Math 241 Final Exam, Spring 2014

Name:
Section number: Solutions
Instructor:

by
D. Yuen

Question	Points	Score
1	15	
2	15	
3	10	
4	15	
5	10	
6	10	
7	10	
8	10	
9	15	
10	10	
11	10	
12	20	
Total:	150	

Read all of the following information before starting the exam.

- Calculators and notes are not allowed.
- Show all work clearly. You may lose points if we cannot see how you arrived at your solution.
- Box or otherwise clearly indicate your final answers.
- This test has 10 pages total including this cover sheet and is worth 150 points. It is your responsibility to make sure that you have all of the pages!

1. (15 points) Compute the following limits or show that they do not exist:

(a)

$$\lim_{t\to 4}\left(\frac{\sqrt{t}-3}{t-9}\right)$$

Limit type $\frac{2-3}{4-9}$
No issue.

$$= \frac{\sqrt{4}-3}{4-9} = \frac{-1}{-5} = \boxed{\frac{1}{5}}$$

(b)

$$\lim_{x\to 0}\frac{x}{\sin 3x}$$

Limit type $\frac{0}{0}$
Idea:
Use $\lim_{x\to 0}\frac{3x}{\sin(3x)} = 1$

$$= \lim_{x\to 0}\frac{1}{3}\,\frac{3x}{\sin(3x)}$$

$$= \frac{1}{3}\cdot 1 = \boxed{\frac{1}{3}}$$

(c)

$$\lim_{x\to\infty}\frac{x\sin(x^2)}{x^2+1}$$

Limit Type $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty}\frac{x\sin(x^2)}{x^2+1}\,\frac{\cdot 1/x^2}{\cdot 1/x^2} = \lim_{x\to\infty}\frac{\frac{\sin(x^2)}{x}}{1+\frac{1}{x^2}}$$

← use sandwich theorem.

We have $-1 \le \sin(x^2) \le 1$.

Then
$$\frac{\frac{-1}{x}}{1+\frac{1}{x^2}} \le \frac{\frac{\sin(x^2)}{x}}{1+\frac{1}{x^2}} \le \frac{\frac{1}{x}}{1+\frac{1}{x^2}}$$

$$\lim_{x\to\infty}\frac{-\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0 \qquad \lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0,$$

By sandwich theorem, $\lim_{x\to\infty}\frac{x\sin(x^2)}{x^2+1} = \boxed{0}$

2. (15 points) Find the derivative of the following functions. Do not simplify.

(a) $f(x) = x^3\sqrt{\sin x}$

$f'(x) = 3x^2\sqrt{\sin x} + x^3 \frac{1}{2}(\sin(x))^{-1/2} \cos(x).$

(b) $g(x) = \tan(2 + x^2)$

$g'(x) = \sec^2(2+x^2) \cdot 2x$

(c) $h(x) = \int_2^{x^3} \frac{\sin x}{x}\, dx$

$h'(x) = \frac{\sin x}{x} \cdot 3x^2$

3. (10 points) Using the **definition of derivative, NOT differentiation rules**, find $f'(2)$ if

$$f(x) = \frac{1}{3x}.$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3(2+h)} - \frac{1}{6}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2 - (2+h)}{6(2+h)}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{-h}{6(2+h)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{6(2+h)}$$

$$= \frac{-1}{6(2+0)}$$

$$= -\frac{1}{12}$$

4. Evaluate the following integrals. Show your work!

(a) (5 points) $\int \sin(x)\cos^3(x)\,dx$

Sub $u = \cos x$
$du = -\sin x\,dx$

$$= -\int \cos^3(x)\,(-\sin x\,dx)$$
$$= -\int u^3\,du$$
$$= -\frac{1}{4}u^4 + C$$
$$= -\frac{1}{4}(\cos x)^4 + C$$

