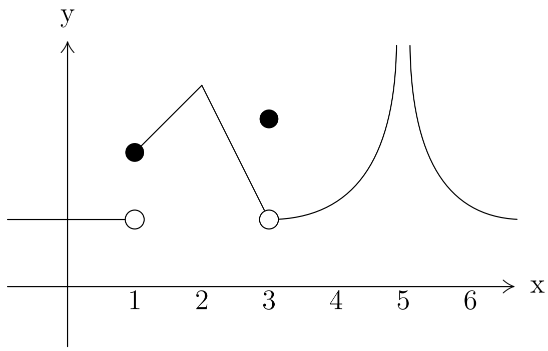
c. $\displaystyle\lim_{x\to 0} \frac{x^2-4}{x-2}$

$$\lim_{x\to 0} \frac{x^2-4}{x-2} = \frac{0-4}{0-2} = \frac{-4}{-2} = \boxed{2}$$

d. $\displaystyle\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta}$

$$\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta} \cdot \frac{2}{2} = \lim_{\theta\to 0} \frac{\sin(2\theta)}{2\theta} \cdot 2 = 2 \lim_{\theta\to 0} \frac{\sin(2\theta)}{2\theta} = 2 \cdot 1 = \boxed{2}$$

Problem 2. (4 points) Below is the graph of $y = f(x)$.



a. Find the values of a for which $\lim_{x \to a^+} f(x)$ is infinite or does not exist.

a=5

b. Find the values of a for which $\lim_{x \to a^-} f(x)$ is infinite or does not exist.

a=5

c. Find the values of a for which $\lim_{x \to a} f(x)$ is infinite or does not exist.

a=1, a=5

d. Find the values of a for which f is not continuous at $x = a$.

a=1, a=3, a=5

Problem 3. (10 points)

a. State the definition of $f'(x)$ as a limit.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

b. Let $f(x) = \sqrt{2x}$. Use the definition of the derivative to calculate $f'(2)$ (do not use differentiation rules).

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{2(2+h)}-\sqrt{2(2)}}{h} = \lim_{h\to 0} \frac{\sqrt{4+2h}-\sqrt{4}}{h} = \lim_{h\to 0} \frac{\sqrt{4+2h}-2}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+2h}-2}{h} \cdot \frac{\sqrt{4+2h}+2}{\sqrt{4+2h}+2} = \lim_{h\to 0} \frac{(\sqrt{4+2h}-2)(\sqrt{4+2h}+2)}{h(\sqrt{4+2h}+2)} = \lim_{h\to 0} \frac{\cancel{4}+2h-\cancel{4}}{h(\sqrt{4+2h}+2)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{4+2h}+2)} = \lim_{h\to 0} \frac{2}{\sqrt{4+2h}+2} = \frac{2}{\sqrt{4+0}+2} = \frac{2}{2+2} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

Alternatively:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} \cdot \frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{2x+2h}-\sqrt{2x})(\sqrt{2x+2h}+\sqrt{2x})}{h(\sqrt{2x+2h}+\sqrt{2x})} = \lim_{h\to 0} \frac{(\cancel{2x}+2h)-(\cancel{2x})}{h(\sqrt{2x+2h}+\sqrt{2x})} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2x+2h}+\sqrt{2x})}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h}+\sqrt{2x}} = \frac{2}{\sqrt{2x+0}+\sqrt{2x}} = \frac{2}{\sqrt{2x}+\sqrt{2x}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2x}} = \frac{1}{\sqrt{2x}}$$

so $f'(2) = \frac{1}{\sqrt{2(2)}} = \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$

Problem 4. (15 points) Find the following derivatives using differentiation rules. You do not have to simplify your answers.

