# Math 241 Final Exam, Spring 2013

**Name:**

**Section number:**

**Instructor:** Solutions by D. Yuen

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 12 | |
| 4 | 10 | |
| 5 | 17 | |
| 6 | 15 | |
| 7 | 6 | |
| 8 | 12 | |
| 9 | 12 | |
| 10 | 14 | |
| 11 | 17 | |
| 12 | 10 | |
| 13 | 5 | |
| Total: | 140 | |

**Read all of the following information before starting the exam.**

- Electronic devices (calculators, cell phones, computers), books and notes are not allowed.
- Show all work clearly. You may lose points if we cannot see how you arrived at your solution.
- You do not have to simplify your arithmetic. But be aware that if your answer looks like you need a calculator, you are probably doing it wrong.
- Box or otherwise clearly indicate your final answers.
- This test has 12 pages total including this cover sheet and is worth 140 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (5 points) Compute the following limit

$$\lim_{t\to 4}\left(\frac{t-4}{\sqrt{t}-2}\right)\cdot\frac{\sqrt{t}+2}{\sqrt{t}+2}$$

Limit type $\frac{0}{0}$
Do more work.

$$=\lim_{t\to 4}\frac{(t-4)(\sqrt{t}+2)}{t-4}$$

$$=\lim_{t\to 4}(\sqrt{t}+2) = 2+2 = \boxed{4}$$

2. (5 points) Compute the following limit, making sure to **justify all steps.**

$$\lim_{x\to\infty}\frac{3x\sin(x^2)+1}{x^2+4}\quad\frac{\cdot 1/x^2}{\cdot 1/x^2}$$

Limit type $\frac{\infty}{\infty}$
Divide by
highest power
from denominator

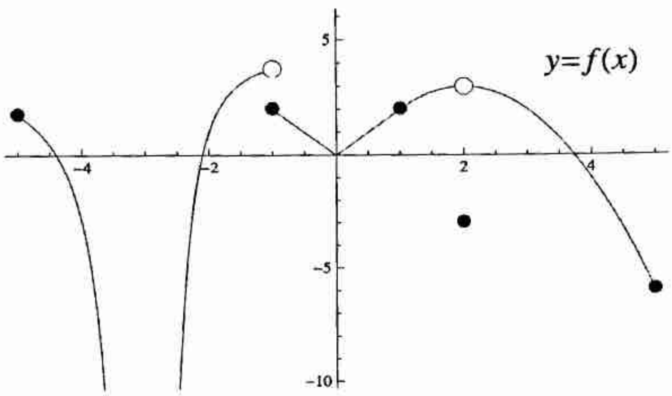
$$=\lim_{X\to\infty}\frac{\frac{3\sin(X^2)}{X}+\frac{1}{X^2}}{1+\frac{4}{X^2}} = \frac{0+0}{1+0} = \boxed{0}$$

Note $-\frac{3}{X}\le\frac{3\sin(x^2)}{X}\le\frac{3}{X}$

and $\lim_{X\to\infty}\frac{-3}{X} = 0 = \lim_{X\to\infty}\frac{3}{X}$

and so $\lim_{X\to\infty}\frac{3\sin(x^2)}{X} = 0$ .

3. The graph of the function $y = f(x)$ is given below:




You will not receive full credit without an explanation at each point.

(a) (4 points) At which points in $[-5, 5]$ does the limit of the function **not exist**? Explain why.

$x = -3 \quad \lim_{x \to -3} f(x) = -\infty$

$x = -1$ Jump discontinuity $\lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x)$

(b) (4 points) At which points in $[-5, 5]$ is the function **discontinuous**? Explain why.

$\left. \begin{array}{l} x = -3 \\ x = -1 \end{array} \right\}$ because limit does not exist

$x = 2$ because $\lim_{x \to 2} f(x) \neq f(2)$

(c) (4 points) At which points in $[-5, 5]$ is the function **not differentiable**? Explain why.

