Math 241, Fall 2018, Final Exam

Name and section number:

Instructor name:

Question	Points	Score
1	20	
2	6	
3	4	
4	20	
5	6	
6	6	
7	10	
8	10	
9	12	
10	10	
11	18	
12	8	
13	10	
14	10	
Total:	150	

- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work.
- You do **not** need to simplify your answers.
- Good luck!

1. Calculate the following limits. **Do not** use L'Hospital's rule. If the limit is positive or negative infinity, say which.

(a) (5 points) $\lim\limits_{x\to 3} \frac{x^2-4x+3}{x^2-9}$. $\xrightarrow[\text{in}]{\text{plug}} \frac{0}{0}$ more work!

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+3)} = \lim_{x\to 3} \frac{x-1}{x+3} = \frac{3-1}{3+3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

(b) (5 points) $\lim\limits_{x\to 2^+} \frac{(x+1)^2}{2-x}$.

$$\lim_{x\to 2^+} \frac{(x+1)^2}{2-x} \longrightarrow \frac{3^2}{0^-} \longrightarrow \boxed{-\infty}$$

(c) (5 points) $\lim\limits_{x\to 0} \frac{\sin 3x}{2x(x-3)}$. $\rightarrow \frac{0}{0}$ more work!

$$\lim_{x\to 0} \frac{\sin(3x)}{2x(x-3)} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{1}{2(x-3)} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{3}{2(x-3)}$$

$$= 1 \cdot \left(\frac{3}{2(-3)}\right) = 1\left(\frac{-1}{2}\right) = \boxed{\frac{-1}{2}}$$

(d) (5 points) $\lim\limits_{x\to\infty} \frac{x^2+1}{2x^2+\sin x}$. $\rightarrow \frac{\infty}{\infty}$ more work!

$$-1 \le \sin x \le 1 \Rightarrow 2x^2-1 \le 2x^2+\sin x \le 2x^2+1 \qquad \left(\begin{array}{c} x\to\infty \text{ so can assume} \\ 2x^2-1>0 \end{array}\right)$$

$$\Rightarrow \frac{1}{2x^2-1} \ge \frac{1}{2x^2+\sin x} \ge \frac{1}{2x^2+1} \Rightarrow \frac{x^2+1}{2x^2-1} \ge \frac{x^2+1}{2x^2+\sin x} \ge \frac{x^2+1}{2x^2+1}$$

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-1} = \lim_{x\to\infty} \frac{x^2/x^2+1/x^2}{2x^2/x^2-1/x^2} = \lim_{x\to\infty} \frac{1+1/x^2}{2-1/x^2} = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2+1} = \lim_{x\to\infty} \frac{x^2/x^2+1/x^2}{2x^2/x^2+1/x^2} = \lim_{x\to\infty} \frac{1+1/x^2}{2+1/x^2} = \frac{1}{2}$$

So by squeeze theorem

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2+\sin x} = \boxed{\frac{1}{2}}$$

2. (6 points) **Using the definition of the derivative as a limit**, compute $f'(2)$ if $f(x) = \sqrt{2x+1}$. (Warning: you will get no credit if you use the rules of differentiation).

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{2(2+h)+1} - \sqrt{2(2)+1}}{h} = \lim_{h\to 0} \frac{\sqrt{4+2h+1} - \sqrt{5}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{5+2h}-\sqrt{5}}{h} \cdot \frac{\sqrt{5+2h}+\sqrt{5}}{\sqrt{5+2h}+\sqrt{5}} = \lim_{h\to 0} \frac{(5+2h)-5}{h(\sqrt{5+2h}+\sqrt{5})} = \lim_{h\to 0} \frac{\cancel{5}+2h-\cancel{5}}{h(\sqrt{5+2h}+\sqrt{5})}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{5+2h}+\sqrt{5})} = \lim_{h\to 0} \frac{2}{\sqrt{5+2h}+\sqrt{5}} = \frac{2}{\sqrt{5}+\sqrt{5}} = \frac{2}{2\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}}$$

3. (4 points) Let $f(x) = \begin{cases} Ax & x \leq -1 \\ x^2 - 3Ax + 3 & x > -1 \end{cases}$.