a. $\dfrac{d}{dx}\Big(\sin(x)\tan(x^2)\Big)$

$$\frac{d}{dx}(\sin x \tan(x^2)) = \left(\frac{d}{dx}\sin x\right)\tan(x^2) + \sin x\left(\frac{d}{dx}\tan(x^2)\right)$$
$$= \cos x \tan(x^2) + \sin x\left(\sec^2(x^2)\frac{d}{dx}(x^2)\right)$$
$$= \boxed{\cos x \tan(x^2) + \sin x \sec^2(x^2)(2x)}$$

b. $\dfrac{d}{dx}\Big(\dfrac{x}{x^3-1}\Big)$

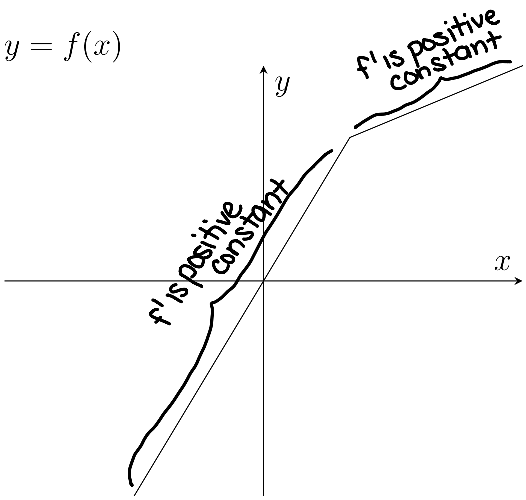
$$\frac{d}{dx}\left(\frac{x}{x^3-1}\right) = \frac{(x^3-1)\,{}^{d}/_{dx}(x) - x\,{}^{d}/_{dx}(x^3-1)}{(x^3-1)^2} = \boxed{\frac{(x^3-1)(1) - x(3x^2)}{(x^3-1)^2}}$$

c. $\dfrac{d}{dx}\Big(\sqrt{\cos(2x+1)}\Big)$

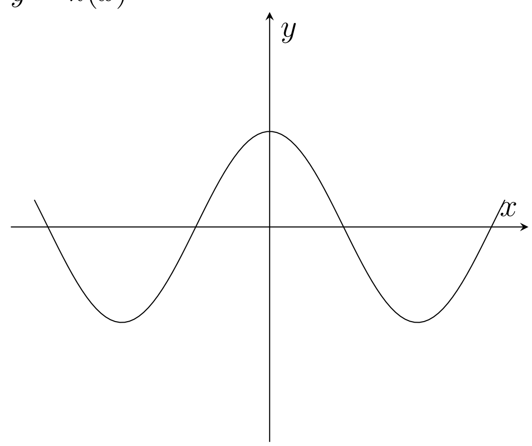
$$\frac{d}{dx}\left(\sqrt{\cos(2x+1)}\right) = \frac{d}{dx}\left((\cos(2x+1))^{1/2}\right) = \frac{1}{2}(\cos(2x+1))^{-1/2}\frac{d}{dx}(\cos(2x+1))$$
$$= \frac{1}{2}(\cos(2x+1))^{-1/2}\left(-\sin(2x+1)\frac{d}{dx}(2x+1)\right) = \boxed{\frac{1}{2}(\cos(2x+1))^{-1/2}(-\sin(2x+1)(2))}$$

Problem 5. (3 points) Decide which function on the left has which derivative on the right.

