

Math 241 Spring 2016 Common Final

Name and section number:

Solutions

by
D. Yuen

Question	Points	Score
1	12	
2	7	
3	12	
4	12	
5	9	
6	6	
7	6	
8	10	
9	11	
10	15	
11	6	
12	14	
13	6	
Total:	126	

You may not use notes or electronic devices on the test.

Show all your work.

1(12 points). Compute the following limits. Simplify your answers. Be as specific as possible, e.g., $\infty, -\infty$ instead of "d.n.e." when possible.

(a)(4 points) $\lim_{x \rightarrow 1^-} \frac{x+1}{x^2-1}$

Limit type $\frac{2}{0}$

Analyzing the denominator further

$$= \lim_{x \rightarrow 1^-} \frac{x+1}{(x+1)(x-1)}$$

$$= \lim_{x \rightarrow 1^-} \frac{1}{x-1}$$

Limit type $\frac{1}{0^-}$

$$= \boxed{-\infty}$$

(b)(4 points) $\lim_{x \rightarrow \infty} \frac{x+1}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

Limit type $\frac{\infty}{\infty}$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$$

$$= \frac{0+0}{1-0} = \boxed{0}$$

(c)(4 points) $\lim_{x \rightarrow 0} \frac{\tan(x)}{x}$

Limit type $\frac{0}{0}$

Idea: Use $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = \boxed{1}$$

2(7 points). For the function $f(x) = 1 - 2x$, use the definition of the derivative to prove that $f'(x) = -2$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1 - 2(x+h)) - (1 - 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - 2x - 2h - 1 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{h} \\ &= \lim_{h \rightarrow 0} (-2) \\ &= -2. \end{aligned}$$

3(12 points). Find the following derivatives. Algebraic simplification is not required.

(a)(4 points) $f(x) = \frac{x+1}{x}$, $f'(1) = ?$

$$f'(x) = \frac{1(x) - (x+1)1}{x^2}$$

$$f'(x) = \frac{x - x - 1}{x^2}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f'(1) = -\frac{1}{1^2} = -1$$

(b)(4 points) $g(t) = \sqrt{1 + \cos^2(t)}$, $g'(t) = ?$

$$g'(t) = \frac{1}{2}(1 + \cos^2(t))^{-1/2} \cdot 2 \cos(t) \cdot (-\sin(t))$$

(c)(4 points) $y = (1-x)^{10}$, $\frac{d^2y}{dx^2} = ?$

$$\frac{dy}{dx} = 10(1-x)^9(-1)$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 10 \cdot 9(1-x)^8 \cdot (-1)(-1) \\ &= 90(1-x)^8. \end{aligned}$$

4(12 points). Find the following integrals.

(a)(4 points) $\int t^9(t^2 - 1) dt$

$$\begin{aligned} &= \int (t^{11} - t^9) dt \\ &= \frac{1}{12} t^{12} - \frac{1}{10} t^{10} + C \end{aligned}$$

(b)(4 points) $\int_0^1 t(t^2 - 1)^9 dt$ Sub $u = t^2 - 1$

$$\begin{aligned} \int t(t^2 - 1)^9 dt &= \int \frac{1}{2} (t^2 - 1)^9 2t dt \quad du = 2t dt \\ &= \int \frac{1}{2} u^9 du = \frac{1}{2} \cdot \frac{1}{10} u^{10} + C = \frac{1}{20} (t^2 - 1)^{10} + C \\ &= \frac{1}{20} (t^2 - 1)^{10} \Big|_0^1 \\ &= \frac{1}{20} (0^{10} - 1^{10}) = -\frac{1}{20} \end{aligned}$$

(c)(4 points) $\int \sin(\pi t) \cos(\pi t) dt$

Sub $u = \sin(\pi t)$ [$u = \cos(\pi t)$ also works]
 $du = \pi \cos(\pi t) dt$

$$\begin{aligned} &= \int \frac{1}{\pi} \sin(\pi t) \underbrace{\pi \cos(\pi t) dt}_{du} \\ &= \int \frac{1}{\pi} u du \\ &= \frac{1}{\pi} \frac{1}{2} u^2 + C \\ &= \frac{1}{2\pi} (\sin(\pi t))^2 + C \end{aligned}$$

5(9 points). Piecewise functions and the Fundamental Theorem. Recall that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

(a)(3 points) $f(x) = \int_0^x |t| dt$. Find $f'(x)$. Write using a single formula.

$$f'(x) = |x|$$

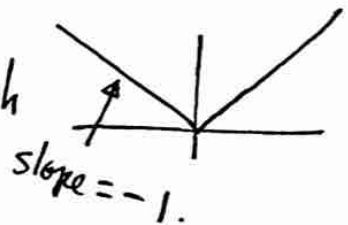
(b)(3 points) $g(x) = |x|$. Find $g'(-2)$.

