Math 241, Spring 2015, Final Exam

Name and section number:

Solutions,

by D. Yuen

Question	Points	Score
1	16	
2	8	
3	16	
4	10	
5	6	
6	10	
7	6	
8	13	
9	4	
10	3	
11	3	
12	8	
13	9	
14	12	
15	6	
16	10	
Total:	140	

- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- Show all your work.
- Good luck!

1. Compute the following limits. **Do not** use L'Hospital's rule. If the limit is either positive or negative infinity say which. Simplify your answers.

(a) (4 points) $\lim_{x \to 0^-} \frac{|x+1|}{2+\sin^2 x}$.

Type $\frac{1}{2+0} \to \frac{1}{2}$

The function is continuous at $x=0$, so there is no issue.

$$= \frac{1}{2}$$

(b) (4 points) $\lim_{t \to -2} \frac{t+2}{t^2-4}$

Type $\frac{0}{0}$ Try factoring

$$= \lim_{t \to -2} \frac{t+2}{(t-2)(t+2)} = \lim_{t \to -2} \frac{1}{t-2} = \frac{1}{-4}$$

(c) (4 points) $\lim_{s \to 0} f(s)$, where $f(s) = \begin{cases} s\left(3 - \frac{5}{s}\right), & s \neq 0; \\ 0, & s = 0. \end{cases}$ ← irrelevant for $\lim_{s \to 0}$

$$= \lim_{s \to 0} s\left(3 - \frac{5}{s}\right) = \lim_{s \to 0} (3s - 5) = -5$$

(d) (4 points) $\lim_{x \to \infty} \frac{2x}{x + 7\cos x}$.

Type $\frac{\infty}{\infty}$ Try dividing numerator and denominator both by highest term in denominator which is x.

$$= \lim_{x \to \infty} \frac{\frac{2x}{x}}{\frac{x}{x} + \frac{7\cos x}{x}}$$

$$= \lim_{x \to \infty} \frac{2}{1 + 7\frac{\cos x}{x}}$$

$$= \frac{2}{1+0} = 2$$

Note $\lim_{x \to \infty} \frac{\cos x}{x} = 0$ by sandwich theorem because $-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$ and $\lim_{x \to \infty} \frac{-1}{x} = 0$, $\lim_{x \to \infty} \frac{1}{x} = 0$.

2. (8 points) Let r be the function $r(x) = \sqrt{2x+1}$. **Using the definition of the derivative as a limit**, show that $r'(0) = 1$.

(Warning: you get no credit for using the rules of differentiation).

$$r'(0) = \lim_{h\to 0} \frac{r(0+h) - r(0)}{h} = \lim_{h\to 0} \frac{r(h) - r(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2h+1} - 1}{h} \cdot \frac{\sqrt{2h+1} + 1}{\sqrt{2h+1} + 1}$$

Try multiplying by the conjugate

$$= \lim_{h\to 0} \frac{(2h+1) - 1}{h\,(\sqrt{2h+1} + 1)}$$

$$= \lim_{h\to 0} \frac{2h}{h\,(\sqrt{2h+1} + 1)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2h+1} + 1}$$

(No more division by 0)

$$= \frac{2}{\sqrt{0+1} + 1} = \frac{2}{2} = 1$$

3. Differentiate the following functions. **Do not** simplify your answers.

(a) (4 points) $y = 3x^5 - 2x^3 - 7x + \pi$.

$$y' = 15x^4 - 6x^2 - 7$$

(b) (4 points) $z = \dfrac{2x \sin x}{5x+3}$.

Quotient rule with product rule for derivative of numerator

$$\frac{dz}{dx} = \frac{(2x\sin x)'(5x+3) - 2x\sin x\,(5x+3)'}{(5x+3)^2}$$

$$= \frac{(2\sin x + 2x\cos x)\,5x+3 - 2x\sin x\,(5)}{(5x+3)^2}$$

(c) (4 points) $w = \sqrt{x} + (3 + x^2)^7$.

$$\frac{dw}{dx} = \frac{1}{2}x^{-1/2} + 7(3+x^2)^6(2x)$$

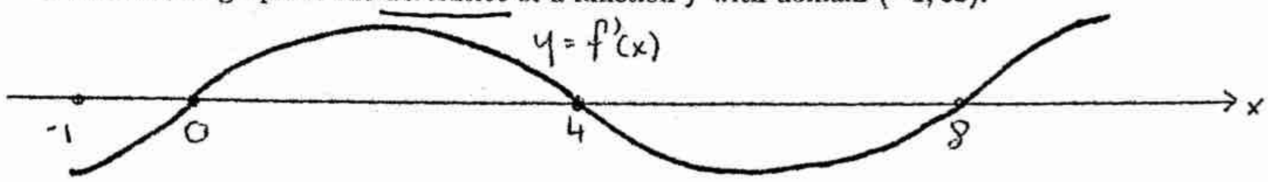
(d) (4 points) $R(t) = \displaystyle\int_0^{t+2} \cos(1 + x^2)\, dx$.