(b) (5 points) $\int_0^1 (t^2+1)(4t-1)\,dt$

Expand

$$= \int_0^1 (4t^3 - t^2 + 4t - 1)\,dt$$
$$= t^4 - \frac{1}{3}t^3 + 2t^2 - t\Big|_0^1$$
$$= \left(1 - \frac{1}{3} + 2 - 1\right) - (0 - 0 + 0 - 0)$$
$$= 2 - \frac{1}{3} = \frac{5}{3}$$

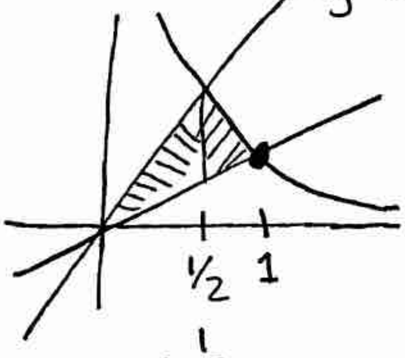
(c) (5 points) $\int x^2\sqrt{x+1}\,dx$

Sub $u = x+1 \Rightarrow u - 1 = x$
$du = dx$

$$= \int (u-1)^2\sqrt{u}\,du$$
$$= \int (u^2 - 2u + 1)\,u^{1/2}\,du$$
$$= \int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$
$$= \frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C$$

5. (10 points) Find the area of the region bounded by the lines $y = x$, $y = 8x$ and the curve $y = 1/x^2$.

$y=8x$ intersect with $y=\frac{1}{x^2}$ at $8x=\frac{1}{x^2} \Rightarrow x^3=\frac{1}{8}$
$\Rightarrow x = 1/2$



$y=x$
$y=\frac{1}{x^2}$ } intersect $x=\frac{1}{x^2} \Rightarrow x^3=1 \Rightarrow x=1$

$$A=\int_0^{\frac{1}{2}}(8x-x)dx + \int_{\frac{1}{2}}^{1}\left(\frac{1}{x^2}-x\right)dx$$

$$= \frac{7}{2}x^2\Big|_0^{1/2} + \frac{x^{-1}}{-1} - \frac{1}{2}x^2\Big|_{\frac{1}{2}}^{1}$$

$$= \frac{7}{8} - 0 + \left(-1-\frac{1}{2}\right) - \left(-2-\frac{1}{8}\right) = \frac{3}{2}$$

6. (10 points) Find an equation for the tangent line to the curve defined by

$$x^3 + y^3 = 9xy$$

at the point $(4, 2)$. Show your work! Implicit diff.

$$3x^2 + 3y^2y' = 9\cdot 1y + 9xy'.$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$(3y^2-9x)\,y' = 9y - 3x^2$$

$$y' = \frac{9y - 3x^2}{3y^2 - 9x}$$

At $x=4, y=2$.

$$y' = \frac{9\cdot 2 - 3\cdot 16}{3\cdot 4 - 9\cdot 4} = \frac{18-48}{12-36} = \frac{-30}{-24} = \frac{5}{4}$$

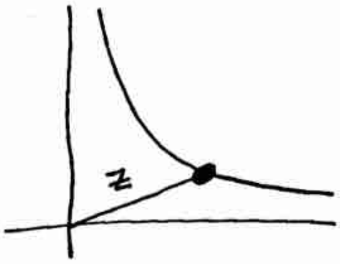
Tangent line

$$y - 2 = \frac{5}{4}(x-4).$$

7. (10 points) Consider a point $P = (x, y)$ that moves along the graph of the function $y = 8/x$ with a horizontal velocity of 3 units per second. (This means that $dx/dt = 3$.) At what rate does the distance between P and the origin $(0, 0)$ change as the point passes through $(4, 2)$?

Relate rates Given $\frac{dx}{dt} = 3$.