For which values of A is the function f continuous?

want $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (Ax) = A(-1) = -A$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x^2-3Ax+3) = (-1)^2-3A(-1)+3 = 1+3A+3 = 4+3A$

$f(-1) = A(-1) = -A$

want $-A = 4+3A \Rightarrow -4A = 4 \Rightarrow \boxed{A = -1}$

4. Differentiate the following functions. You do not need to simplify your answers.

(a) (5 points) $f(x) = \dfrac{x^3}{2x^2 - 5}$ quotient rule

$$f'(x) = \boxed{\frac{(2x^2-5)(3x^2)-(x^3)(4x)}{(2x^2-5)^2}}$$

(b) (5 points) $g(x) = 3x\sin(x^2)$

$$g'(x) = \boxed{3\sin(x^2) + 3x\cos(x^2)(2x)}$$

(product and chain rule)

(c) (5 points) $h(x) = (\sqrt{x} - \dfrac{1}{x^4} + \pi^3)^5$

$h(x) = (x^{1/2} - x^{-4} + \pi^3)^5$

$h'(x) = 5(x^{1/2} - x^{-4} + \pi^3)^4\left(\frac{1}{2}x^{-1/2} + 4x^{-5}\right)$

(Chain rule)

(d) (5 points) $R(x) = \displaystyle\int_1^{2x} (t + t^4)^3\, dt$

let $u = 2x \Rightarrow \dfrac{du}{dx} = 2$

$$R'(x) = \left(\frac{d}{du}\int_1^u (t+t^4)^3 dt\right)\frac{du}{dx} = (u+u^4)^3(2) = \boxed{2(2x+(2x)^4)^3}$$

5. (a) (5 points) Use linear approximation and the fact that $\sqrt{4} = 2$ to find an approximation to $\sqrt{3.99}$.

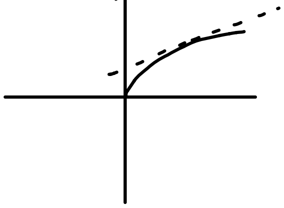
$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad a = 4$

$L(x) = f(a) + f'(a)(x-a) = \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4) = 2 + \frac{1}{4}(x-4)$

$L(3.99) = 2 + \frac{3.99-4}{4} = \boxed{2 - \frac{0.01}{4}}$

(b) (1 point) Is the exact value for $\sqrt{3.99}$ more or less than the number you calculated in the previous part?

Graph of $y = \sqrt{x}$



concave down so tangent line lies above

$\Rightarrow$ $\boxed{\text{less}}$

6. (6 points) Find an equation for the tangent line to the graph of $x^3 - 3x^2y + 2xy^2 = 0$ at the point $(1, 1)$.

product rules

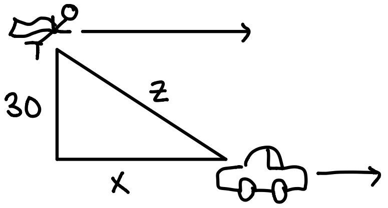
$$x^3-3x^2y+2xy^2=0 \Rightarrow 3x^2-\overbrace{\left(6xy+3x^2\frac{dy}{dx}\right)}+\overbrace{\left(2y^2+2x\left(2y\frac{dy}{dx}\right)\right)}=0$$

plug in $x=1$, $y=1$

$3-\left(6+3\frac{dy}{dx}\right)+\left(2+4\frac{dy}{dx}\right)=0 \Rightarrow 3-6-3\frac{dy}{dx}+2+4\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}-1=0 \Rightarrow \frac{dy}{dx}=1$

$\boxed{y-1=1(x-1)}$

7. (10 points) Superman is chasing a villain who is driving along a straight highway in a car. Superman flies at a speed of 200 feet per second, and at a constant height of 30 feet. The villain is driving at a speed of 100 feet per second. What is the rate of change of the distance between Superman and the villain when Superman is directly above a point that is 40 feet behind the villain's car?



$x^2+(30)^2=z^2$

Given: $\frac{dx}{dt}=200-100=100$

want $\frac{dz}{dt}$ when $x=40$

when $x=40$: $(40)^2+(30)^2=z^2 \Rightarrow 1600+900=z^2 \Rightarrow 2500=z^2 \Rightarrow z=50$

$x^2+(30)^2=z^2 \Rightarrow 2x\frac{dx}{dt}+0=2z\frac{dz}{dt} \Rightarrow 2(40)(100)=2(50)\frac{dz}{dt}$

$\Rightarrow \frac{dz}{dt}=\frac{2(40)(100)}{100}=\boxed{80\ ft/sec}$

8. (10 points) A landscape artist plans to create a rectangular garden whose area is $10\,m^2$. She plans to enclose three sides of the rectangle using trees that cost \$25 per meter, and to use fencing which costs \$20 per meter on the fourth side. Find the dimensions of the garden that will minimize her cost.