$R'(t) =$

$$= \cos(1 + (t+2)^2) \cdot (t+2)'$$

$$= \cos(1 + (t+2)^2) \cdot 1$$

4. Here is the graph of the *derivative* of a function f with domain $(-1, \infty)$.



(a) (5 points) Which is larger, $f(2)$ or $f(4)$ (1 of 4 pts)? You must explain (4 of 5 pts).

$f(4)$

because f is increasing from 2 to 4

because $f'(x) > 0$ for $2 < x < 4$.

(b) (5 points) At what value(s) of x does f have a local minimum (2 pts)? Explain (3 pts).

1ST derivative test $f'(x)$: $- -$ 0 $+++$ 0 $- -$ 0 $++$

x: 0 4 8

f $\searrow$ LOCAL MIN $\nearrow$ LOCAL MAX $\searrow$ LOCAL MIN $\nearrow$

Local min at $x = 0, 8$.

5. Let f be the function $f(x) = (4 - 5x)^3$.

(a) (6 points) Find the equation of the tangent line to the graph $y = f(x)$ at the point $(1, -1)$.

$$f(1) = (-1)^3 = -1$$

$$f'(x) = 3(4-5x)^2(-5) = -15(4-5x)^2$$

$$f'(1) = -15(-1)^2 = -15.$$

Tangent line is: $y = f(1) + f'(1)(x-1)$

$$\boxed{y = -1 - 15(x-1)}$$

6. A point moves along the curve $y = x^2 + 2x - 2$ in such a way that when it is at $(1, 1)$ the x-coordinate is **decreasing** at a rate of 1 unit/second. At this time what is

(a) (7 points) the rate of change of the y-coordinate of the point?

Related rates; Implicitly differentiate $\frac{d}{dt}$

$$\frac{dy}{dt} = (2x+2)\frac{dx}{dt}$$

Given at $x=1$, $\frac{dx}{dt} = -1$.

So $\frac{dy}{dt} = (2\cdot 1 + 2)(-1) = -4$

(b) (3 points) the rate of change of the distance $d = \sqrt{x^2 + y^2}$ from the point to the origin?

$\frac{d}{dt}$ again $\quad \frac{dd}{dt} = \frac{1}{2}(x^2+y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$

plug in $x=1$, $y=1$, $\frac{dx}{dt} = -1$, $\frac{dy}{dt} = -4$ from part (a)

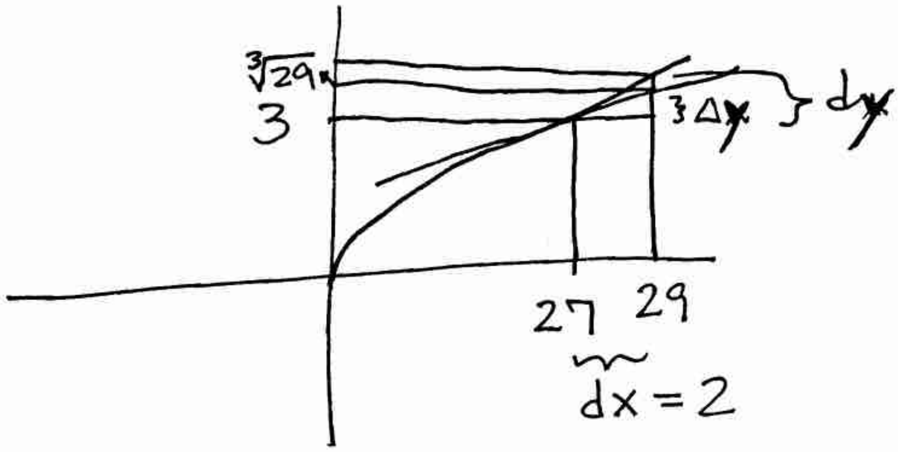
$$\frac{dd}{dt} = \frac{1}{2}(1^2+1^2)^{-1/2}\left(2\cdot 1(-1) + 2(1)(-4)\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}(-10) = \frac{-10}{2\sqrt{2}} = -\frac{5}{\sqrt{2}}$$

7. Starting with $(27)^{1/3} = 3$,

(a) (6 points) show how differentials/linear approximation can be used to approximate $(29)^{1/3}$. Sketch a picture illustrating your computation.