Want $\frac{dz}{dt} = ?$ when $x = 4$, $y = 2$



$$z = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{8}{x}\right)^2} = \sqrt{x^2 + \frac{64}{x^2}}$$

$$\frac{dz}{dt} = \frac{1}{2}\left(x^2 + \left(\frac{8}{x}\right)^2\right)^{-1/2}\left(2x + 64(-2)x^{-3}\right)\frac{dx}{dt}$$

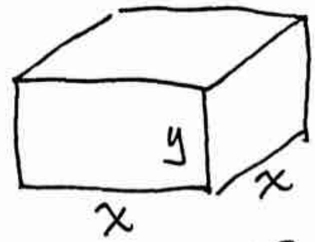
At $x = 4$:

$$\frac{dz}{dt} = \frac{1}{2}(16 + 4)^{-1/2}\left(8 + 64(-2)\frac{1}{64}\right)(3)$$

$$= \frac{1}{2}\frac{1}{\sqrt{20}} \cdot \cancel{10}\,6 \cdot 3$$

8. (10 points) A rectangular box with volume 6 cubic feet is to be built with a square base and no top. The material used for the bottom panel costs \$3.00 per square foot and the material for the side panels costs \$2.00 per square foot. Find the minimum cost of such a box. **Justify your answer using the methods of calculus.**

Constraint $x^2 y = 6 \Rightarrow y = 6/x^2$.



Cost is

$$C = \underset{0}{\overset{\text{TOP}}{\cancel{\ \ }}} + \underset{\text{cost}}{\overset{\text{BOTTOM}}{3x^2}} + \underset{\text{cost}}{\overset{\text{4 sides}}{4 \cdot 2\, xy}}$$

$$C = 3x^2 + 8xy = 3x^2 + 8x\left(\frac{6}{x^2}\right) = 3x^2 + \frac{48}{x}$$

Domain $x > 0$.

$$\frac{dC}{dx} = 6x - \frac{48}{x^2} \overset{\text{SET}}{=} 0$$

$$= \frac{6x^3 - 48}{x^2} \leftarrow \text{set} = 0$$

$6x^3 - 48 = 0$

$x^3 - 8 = 0$

$x = 2$

$\frac{dC}{dx}$: $-\,-$ 0 $+\,+$

x: 0, 2

C: ABS MIN

MIN at

$x = 2$

$y = \frac{6}{2^2} = \frac{3}{2}$

9. Let $f(x) = x(x-3)^2 = x^3 - 6x^2 + 9x$.

(a) (3 points) Compute the first and second derivatives:

$f'(x) = 3x^2 - 12x + 9$ $\qquad$ $f''(x) = 6x - 12$

$= 3(x^2 - 4x + 3)$ $\qquad$ $= 6(x-2)$

$= 3(x-1)(x-3)$

(b) (3 points) Find the interval(s) where f is increasing and those where f is decreasing.

$f'(x)$ + + + 0 − − − 0 + + +

x 1 3

f is inc for $x < 1$ and $x > 3$, dec for $1 < x < 3$.

(c) (4 points) Find the local and global maxima and minima on the interval $[-1, 3.5]$. Give both the values and where they are attained.

By EVT, they occur at either critical pt or endpt.

$x = 1$ $\quad f(1) = 1(-2)^2 = 4$ ⟵ Max at $(1, 4)$

$x = 3$ $\quad f(3) = 3(0)^2 = 0$

$x = -1$ $\quad f(-1) = -1(-4)^2 = -16$ ⟵ Min at $(-1, -16)$

$x = 3.5$ $\quad f(3.5) = 3.5(.5)^2 = 3.5(.25) < 4$

(d) (3 points) Find the intervals where f is concave up and those where it is concave down.

$f''(x)$ − − 0 + +

x 2

f

Concave up for $x > 2$, Concave down for $x < 2$.

(e) (2 points) Find the inflection points of f.

Inflection point at $x = 2$

$(y = 2$ here$)$

10. (10 points) Suppose $f(x)$ and $g(x)$ are both positive, increasing, and concave upward on an interval I. Also suppose that $f(x)$ and $g(x)$ are twice-differentiable; that is, suppose $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$ all exist. Show that $f(x)g(x)$ is also concave upward on I.

We need to show $(f(x)g(x))'' > 0$ on I.

We have $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Then $(f(x)g(x))'' = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)$

> 0 because we are given that $f(x), g(x), f'(x), g'(x), f''(x), g''(x)$ all > 0.