$x = -3, -1, 2$ because not continuous

$x = 0$ because of a "corner"
(left hand slope $\neq$ right hand slope)

4. (10 points) Using the **definition of derivative, NOT differentiation rules**, find $f'(2)$ if

$$f(x) = \frac{1}{x+1}.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+1) - (x+h+1)}{(x+h+1)(x+1)}}{\frac{h}{1}}$$

invert and multiply

$$= \lim_{h \to 0} \frac{-h}{(x+h+1)(x+1)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)}$$

$$= \frac{-1}{(x+0+1)(x+1)}$$

$$= \frac{-1}{(x+1)^2}$$

5. Compute the following integrals

(a) (5 points) $\int \sin(3x+1)dx$

Sub $u=3x+1$
$du=3dx$

$$= \frac{1}{3}\int \sin(3x+1)\, 3dx$$
$$= \frac{1}{3}\int \sin(u)\, du$$
$$= \frac{1}{3}(-\cos(u)) + C$$
$$= -\frac{1}{3}\cos(3x+1) + C$$

(b) (5 points) $\int_{-1}^{2} t^3 - t\,dt$

$$\frac{1}{4}t^4 - \frac{1}{2}t^2 \Big|_{-1}^{2}$$
$$= (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{9}{4}$$

(c) (7 points) $\int_0^1 x(x-1)^{2013}dx$

Sub $u = x-1 \Rightarrow u+1 = x$
$du = dx$

$$\int (u+1)\, u^{2013}\, du$$
$$= \int (u^{2014} + u^{2013})\, du$$
$$= \frac{u^{2015}}{2015} + \frac{u^{2014}}{2014} + C = \frac{(x-1)^{2015}}{2015} + \frac{(x-1)^{2014}}{2014} + C$$
$$= \frac{(x-1)^{2015}}{2015} + \frac{(x-1)^{2014}}{2014}\Big|_0^1 = (0+0) - \left(\frac{-1}{2015} + \frac{1}{2014}\right)$$

6. (15 points) This question refers to the curve defined by

$$2x^3 + xy = y^2.$$

(a) Find $dy/dx$ at the point $(1, 2)$.

$6x^2 + 1y + xy' = 2yy'$

At $x = 1$, $y = 2$

$6 + 2 + y' = 4y'$

$8 = 3y'$

$8/3 = y'$

(b) Find the tangent line to the curve at $(1, 2)$.

$y - 2 = \frac{8}{3}(x - 1)$

(c) Using the tangent line, estimate the value of $y$ at $x = 1.2$. [You do not have to simplify the arithmetic.]

Tangent line is $y = L(x) = 2 + \frac{8}{3}(x - 1)$

$L(1.2) = 2 + \frac{8}{3}(1.2 - 1)$

$= \boxed{2 + \frac{8}{3}(.2)}$

7. (6 points) Does the equation $x^3 - 4x^2 + 2 = 1$ have a solution on the interval $[-1, 1]$? Make sure you justify your answer, and clearly state any theorems that you are using.

Call $f(x) = x^3 - 4x^2 + 2$.
Note $f$ is continuous everywhere.

Try $f(-1) = -1 - 4 + 2 = -3$
$f(1) = 1 - 4 + 2 = -1$

unfortunately 1 is not between these values.

Try $f(0) = 0 - 0 + 2 = 2$.

So $f(-1) < 1 < f(0)$.

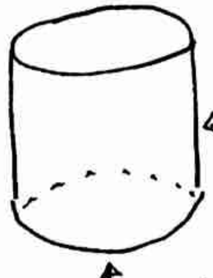
By I.V.T.,
there must be a $-1 < c < 0$ such that $f(c) = 1$.

[You can also use the fact that $f(0) > 1 > f(1)$ to conclude $f(c) = 1$ for some $0 < c < 1$.]

[Alternatively, you could instead let $f(x) = x^3 - 4x^2 + 1$ and try to show $f(x) = 0$ somewhere in $[-1, 1]$.]

8. (12 points) Bob wants to use 12 square inches of metal to make a cylindrical can of height $h$ and radius $r$. Bob just wants the sides and bottom of the can to be included, but not the top. What is the maximum volume of the can that Bob can make?

*Hint: the volume of a cylinder as given is* $V = \pi r^2 h$.