Use function $f(x) = x^{1/3}$



$$29^{1/3} = f(29) = f(27) + \Delta y$$
$$\approx f(27) + dy$$
$$= 3 + dy$$

$$y = f(x) = x^{1/3}$$
$$dy = \tfrac{1}{3} x^{-2/3}\, dx$$

At $x = 27$, $dx = 2$,

$$dy = \tfrac{1}{3}(27)^{-2/3}(2)$$
$$= \tfrac{1}{3}(3)^{-2}(2)$$
$$= \frac{2}{27}$$

Thus

$$29^{1/3} \approx 3 + \frac{2}{27}$$

8. Let $f(x) = \dfrac{(x+1)(x+3)}{x^2+3}$. Then $f'(x) = \dfrac{4(3-x^2)}{(x^2+3)^2}$ and $f''(x) = \dfrac{8x(x^2-9)}{(x^2+3)^3}$.
No need to check; you can trust me!

(a) (2 points) List all $x-$intercepts and $y-$intercepts.

$f(x)=0$ when $x=-1,-3$ (x-intercepts)

$f(0) = \frac{3}{3} = 1$ (y-intercept)

(b) (3 points) List all intervals where f increases.

$f'(x)=0$ when $3-x^2=0$

$3 = x^2$

$\pm\sqrt{3} = x$

$f'(x)$: $- - -$ 0 $+ + + +$ 0 $- - -$

x: $-\sqrt{3}$, $\sqrt{3}$

f: $\searrow$ LOCAL MIN $\nearrow$ LOCAL MAX $\searrow$

Increasing on $[-\sqrt{3}, \sqrt{3}]$

(c) (3 points) List all inflection points and intervals where the graph is concave down.

$f''(x)$: $- -$ 0 $+ + +$ 0 $- - -$ 0 $+ +$

x: -3, 0, 3

f: $\frown$ $\smile$ $\frown$ $\smile$

$f''(x)=0$ when $8x(x^2-9)=0$

$x = 0, -3, 3$

Concave down on $(-\infty, -3)$ and $(0, 3)$

(d) (2 points) List any asymptotes (vertical or horizontal).

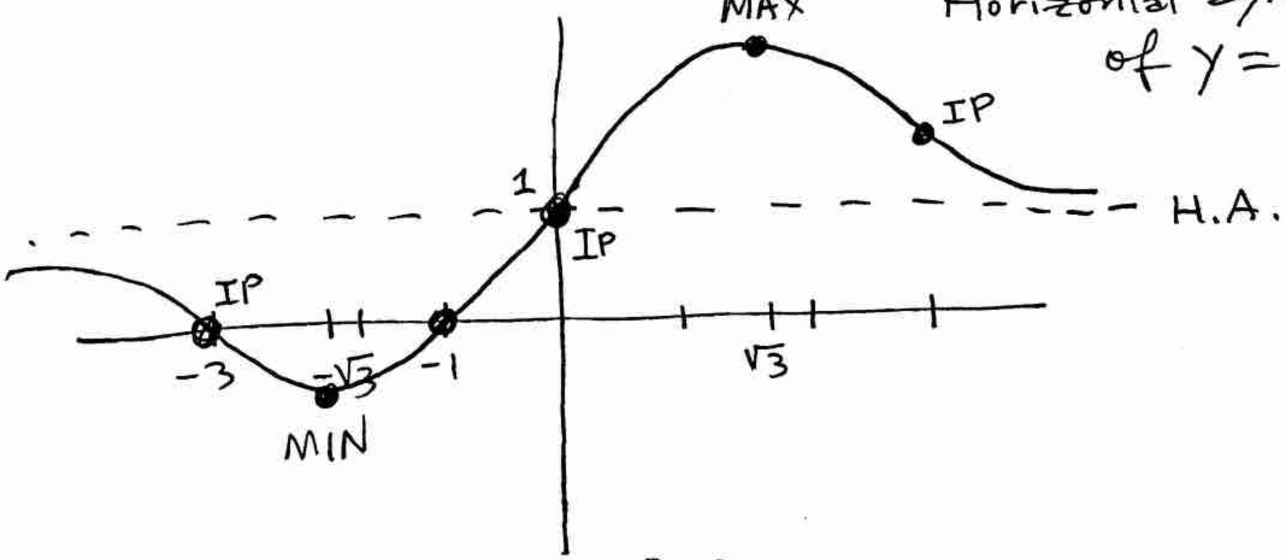
No vertical asymptotes

$$\lim_{x\to\infty} \frac{(x+1)(x+3)}{x^2+3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)\left(1+\frac{3}{x}\right)}{1+\frac{3}{x^2}} = \frac{(1+0)(1+0)}{1+0} = 1$$

Same as $x \to -\infty$

Horizontal asymptote of $y=1$

(e) (3 points) Sketch the graph $y = f(x)$.