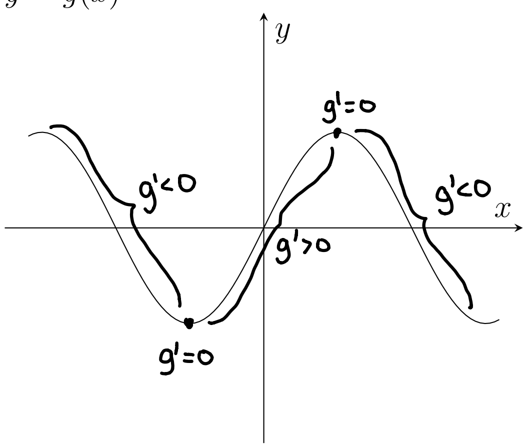
1. $y = f(x)$



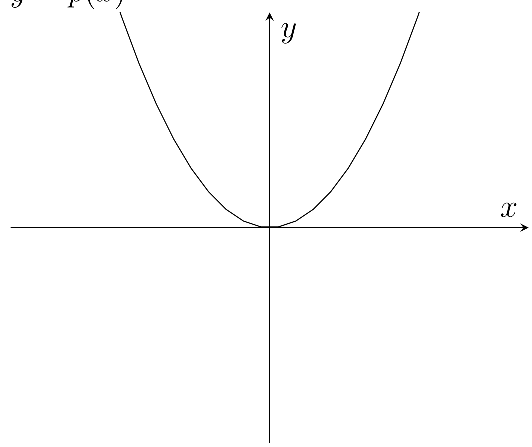
a. $y = k(x)$



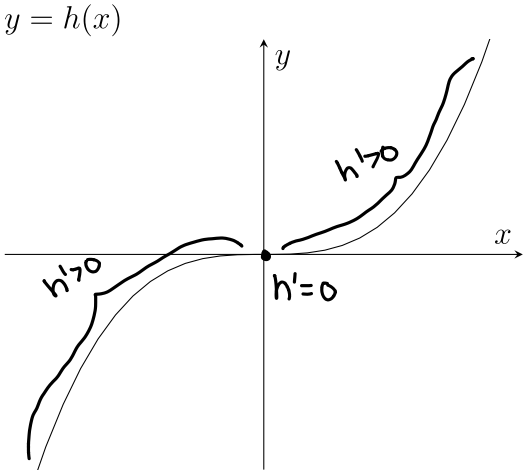
2. $y = g(x)$



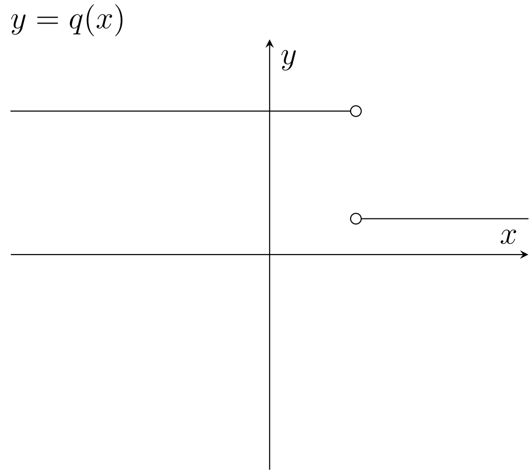
b. $y = p(x)$



3. $y = h(x)$

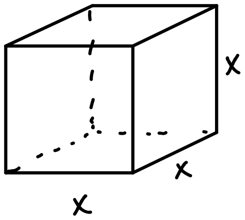


c. $y = q(x)$



1. $f'(x) =$ ☐ $k(x)$ ☐ $p(x)$ ☑ $q(x)$
2. $g'(x) =$ ☑ $k(x)$ ☐ $p(x)$ ☐ $q(x)$
3. $h'(x) =$ ☐ $k(x)$ ☑ $p(x)$ ☐ $q(x)$

Problem 6. (12 points) A cube of ice is melting evenly at a rate of 12 $\mathrm{cm^3/hour}$. How fast is the side length of the cube changing when the side length is 4 cm?



$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

Given $\frac{dV}{dt} = -12$ and $x = 4$, want $\frac{dx}{dt}$

$-12 = 3(4)^2 \frac{dx}{dt} \Rightarrow -12 = 48 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-12}{48} = \boxed{\frac{-1}{4} \text{ cm/hr}}$

Problem 7. (12 points) Let $f(x) = x^4 - 2x^3$.

a. Find the critical points of f and classify them as local minima, local maxima or neither.