← side has surface area $(2\pi r)h$

Bottom has area $\pi r^2$

Constraint is $\pi r^2 + 2\pi r h = 12.$

Solve for $h$:

$$2\pi r h = 12 - \pi r^2$$

$$h = \frac{12 - \pi r^2}{2\pi r}$$

$$V = \pi r^2 h = \pi r^2 \left(\frac{12 - \pi r^2}{2\pi r}\right)$$

NOTE

Domain is

$$12 - \pi r^2 \geq 0$$

$$12 \geq \pi r^2$$

$$\frac{12}{\pi} \geq r^2$$

$$V = \frac{r(12 - \pi r^2)}{2}$$

$$V = 6r - \frac{1}{2}\pi r^3$$

$$\frac{dV}{dr} = 6 - \frac{3}{2}\pi r^2 \overset{\text{set}}{=} 0$$

$$6 = \frac{3}{2}\pi r^2$$

$$\frac{12}{3\pi} = r^2 \text{ (in domain)}$$

$$\sqrt{\frac{12}{3\pi}} = r$$

$\frac{dV}{dr}$: ++ 0 --

$r$: $0$, $\sqrt{\frac{12}{3\pi}}$, $\sqrt{\frac{12}{\pi}}$

$V$: ↗ ↘

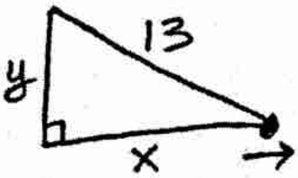
Max Volume at $r = \sqrt{\frac{12}{3\pi}}$

Max volume is $\boxed{6\sqrt{\frac{12}{3\pi}} - \frac{1}{2}\pi\left(\sqrt{\frac{12}{3\pi}}\right)^3}$

9. A 13 foot ladder is leaning against a wall when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) (4 points) How fast is the top of the ladder sliding down the wall at that time?

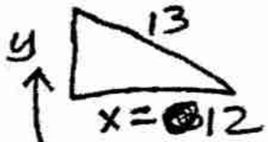
Units




Evaluate at $x = 12$, $\frac{dx}{dt} = 5$ ft, s

$x^2 + y^2 = 13^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$




$y = \sqrt{13^2 - 12^2} = 5$

Plug in $2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \boxed{-12.\ \text{ft/s}}$

(b) (4 points) At what rate is the area of the triangle formed by the ladder, wall and ground changing at that time?

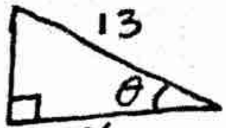
$A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\frac{dx}{dt}\cdot y + \frac{1}{2}x\frac{dy}{dt}.$

$= \frac{1}{2}(5)(5) + \frac{1}{2}(12)(-12) = \frac{1}{2}(25 - 144)$ ft²/s

Plug in

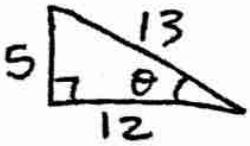
(c) (4 points) At what rate is the angle $\theta$ between the ladder and the ground changing at that time?




$\cos\theta = \frac{x}{13}$

$13\cos\theta = x$

$-13\sin\theta\frac{d\theta}{dt} = \frac{dx}{dt}$




$\sin\theta = \frac{5}{13}$

$-13\cdot\frac{5}{13}\frac{d\theta}{dt} = 5$

$\frac{d\theta}{dt} = \boxed{-1\ \text{rad/s}}$

critical points
$x=0, x=3$

possible IPs
at $x=0,2$

10. Let $f(x) = x^4 - 4x^3 + 10$. Then $f'(x) = 4x^2(x-3)$, $f''(x) = 12x(x-2)$.

(a) (4 points) Find the intervals where $f$ is increasing, and those where it is decreasing.

$f'(x)$ $- - -$ $0$ $- - - -$ $0$ $+ +$

$x$ $0$ $3$

Local and global min

$f$ increasing on $(3, \infty)$
decreasing on $(-\infty, 3)$

(b) (4 points) Find the local and global maxima and minima. Give both the values, and where they are attained.

Local and global min at $\left(x = 3, y = 81 - 108 + 10 = -17\right)$

No local or global max

(c) (4 points) Find the intervals where $f$ is concave up, and those where it is concave down.

$f''(x)$: $+ +$ $0$ $- -$ $0$ $+ +$

$x$: $0$ $2$

$f$ IP IP

concave up for $x < 0$ and $x > 2$
Concave down for $0 < x < 2$.