Around $x = -2$, $g(x) = -x$ for $x < 0$.

So $g'(x) = -1$ for $x < 0$.

Then $g'(-2) = -1$.

Can also deduce this by looking at the graph



(c)(3 points) $g(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$. Find $\int_{-1}^2 g(t) dt$.

$$= \int_{-1}^0 g(t) dt + \int_0^2 g(t) dt$$

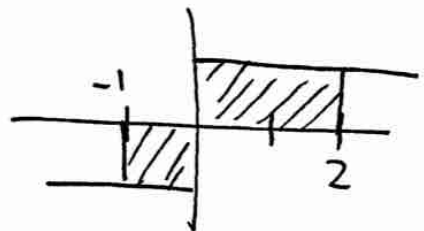
$$= \int_{-1}^0 (-1) dt + \int_0^2 1 dt$$

$$= -x \Big|_{-1}^0 + x \Big|_0^2$$

$$= (0 - (+1)) + (2 - 0)$$

$$= 1.$$

OR



Net area is

$$-1 + 2 = 1.$$

6(6 points). Given a curve with equation $x^2 - xy + 2y^2 = 2$, find the equation for the line tangent to the curve at the point $(0, 1)$. Hint, use implicit differentiation.

$$2x - 1y - xy' + 4yy' = 0$$

$$y'(-x + 4y) = -2x + y$$

$$y' = \frac{-2x + y}{-x + 4y}$$

at $x=0, y=1$

$$y' = \frac{-0 + 1}{-0 + 4} = \frac{1}{4}$$

Tangent line is $y - 1 = \frac{1}{4}(x - 0)$.

7(6 points). The volume of a sphere of radius r is $V = \frac{4\pi}{3}r^3$. Use linear approximation (differential approximation) to estimate the change in volume of a ball if the radius changes from 10 inches to 10.01 inches.

$$dV = 4\pi r^2 dr$$

At $r=10, dr=.01$, we have

Change in volume is approximately

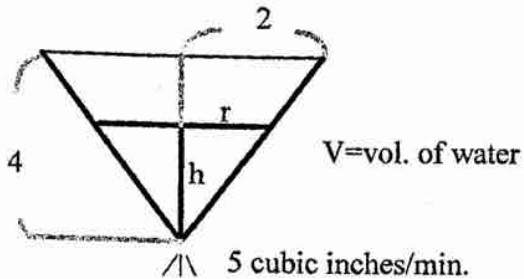
$$dV = 4\pi (10)^2 (.01) = 4\pi \text{ in}^3$$

8(10 points). A conical cup has radius 2" and height 4".

It leaks water at a rate of 5 cubic inches/min.

Find the rate of change of the water level when the level is 2".

The volume V of a cone of radius r , height h is $V = \frac{\pi}{3}r^2h$.



By similar Δ s,

$$\frac{2}{4} = \frac{r}{h}$$

$$\frac{1}{2}h = r$$

$$V = \frac{\pi}{3}r^2h$$

$$V = \frac{\pi}{3} \cancel{r^2} \left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

Evaluate at $h=2$, $\frac{dV}{dt} = -5 \text{ in}^3/\text{min}$

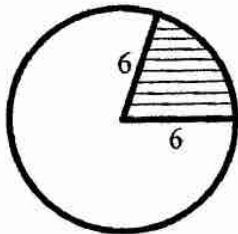
$$-5 = \frac{\pi}{4}(2)^2 \frac{dh}{dt}$$

$$-5 = \pi \frac{dh}{dt}$$

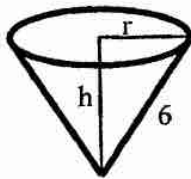
$$-\frac{5}{\pi} = \frac{dh}{dt}$$

Water level is decreasing at $\frac{5}{\pi}$ in/min.

9(11 points). You have a 6 inch radius circle of paper. You remove a pie-shaped wedge and form the remaining paper into a cone-shaped cup of some height h and radius r . The volume V of a cone of radius r , height h is $V = \frac{\pi}{3}r^2h$.



paper radius = 6 inches



$V =$ volume (maximize)

Want r, h

$$\begin{aligned} r^2 + h^2 &= 36 \\ r^2 &= 36 - h^2 \\ r &= \sqrt{36 - h^2} \end{aligned}$$

(a)(4 points) Write the volume V as a function of the height h .