$f'(x) = 4x^3 - 6x^2$

f' never undefined

critical pts

$f'(x) = 0 \Rightarrow 4x^3 - 6x^2 = 0 \Rightarrow 2x^2(2x-3) = 0 \Rightarrow x = 0, 3/2$

f': $-$ | 0 | $-$ | $3/2$ | $+$

f has a local min at $x = 3/2$

f has neither at $x = 0$

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - 2\left(\frac{27}{8}\right) = \frac{81}{16} - \frac{27}{4} = \frac{81-108}{16} = \frac{-27}{16}$

$f(0) = 0 - 0 = 0$

b. On which intervals is f increasing and on which is f decreasing?

f is decreasing on $(-\infty, 0), (0, 3/2)$

f is increasing on $(3/2, \infty)$

c. Find the inflection points of f and the intervals on which it is concave up and those on which it is concave down.

$f''(x) = 12x^2 - 12x$

f'' never undefined

$f''(x) = 0 \Rightarrow 12x^2 - 12x = 0 \Rightarrow 12x(x-1) = 0 \Rightarrow x = 0, 1$

f'': $+$ | 0 | $-$ | 1 | $+$

f is concave up on $(-\infty, 0), (1, \infty)$

f is concave down on $(0, 1)$

Inflection points at $x = 0$ $(0,0)$ and $x = 1$ $(1,-1)$

d. Find the absolute maximum and the absolute minimum of f on the interval $[-1, 1]$.

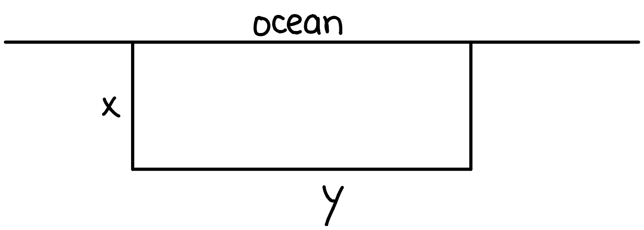
$f(-1) = (-1)^4 - 2(-1)^3 = 1 + 2 = 3 \leftarrow$ abs max

$f(1) = (1)^4 - 2(1)^3 = -1 \leftarrow$ abs min

$f(0) = 0$

$\cancel{f\left(\frac{3}{2}\right) = \frac{-27}{16}}$ not in interval

Problem 8. (10 points) A rectangular section of a beach reserved for monk seals is being fenced off on three sides (the fourth side borders on the ocean and does not require fencing). If there are $100m$ of fencing, what is the largest area that can be fenced off?



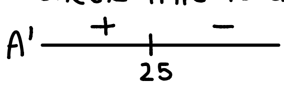
$100 = 2x + y \Rightarrow y = 100 - 2x$

$A = xy = x(100 - 2x) = 100x - 2x^2$

$A'(x) = 100 - 4x$

$0 = 100 - 4x \Rightarrow 4x = 100 \Rightarrow x = 25$

Check this is a max



max at $x = 25$

$A(25) = 25(100 - 2(25)) = 25(50) = \boxed{1250 m^2}$

Problem 9. (10 points) Find an equation for the tangent line to the curve $x^2y^2 = 9$ at the point $(3, -1)$.

$x^2y^2=9 \Rightarrow 2xy^2+x^2\left(2y\frac{dy}{dx}\right)=0$

plug in $x=3$, $y=-1$

$2(3)(-1)^2+(3)^2(2)(-1)\frac{dy}{dx}=0$

$\Rightarrow 6-18\frac{dy}{dx}=0 \Rightarrow 18\frac{dy}{dx}=6 \Rightarrow \frac{dy}{dx}=\frac{6}{18}=\frac{1}{3}$

so $m=\frac{1}{3}$

$y-(-1)=\frac{1}{3}(x-3) \Rightarrow \boxed{y+1=\frac{1}{3}(x-3)}$

Problem 10. (10 points) Show that $f(x) = 2x - \cos(x)$ has exactly one zero in the interval $[-\pi, \pi]$.

a. Show that $f(x)$ has a zero.