$$C = 25x + 25y + 25x + 20y = 50x + 45y$$

$$10 = xy \Rightarrow y = \frac{10}{x}$$

$$C(x) = 50x + 45\left(\frac{10}{x}\right) = 50x + \frac{450}{x} = 50x + 450x^{-1}$$

$$C'(x) = 50 - 450x^{-2} = 50 - \frac{450}{x^2}$$

$$C'(x) = 0 \Rightarrow 50 = \frac{450}{x^2} \Rightarrow 50x^2 = 450 \Rightarrow x^2 = 9 \Rightarrow x = 3 \Rightarrow y = \frac{10}{3}$$

$$\boxed{3\text{ m} \times {}^{10}/_{3}\text{ m}}$$

9. Let $f(x) = \dfrac{1}{x^2 - 1}$. You may use that $f' = \frac{-2x}{(x^2-1)^2}$ and $f'' = \frac{6x^2+2}{(x^2-1)^3}$.

(a) (2 points) Find the vertical asymptotes of the graph of f.

$f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)} \Rightarrow$ V.A are $\boxed{x=1, x=-1}$

(b) (2 points) Find the horizontal asymptotes of the graph.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1}{x^2-1} = 0$ $\qquad$ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{1}{x^2-1} = 0$

so H.A. is $\boxed{y=0}$

(c) (2 points) Find the intervals where f is increasing.

increasing when $f' > 0$ $\qquad$ $f'(x) = 0 \Rightarrow x = 0$

$f'(x)$ undef. when $(x^2-1)^2 = 0 \Rightarrow (x-1)^2(x+1)^2 = 0 \Rightarrow x = \pm 1$

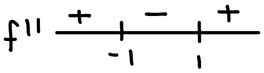
$$f' \quad \begin{array}{ccccccc} + & & + & & - & & - \\ \hline & -1 & & 0 & & 1 & \end{array} \Rightarrow \boxed{(-\infty, -1), (-1, 0)}$$

Recall that $f' = \frac{-2x}{(x^2-1)^2}$ and $f'' = \frac{6x^2+2}{(x^2-1)^3}$

(d) (2 points) Find the intervals where f is concave up.

concave up when $f''>0$

$f''(x)=0$ never, $f''(x)$ undef when $(x^2-1)^3=0 \Rightarrow (x-1)^3(x+1)^3=0 \Rightarrow x=\pm 1$



$\Rightarrow \boxed{(-\infty,-1),(1,\infty)}$

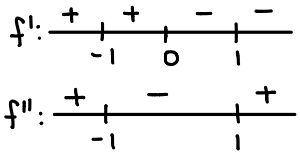
(e) (2 points) Find the maximal value of f in the interval $[4,6]$

cr # = 0 (outside int.)

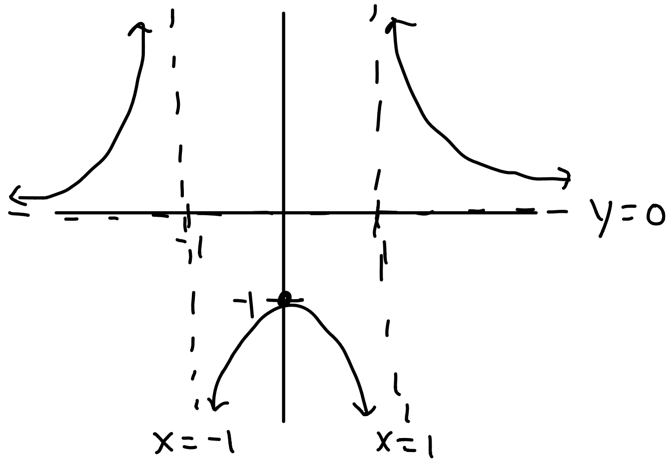
$f(4)=\frac{1}{16-1}=\frac{1}{15}$ $f(6)=\frac{1}{36-1}=\frac{1}{35}$

max is $\boxed{1/15}$

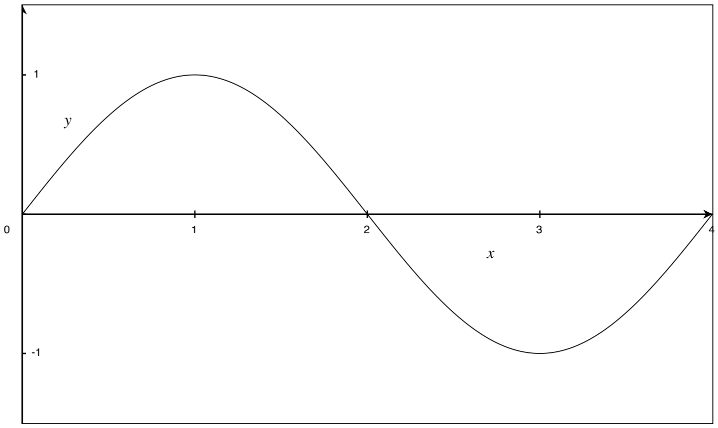
(f) (2 points) Sketch of the graph of $y = f(x)$.