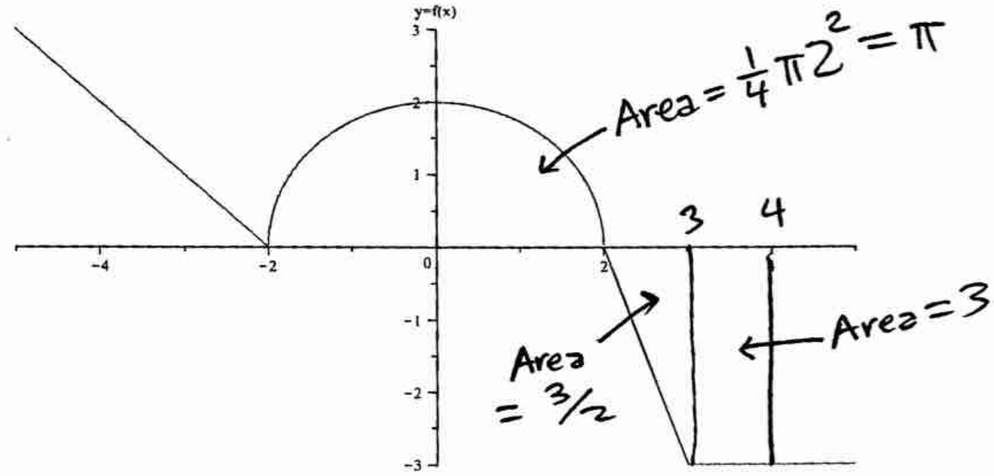
(d) (2 points) Find the inflection points of $f$.

IPs at $(x = 0, y = 10)$

and at $\left(x = 2, y = 16 - 32 + 10 = -6\right)$

Not asked for

11. Below is a plot of a function $y = f(x)$. It is linear on the intervals $[-5,-2]$, $[2,3]$, and $[3,5]$, and a radius 2 semicircle on the interval $[-2,2]$:




Define $F(x) = \int_0^x f(t)\,dt$ for $x$ in the interval $[-5,5]$.

(a) (8 points) Evaluate the following

$$
\begin{aligned}
F(4) &= \int_0^4 f(t)dt = \pi - \tfrac{3}{2} - 3 \\
F'(4) &= -3 \quad (= f(4)) \\
F''(4) &= 0 \quad (= f'(4)) \\
F(-2) &= \int_0^{-2} f(t)\,dt = -\int_{-2}^{0} f(t)dt = -\pi
\end{aligned}
$$

(b) (4 points) On what interval(s) is the function $F$ increasing?

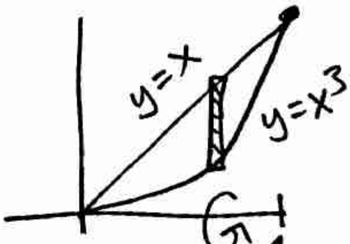
$F'(x) \geq 0$ for $-5 \leq x \leq 2$.

(c) (5 points) Another function $G(x)$ is defined as $G(x) = \int_0^{x^2} f(t)dt$. Find $G'(2)$.

$$
\begin{aligned}
G'(x) &= f(x^2)\cdot 2x \\
G'(2) &= f(4)\cdot 2\cdot 2 \\
&= -3\cdot 2\cdot 2 \\
&= -12.
\end{aligned}
$$

12. Let $f(x) = x$ and $g(x) = x^3$. Let $R$ be the region between the graphs of $f$ and $g$ as $x$ ranges over the interval $[0, 1]$.

(a) (5 points) Find the volume of the region arrived at by rotating $R$ about the x-axis.



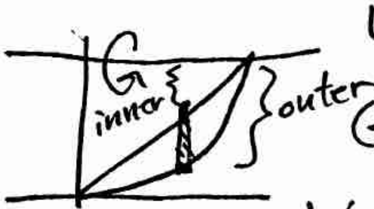

Use dx

Get washers.

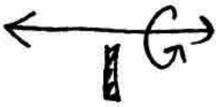
$$V = \pi\int_0^1 ((x)^2 - (x^3)^2)\,dx$$

$$= \pi\int_0^1 (x^2 - x^6)\,dx = \pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7\right)\Big|_0^1 = \pi\left(\left(\tfrac{1}{3} - \tfrac{1}{7}\right) - 0\right)$$

(b) (5 points) Set up, but do not integrate, an integral for the volume when $R$ is rotated about the axis $y = 1$.




Use dx



Get washers.

$$V = \pi\int_0^1 (\underbrace{(1-x^3)^2}_{\text{outer}} - \underbrace{(1-x)^2}_{\text{inner}})\,dx$$

13. (5 points) Leslie would like to approximate the area under the following curve on the interval $[-1, 1]$ using a Riemann sum and intervals of length 0.50. Sketch the areas she should compute in order to guarantee that the approximation is an **overestimate**.