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(36 - h^2)h = \frac{\pi}{3}(36h - h^3)$$

$$0 \leq h \leq 6$$

(b)(4 points) Find height and radius which gives the cup of largest volume.

$$\frac{dV}{dh} = \frac{\pi}{3}(36 - 3h^2) \stackrel{\text{SET}}{=} 0$$

$$36 = 3h^2$$

$$12 = h^2$$

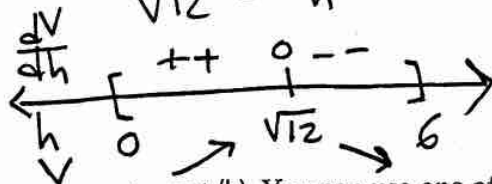
$$\pm\sqrt{12} = h$$

$$\sqrt{12} = h$$

Abs max volume
at $h = \sqrt{12}$

$$r = \sqrt{36 - 12} = \sqrt{24}$$

1st derivative
diagram



(c)(3 points) Justify your answer to part (b). You can use one of the derivative tests or the Extreme Value Theorem (for a continuous function on a closed interval, the largest extreme value is the absolute maximum, the smallest is the absolute minimum).

10(15 points). Given $f(x) = \frac{x(3-x)}{(x+3)^2}$, $f'(x) = \frac{-9(x-1)}{(x+3)^3}$, $f''(x) = \frac{18(x-3)}{(x+3)^4}$

You don't need to check the derivatives.

(a)(2 points) Find the x-intercepts and the y-intercept.

x-intercept ($y=0$) when $x=0, 3$

y-intercept ($x=0$) when $y=0$

(c)(1 points) Find the equation for the vertical asymptote

$$x = -3$$

(d)(2 points) Find the equation for the horizontal asymptote

$$y = -1$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{3x - x^2}{(x+3)^2} = -1. \quad \text{Same } \lim_{x \rightarrow -\infty} f(x) = -1$$

(e)(2 points) Find both coordinates of the point with maximum value

$f'(x)$ ~~0~~ DNE ++ 0 ---
 x ~~0~~ -3 1

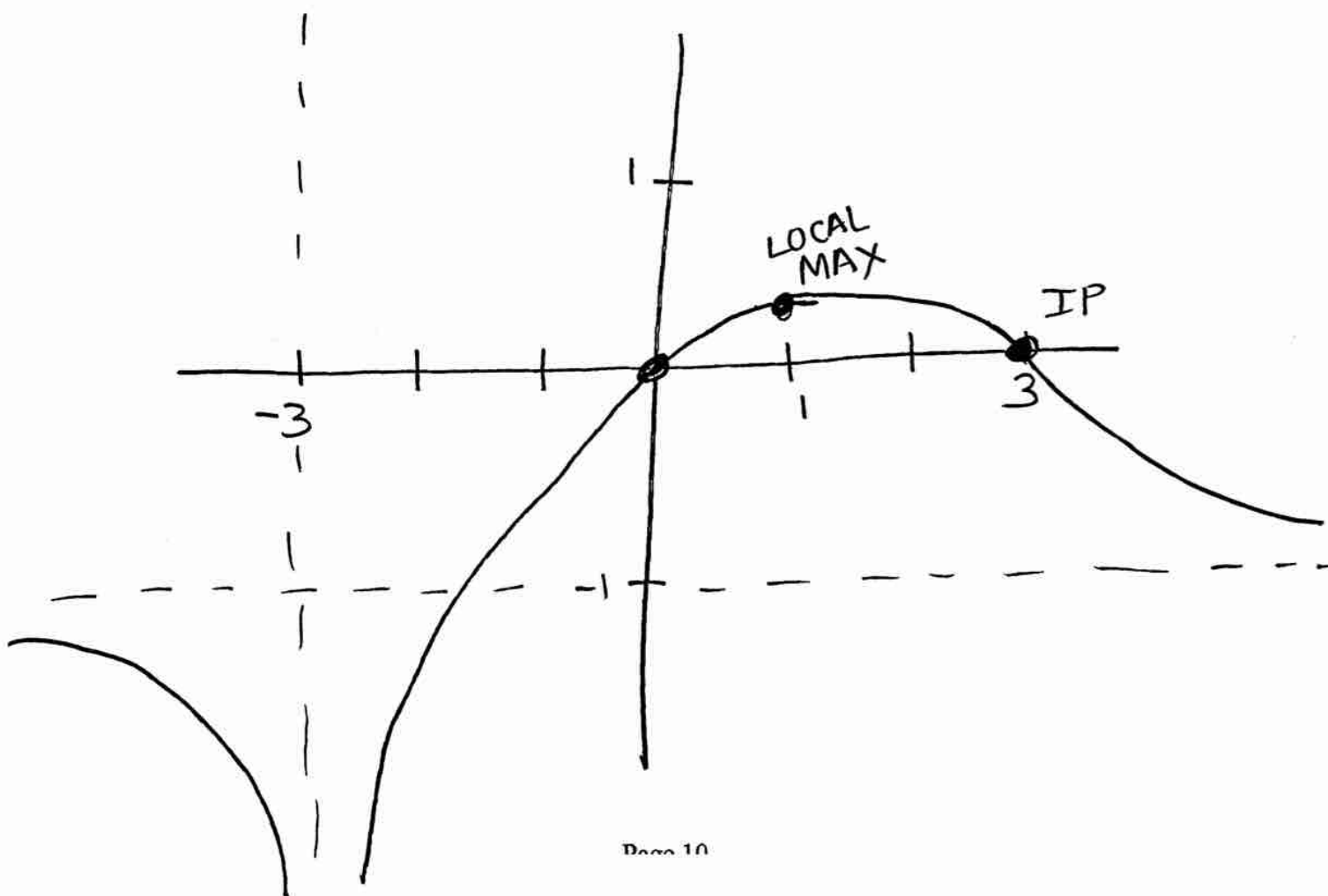
Max at $x=1$, $f(1) = \frac{2}{16} = \frac{1}{8}$