(0,-1) rel max



10. Below is the graph of the **derivative** of the function f in the interval $0 \leq x \leq 4$.



(a) (3 points) Find the intervals in which f is increasing.

when $f'>0$ (above x-axis): $\boxed{(0,2)}$

(b) (3 points) Find the intervals in which f is concave down.

when $f''<0$ (graph dec.): $\boxed{(1,3)}$

(c) (3 points) At which x between 0 and 4 does f attain its maximal value? Explain your answer.

graph changes from above x-axis to below (i.e. f' changes from + to −) at $x=2$ (only sign change) so $\boxed{x=2}$

(d) (1 point) Is it possible for the equation $f(x) = 0$ to have 3 solutions in the interval $1 \leq x \leq 3$?

$\boxed{\text{no}}$ by MVT $\Rightarrow f'=0$ at least twice

11. Compute each of the following.

(a) (6 points) $\int (\sqrt{x} - x^{\frac{3}{2}} - \frac{4}{x^2})\, dx$

$$\int (\sqrt{x} - x^{3/2} - \frac{4}{x^2})\, dx = \int (x^{1/2} - x^{3/2} - 4x^{-2})\, dx$$

$$= \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} - \frac{4x^{-1}}{-1} + C = \boxed{\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + 4x^{-1} + C}$$

(b) (6 points) $\int_0^{\pi} (\sin x)(\cos x + 2)^3\, dx$

$$u = \cos x + 2 \Rightarrow du = -\sin x\, dx \Rightarrow dx = \frac{du}{-\sin x}$$

$$x = 0 \Rightarrow u = 1 + 2 = 3, \quad x = \pi \Rightarrow u = -1 + 2 = 1$$

$$\int_3^1 \cancel{\sin x}\, u^3 \frac{du}{-\cancel{\sin x}} = -\int_3^1 u^3\, du = \frac{-u^4}{4}\Big|_3^1 = \frac{-1}{4} - \frac{-81}{4} = \frac{-1+81}{4} = \frac{80}{4} = \boxed{20}$$

(c) (6 points) $\int \frac{x}{\sqrt{x^2+1}}\, dx$

$$u = x^2 + 1 \Rightarrow du = 2x\, dx \Rightarrow dx = \frac{du}{2x}$$

$$\int \frac{\cancel{x}}{\sqrt{u}} \frac{du}{2\cancel{x}} = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \boxed{\sqrt{x^2+1} + C}$$

12. A ball is thrown upwards from a height of 20 meters, at a speed of $15m/s$. The gravity of the earth causes the ball to accelerate **downwards** at a rate of $10m/s^2$.

   (a) (4 points) Write a function f that describes the height of the ball at time t.

$s(0)=20$

$v(0)=15$

$a(t)=-10$

$v(t)=\int a(t)dt=-10t+C$

$v(0)=15 \Rightarrow C=15 \Rightarrow v(t)=-10t+15$

$s(t)=\int v(t)dt=\frac{-10t^2}{2}+15t+D=-5t^2+15t+D$

$s(0)=20 \Rightarrow D=20 \Rightarrow \boxed{s(t)=-5t^2+15t+20}$

   (b) (2 points) When will the ball reach its highest point?

when $v(t)=0$

$-10t+15=0 \Rightarrow 10t=15 \Rightarrow t=\frac{15}{10}=\boxed{\frac{3}{2}\text{ sec}}$

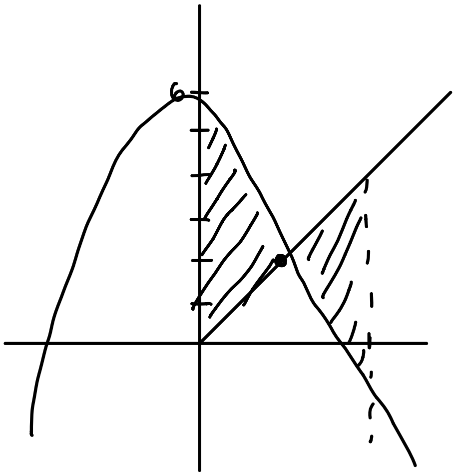
   (c) (2 points) When will the ball hit the ground?