$f(-\pi)=2(-\pi)-\cos(-\pi) = -2\pi-(-1)=-2\pi+1<0$
$f(\pi)=2\pi-\cos(\pi)=2\pi-(-1)=2\pi+1>0$

f is continuous so by the intermediate value theorem there is a zero in $[-\pi,\pi]$

b. Use Rolle's Theorem to show that it has exactly one zero.

f is differentiable on $(-\pi,\pi)$ so by the mean value theorem (or Rolle's Theorem) if there is more than one zero then $f'(x)=0$ at some point in $[-\pi,\pi]$

$f'(x)=2+\sin x$
$2+\sin x=0 \Rightarrow \sin x=-2$ which is impossible $(-1\leq \sin x\leq 1)$

Thus f must have exactly one zero

Problem 11. (18 points) Evaluate the following integrals.

a. $\displaystyle\int_0^1 2x\sqrt{x^2+3}\,dx$

$u = x^2+3 \Rightarrow du = 2x\,dx \Rightarrow dx = \dfrac{du}{2x}$

new bounds: $x=0 \Rightarrow u=3$
$x=1 \Rightarrow u=4$

$$\int_0^1 2x\sqrt{x^2+3}\,dx = \int_3^4 \cancel{2x}\sqrt{u}\,\frac{du}{\cancel{2x}} = \int_3^4 u^{1/2}\,du = \frac{u^{3/2}}{3/2}\Big|_3^4 = \frac{2}{3}u^{3/2}\Big|_3^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(3)^{3/2}$$

$$= \frac{2}{3}(8) - 2\sqrt{3} = \boxed{\frac{16}{3} - 2\sqrt{3}}$$

b. $\displaystyle\int \sin^2(x)\cos(x)\,dx$

$u = \sin x \Rightarrow du = \cos x\,dx \Rightarrow dx = \dfrac{du}{\cos x}$

$$\int \sin^2 x \cos x\,dx = \int u^2 \cancel{\cos x}\,\frac{du}{\cancel{\cos x}} = \int u^2\,du = \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$$

c. Find $f(x)$ such that $f'(x) = \dfrac{2}{x^2}$ and $f(1) = 0$.

$$f(x) = \int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = 2\left(\frac{x^{-1}}{-1}\right) + C = \frac{-2}{x} + C$$

$$f(1) = \frac{-2}{1} + C = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 2$$

so $\boxed{f(x) = \dfrac{-2}{x} + 2}$

Problem 12. (6 points) Setup an integral for the area between the curve $y = x^2 + 2x + 1$ and the line $y = x + 1$. You do not need to evaluate the integral.

Find intersections:

$x^2+2x+1 = x+1 \Rightarrow x^2+x = 0 \Rightarrow x(x+1)=0 \Rightarrow x=0,-1$

Check which is bigger

$\left(\frac{-1}{2}\right)^2 + 2\left(\frac{-1}{2}\right)+1 = \frac{1}{4} - 1 + 1 = \frac{1}{4}$

$\left(\frac{-1}{2}\right)+1 = \frac{1}{2}$

so $x+1 > x^2+2x+1$

$$\boxed{A = \int_{-1}^{0} \left[(x+1) - (x^2+2x+1)\right] dx}$$

Problem 13. (6 points) Estimate $\int_{-1}^{2} (x^2 + 1)\, dx$ with a Riemann sum using left endpoints of 3 equal subintervals.

$f(x) = x^2+1$

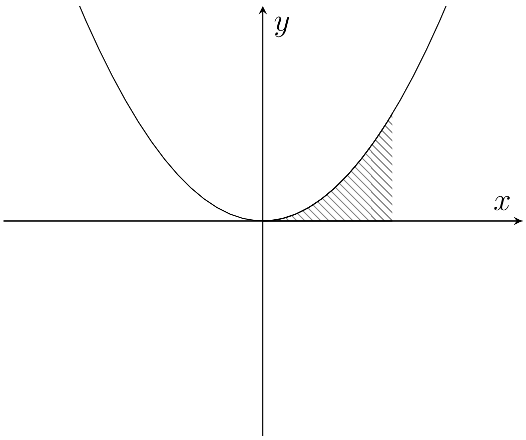
$\Delta x = \frac{2-(-1)}{3} = \frac{3}{3} = 1$