The problems on this page are multiple choice. Circle the letter of the correct answer.

9. (4 points) Consider the integral $I = \int_0^1 \frac{1}{1+x^6}\,dx$. Then

   (a) $I < 1$.

   (b) $I = 1$.

   (c) $I > 1$.

$\frac{1}{1+x^6} < 1$ all $x \neq 0$

10. (3 points) True or False? If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.

    (a) True.

    (b) False.

Can have a "corner"

11. (3 points) True or False? Suppose that f and g are differentiable on $(-\infty, \infty)$ and $f'(x) = g'(x)$ for all x. If $f(0) > g(0)$, then $f(x) > g(x)$ for all x.

    (a) True.

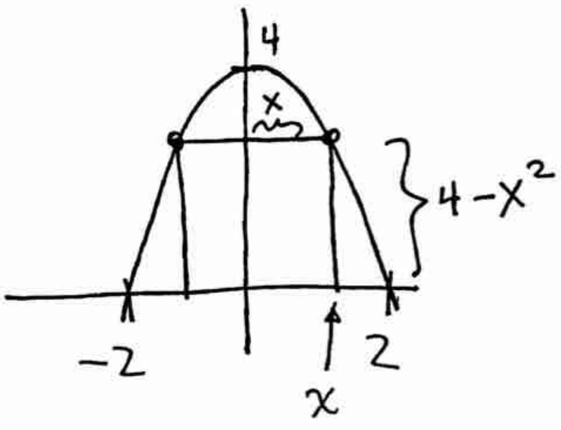
    (b) False.

Then $f(x) = g(x) + C$ some C

$f(0) > g(0)$ implies $C > 0$.

12. (8 points) Find the maximum *perimeter* of a rectangle that has its bottom two corners on the x-axis and its top two corners on the parabola $x^2 + y = 4$. (The perimeter of a rectangle is the sum of lengths of its four edges.)

$y = 4 - x^2$



The perimeter is

$$P = 4x + 2(4 - x^2)$$
$$P = 4x + 8 - 2x^2$$

domain $0 \le x \le 2$

$$\frac{dP}{dx} = 4 - 4x \stackrel{\text{set}}{=} 0$$
$$4 = 4x$$
$$1 = x \quad \text{critical point}$$

1ST derivative diagram

$\frac{dP}{dx}$: [++ 0 --]

x: 0, 1, 2

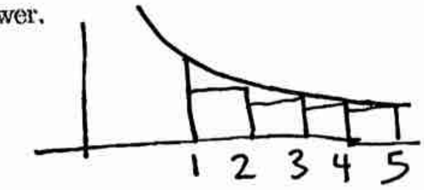
P: ↗ ABS MAX ↘

Absolute max at $x = 1$, which yields

$$P = 4 + 2(3) = \boxed{10}$$

13. Consider the function $f(x) = \dfrac{1}{x}$, $1 \le x \le 5$.

(a) (4 points) Compute a Riemann sum for this function that approximates the integral $\int_1^5 f(x)dx$. Use four equal-width intervals for your Riemann sum, and use the right end-point of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

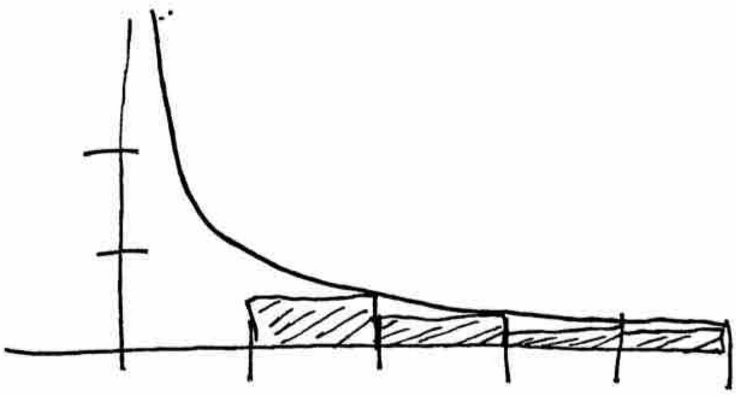


$n = 4$

$\Delta x = \dfrac{b-a}{4} = \dfrac{5-1}{4} = 1$

$$\text{Riemann sum} = f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x$$
$$= \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{5}\cdot 1$$

(b) (4 points) Sketch the graph $y = f(x)$, $1 \le x \le 5$, and the rectangles that correspond to the Riemann sum in part (a).