when $s(t)=0$

$-5t^2+15t+20=0 \Rightarrow t^2-3t-4=0 \Rightarrow (t-4)(t+1)=0 \Rightarrow t=4,-1$

so $\boxed{4\text{ sec}}$

13. (a) (2 points) Sketch the region in the plane bounded by the lines $x = 0$, $x = 4$, $y = x$, and $y = 6 - x^2$.



intersection:

$$x = 6 - x^2 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$$

$$\Rightarrow x = -3, 2$$

(b) (8 points) Calculate the area of the region you sketched in the previous part.

$$A = \int_0^2 ((6 - x^2) - x)\,dx + \int_2^4 (x - (6 - x^2))\,dx$$

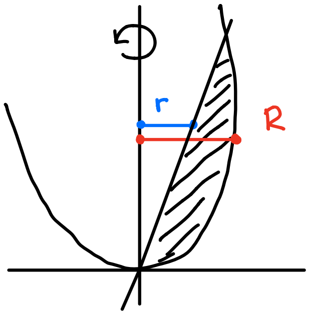
$$= \int_0^2 (6 - x^2 - x)\,dx + \int_2^4 (x - 6 + x^2)\,dx$$

$$= \left(6x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^2 + \left(\frac{x^2}{2} - 6x + \frac{x^3}{3}\right)\Big|_2^4$$

$$= \boxed{\left[\left(6(2) - \frac{(2)^3}{3} - \frac{(2)^2}{2}\right) - \left(6(0) - \frac{(0)^3}{3} - \frac{(0)^2}{2}\right)\right] + \left[\left(\frac{(4)^2}{2} - 6(4) + \frac{(4)^3}{3}\right) - \left(\frac{(2)^2}{2} - 6(2) + \frac{(2)^3}{3}\right)\right]}$$

14. Let R be the region bounded by the graphs $y = x^2$ and $y = 9x$.

(a) (5 points) The region R is rotated about the y-axis. Set up, but **do not evaluate** an integral describing the volume of the resulting shape. You may use any method you like.



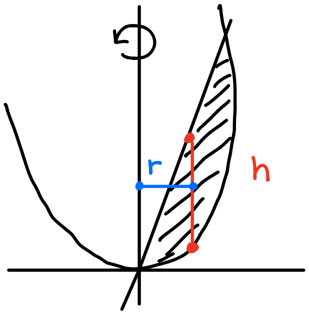
washer:

$x=\sqrt{y}$, $x=y/9$

intersection: $\sqrt{y}=\frac{y}{9} \Rightarrow 9y^{1/2}=y \Rightarrow 9=y^{1/2}$

$\Rightarrow y=81$

$V=\int_0^{81}\pi(R^2-r^2)dy = \boxed{\int_0^{81}\pi\left(\sqrt{y}-\frac{y}{9}\right)dy}$



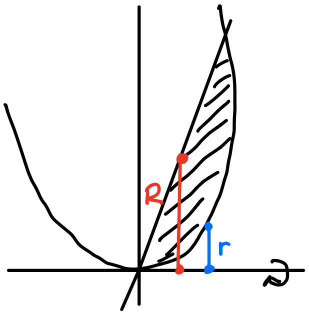
shell:

$h=9x-x^2$, $r=x$

intersection: $x^2=9x \Rightarrow x=9$

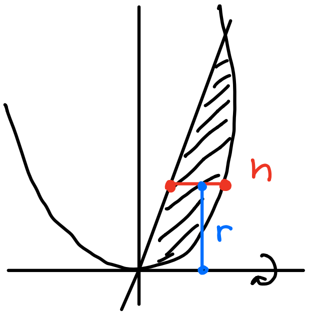
$V=\int_0^{9}2\pi rh\,dx = \boxed{\int_0^{9}2\pi x(9x-x^2)dx}$

(b) (5 points) The region R is rotated about the x-axis. Set up, but **do not evaluate** an integral describing the volume of the resulting shape. You may use any method you like.



washer:

$V=\int_0^{9}\pi(R^2-r^2)dx = \boxed{\int_0^{9}\pi(9x-x^2)dx}$



shell:

$V=\int_0^{81}2\pi rh\,dy = \boxed{\int_0^{81}2\pi y\left(\sqrt{y}-\frac{y}{9}\right)dy}$