(f)(2 points) Find both coordinates of the inflection point.

$f''(x)$ - DNE --- 0 ++
 x -3 3

IP at $x=3$, $f(3) = 0$

(g)(6 points) Draw the graph of the function and its asymptotes.



11(6 points). Let $s(t)$, $v(t)$, $a(t)$ be the position, velocity and acceleration of a point on the y -axis at time t . Find the position $s(t)$ at any time t given that:

$$a(t) = -2t, v(0) = 3, s(0) = 1.$$

$$\hookrightarrow v(t) = -t^2 + C$$

Use $v(0) = 3$

$$3 = v(0) = -0 + C$$

$$3 = C$$

$$\rightarrow v(t) = -t^2 + 3$$

$$s(t) = -\frac{1}{3}t^3 + 3t + C_2$$

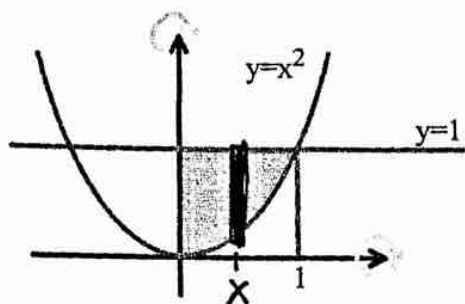
Use $s(0) = 1$

$$1 = s(0) = -0 + 0 + C_2$$

$$1 = C_2$$


$$s(t) = -\frac{1}{3}t^3 + 3t + 1$$

12(12 points). The region below is to the right of $x = 0$, above $y = x^2$, below $y = 1$.



When rotated around the x or y axis, it forms a solid of revolution. You may use the disk method or the shell method.

(a)(6 points) Find the volume of the solid of revolution formed by rotating this region around the x -axis. Set up the integral but do not evaluate it.


Use dx  . Get washers.

$$V = \pi \int_0^1 ((1-0)^2 - (x^2-0)^2) dx$$

Also possible to do as dy -problem, get shells:

$$V = 2\pi \int_0^1 (y-0)(\sqrt{y}-0) dy$$

(b)(6 points) Find the volume of the solid of revolution formed by rotating this region around the y -axis. Set up the integral but do not evaluate it.

Use dx  Get shells.

$$V = 2\pi \int_0^1 (x-0)(1-x^2) dx$$

As a dy -problem, get washers:

$$V = \pi \int_0^1 ((\sqrt{y}-0)^2 - 0^2) dy$$

13(6 points). The Intermediate Value Theorem states that for continuous functions, if $f(a) \leq C \leq f(b)$ then $f(c) = C$ for some c between a and b .

Suppose $f(x) = x^5 - x^2 + 10x - 1$.

(a)(3 points) Show that f has a root between -1 and 1 .

Note f is continuous everywhere.

$$f(-1) = -1 - 1 - 10 - 1 = -13$$

$$f(1) = 1 - 1 + 10 - 1 = 9.$$

Since $f(-1) < 0 < f(1)$,

there there is some $-1 < c < 1$
such that $f(c) = 0$.

The Mean Value Theorem states that for differentiable functions, if a and b are distinct numbers, then $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c between a and b .

(b)(3 points) Show that $f'(c) = 11$ for some c between -1 and 1 .

~~Given~~ Note f is ~~not~~ differentiable everywhere
(note $f'(x) = 5x^4 - 2x + 10$).

Apply MVT to $a = -1$, $b = 1$.

$$f(a) = f(-1) = -13.$$

$$f(b) = f(1) = 9$$

$$\text{We have } \frac{f(b) - f(a)}{b - a} = \frac{9 - (-13)}{1 - (-1)} = \frac{22}{2} = 11.$$

By MVT, there exists a c between -1 and 1
such that $f'(c) = 11$.