(c) (1 point) Does your solution to (a) overestimate or underestimate $\int_1^5 f(x)dx$?

underestimate

14. Compute the following integrals.

(a) (4 points) $\int t^2(t-1)\,dt.$ $= \int (t^3 - t^2)\,dt$

$= \frac{1}{4}t^4 - \frac{1}{3}t^3 + C$

(b) (4 points) $\int_0^1 (2x+1)^4 dx.$ Substitute $u = 2x+1$ $\quad x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=3$

$du = 2dx$

$\frac{1}{2}du = dx$

$= \int_1^3 u^4 \, \frac{1}{2}du$

$= \frac{1}{2}\frac{1}{5}u^5\Big|_1^3 = \frac{1}{10}(3^5 - 1^5) = \frac{3^5-1}{10} \left(= \frac{242}{10}\right)$

(c) (4 points) $\int_0^1 \frac{d}{dx}\left[\frac{x(x+2)}{x^4+5}\right] dx.$ $= \frac{x(x+2)}{x^4+5}\Big|_0^1$

$= \frac{1\cdot 3}{6} - \frac{0\cdot 2}{5} = \frac{3}{6} = \frac{1}{2}$

15. (6 points) Find an exact formula for $f(t)$, given that $f''(t) = 3 - \cos t$, $f'(0) = 5$, and $f(0) = -2$.

$f'(t) = 3t - \sin(t) + C$

$5 = f'(0) = 0 - \sin(0) + C \Rightarrow 5 = C$

$f'(t) = 3t - \sin(t) + 5$

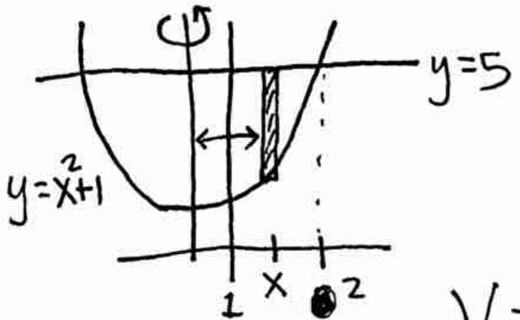
$f(t) = \frac{3}{2}t^2 + \cos(t) + 5t + C_2$

$-2 = f(0) = 0 + \cos 0 + 0 + C_2 \Rightarrow -2 = 1 + C_2$

$-3 = C_2$

$f(t) = \frac{3}{2}t^2 + \cos(t) + 5t - 3$

16. (a) (5 points) Set-up (do not evaluate!) an integral that represents the volume of the solid generated by revolving about the y-axis the region bounded by the curves $x = 1$, $y = 5$, and $y = x^2 + 1$, $1 \le x \le 2$. You may use any method.



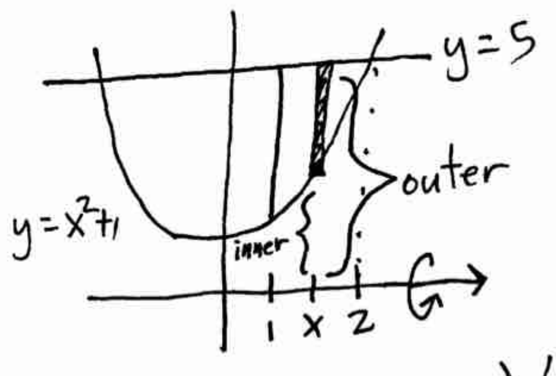
Choose x-problem

Get shells

$$V = 2\pi \int_1^2 (\underbrace{x - 0}_{\text{radius}})(\underbrace{5 - (x^2+1)}_{\text{height}})\,dx$$

$$= 2\pi \int_1^2 x(4 - x^2)\,dx$$

(b) (5 points) Set-up an integral (do not evaluate!) if the solid is now obtained by revolving the above region about the x-axis.



choose x-problem

Get washers

$$V = \pi \int_1^2 \Big((\underbrace{5 - 0}_{\substack{\text{outer}\\\text{radius}}})^2 - (\underbrace{x^2 + 1 - 0}_{\substack{\text{inner}\\\text{radius}}})^2 \Big)\,dx$$

$$= \pi \int_1^2 \left(5^2 - (x^2+1)^2\right)dx$$