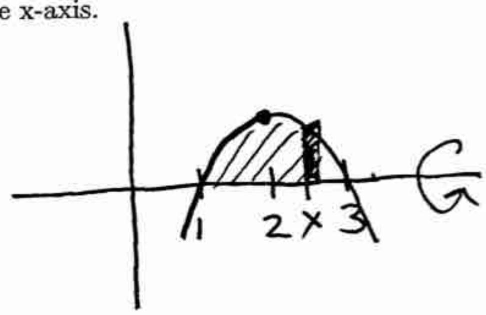
11. (10 points) Let R be the region bounded by the graph of $f(x) = 1 - (x-2)^2$ and the x-axis. Set up but **do not evaluate** integrals for the following:

(a) the volume of the region arrived at by rotating R about the x-axis.

Use vertical ▮ (dx-problem).

Get washers.

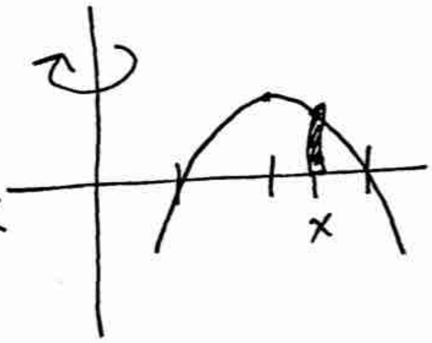
$$V = \pi \int_1^3 \left(\left(1-(x-2)^2\right)^2 - 0^2 \right) dx$$

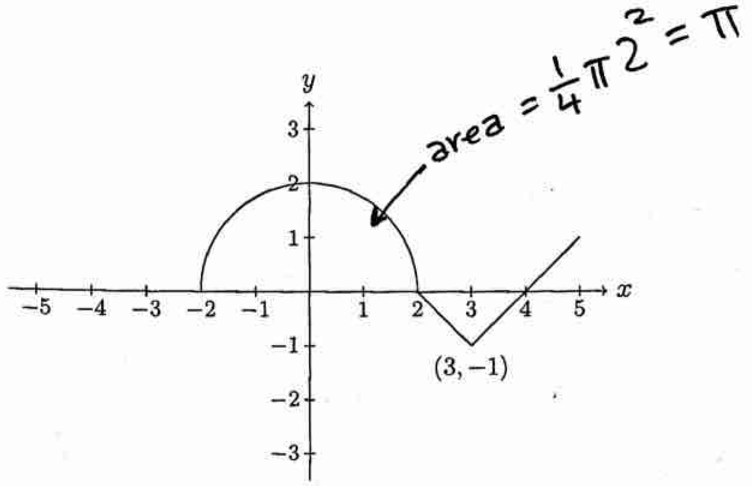


(b) the volume of the region arrived at by rotating R about the y-axis.

Get shells

$$V = 2\pi \int_1^3 (x-0)\left(\left(1-(x-2)^2\right) - 0\right) dx$$





12. (20 points)

The graph of a function f consists of a semicircle and two line segments as shown above. Let g be the function given by

$$g(x) = \int_0^x f(t)dt.$$

(a) Find $g(3)$.

$$= \int_0^3 f(t)dt = \frac{2^2\pi}{4} - \frac{1}{2} = \pi - \frac{1}{2}$$

(b) Find all values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.

$g'(x) = f(x)$

$+++ \quad 0 \quad -- \quad 0 \quad ++$

$0 \quad 2 \quad 4 \quad 5$

g $\nearrow$ local max $\searrow$ local min $\nearrow$

g has rel max at $x = 2$

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

$g'(3) = f(3) = -1$ Note $g(3) = \pi - \frac{1}{2}$

Tangent line is $y - (\pi - \frac{1}{2}) = -1(x - 3)$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

$g''(x) = f'(x)$

$++ \quad 0 \quad -- \quad$ DNE $\quad --$ DNE $\quad ++$

$x \quad 0 \quad 2 \quad 3$

g $\smile$ IP $\frown$ $\frown$ IP $\smile$

IP at $x = 0, x = 3$.