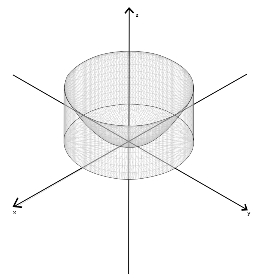
-1 0 1 2

$\int_{-1}^{2} (x^2+1)dx \approx (1)f(-1) + (1)f(0) + (1)f(1) = ((-1)^2+1) + ((0)^2+1) + ((1)^2+1)$

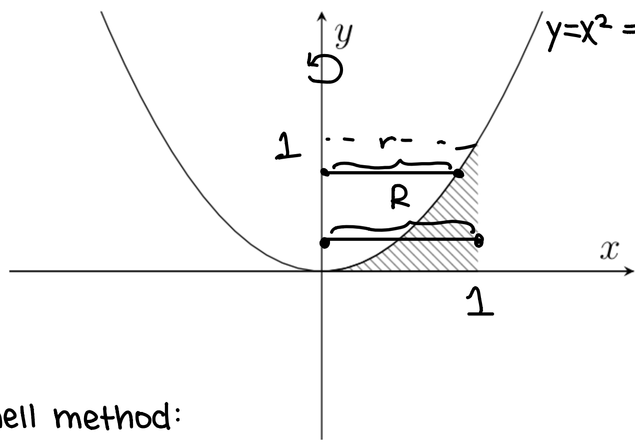
$= 1+1+0+1+1+1 = \boxed{5}$

Problem 14. (10 points) Consider the region between $y = x^2$, the x-axis and the line $x = 1$. Find the volume of the solid that is formed by rotating that region around the y-axis.





washer method:



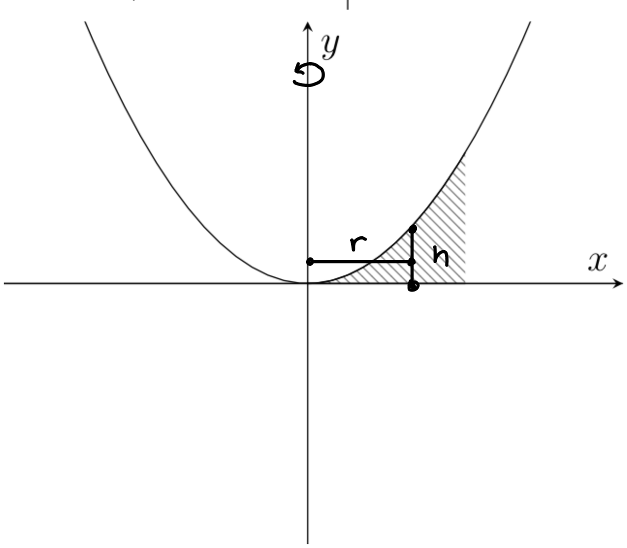
$y=x^2 \Rightarrow x=\sqrt{y}$

$r=\sqrt{y}$, $R=1$

$V=\int_0^1 \pi(1^2-\sqrt{y}^2)\,dy=\int_0^1\pi(1-y)\,dy=\pi(y-\frac{y^2}{2})\Big|_0^1$

$=\pi\left((1-\frac{1}{2})-(0-\frac{0}{2})\right)=\pi(1-\frac{1}{2})=\pi(\frac{1}{2})=\boxed{\frac{\pi}{2}}$

shell method:



$h=x^2$, $r=x$

$V=\int_0^1 2\pi x(x^2)\,dx=\int_0^1 2\pi x^3\,dx$

$=\frac{2\pi x^4}{4}\Big|_0^1=\frac{\pi x^4}{2}\Big|_0^1$

$=\frac{\pi(1)^4}{2}-\frac{\pi(0)^4}{2}=\boxed{\frac{\pi}{2}}$