Math 601: Applied Dynamical Systems

January 9th

Topics to review:

1) Analysis

i) contraction mapping principle:

Definition: let X be a metric space with metric d. If $\varphi: X \to X$ is such that $\exists\, 0 \le c < 1$ with $d(\varphi(x), \varphi(y)) \le c\, d(x,y)$ $\forall x,y \in X$, then φ is a contraction of X into X

Theorem (contraction mapping principle): If X is a complete metric space and if φ is a contraction of X into X, then $\exists!\, x \in X$ s.t. $\varphi(x) = x$

ii) Implicit Function Theorem and Inverse Function Theorem ← used to define manifolds

Informally, the inverse function theorem states that a continuously differentiable, f, is invertible in a neighborhood of any point x at which the linear transformation $f'(x)$ is invertible

Theorem: Suppose f is a C^1-mapping of an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible, and $b = f(a)$ then:

a) $\exists$ open sets U and V in $\mathbb{R}^n$ s.t. $a \in U$, $b \in V$, f is one-to-one on U, and $f(U) = V$

b) If g is the inverse of f defined in V by $g(f(x)) = x$ $(x \in U)$, then $g \in C'(V)$

↳ i.e. If $y = f(x)$, the system of n equations: $y_i = f_i(x_1, \dots, x_n)$ for $1 \le i \le n$, can be solved for $x_1, \dots, x_n$ in terms of $y_1, \dots, y_n$. If we restrict x and y to small neighborhoods of a and b, the solutions are unique and continuously differentiable

Implicit Function Theorem: If f is a continuously differentiable real function in the plane, then the equation $f(x,y) = 0$ can be solved for y in terms of x in a neighborhood of any point (a,b) at which $f(a,b) = 0$ and $\partial f / \partial y \neq 0$

↳ can solve for x in terms of y near (a,b) if $\partial f / \partial x \neq 0$ at (a,b)

iii) Taylor expansions in multiple variables

Definition: The Taylor Series of an infinitely differentiable function f at a is

$$T(x_1, \dots, x_d) = f(a_1, \dots, a_d) + \sum_{j=1}^{d} \frac{\partial f(a_1, \dots, a_d)}{\partial x_j}(x_j - a_j) + \frac{1}{2!} \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f(a_1, \dots, a_d)}{\partial x_j \partial x_k}(x_j - a_j)(x_k - a_k)$$

$$+ \frac{1}{3!} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \frac{\partial^3 f(a_1, \dots, a_d)}{\partial x_j \partial x_k \partial x_\ell}(x_j - a_j)(x_k - a_k)(x_\ell - a_\ell) + \dots = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1 + \dots + n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \right)(a_1, \dots, a_d)$$

Example: In one variable: $T = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$

In two variables: $T(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b) \right)$

2) Linear Algebra

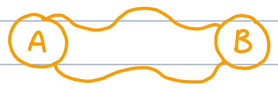
This class will focus on dynamical systems (describing how the state of a system evolves in time)

There are three ingredients to define a dynamical system:

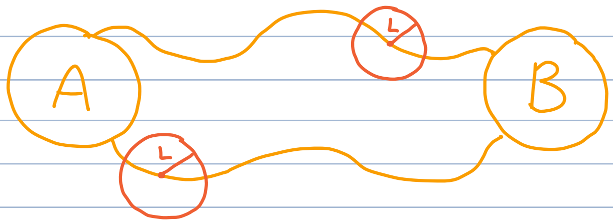
1) The state space: The set of all possible states of the system
2) Time (future/past/discrete/continuous)
3) Evolution operators (a way to describe evolution)

State Space (usually denoted X)

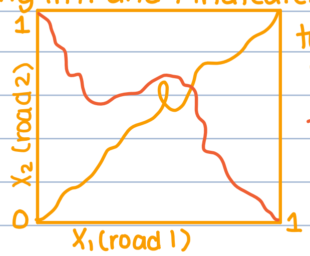
Example: Consider two cities A and B connected by two non-intersecting roads



Suppose that it is known that two cars connected by a rope of length $<2L$ can get from one city to the other without breaking the rope. Is it possible for two circular wagons of radius L, starting at different cities, to pass each other



The position of an object on a road can be described by a number between 0 and 1, where 0 indicates being in A and 1 indicates being in B. Thus the state space is $[0,1]\times[0,1]$



two wagons starting at the same city and travelling to the other connected by a rope of length $<2L$

two wagons starting at opposite cities

Since no matter how they move, the two curves will have to intersect. At this point of intersection the two wagons will be at the same position that two wagons starting from the same city would be in, i.e. they would have to be connected by a rope of length $<2L$. But since each wagon has radius L, this is impossible. Thus the answer is no.

Time

Definition: **Continuous time** is $T=\mathbb{R}$ (or $\mathbb{R}_+$ if you can't look into the past) and **discrete time** is $T=\mathbb{Z}$ (or $\mathbb{Z}_+$) where T denotes the set of all possible times

Example: continuous time is used in physics usually and discrete in biology

Evolution

Need to know the state of the system at time t given a particular state at some time

For any $t\in T$, we have a map $\varphi^t: X\to X$, where X is the state space, such that if x_0 is the state at time 0, then $\varphi^t(x_0)$ is the state at time t

↳ Definition: For a fixed t, φ^t is called an **evolution operator** and the family $\{\varphi^t\}_{t\in T}$ is called the **flow**

↳ The maps φ^t can be defined for both positive and negative t (i.e. φ^t is invertible i.e. you can look into the past) or just for positive t (i.e. φ^t is non-invertible)

↳ we mostly focus on invertible ones

We also need φ^t to satisfy the following properties:

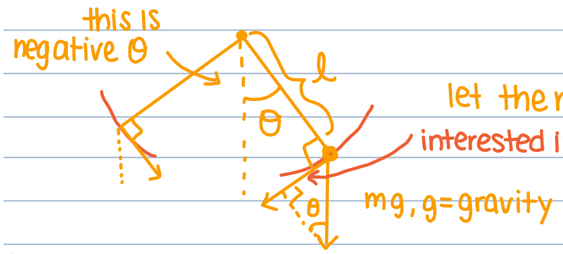
1) $\varphi^0 = id_X$ (i.e. $\varphi^0(x)=x$)

2) $\varphi^{t+s}=\varphi^s\circ\varphi^t=\varphi^t\circ\varphi^s$ (i.e. to get to $t+s$, you first evolved to time t, then evolved the remainder time)

Definition: A **dynamical system** is a triple (X, T, φ^t) where X is a complete metric space
↳ T is the **time set** $(\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+)$ and $\{\varphi_t\}_{t\in T}$ is the **family of evolution operators** (i.e. maps satisfying the above two conditions)

January 11th

Example: set up a dynamical system to describe a typical pendulum



let the mass of the object be m
interested in the tangential component of the force

let T be continuous. It seems reasonable to let the state space be a circle where position is determined by θ (we will see that this is not enough)
If moving left, θ is decreasing
the gravity force will always have sign opposite to acceleration (as we go down, we accelerate faster and faster

by Newton's law of motion, $F=ma \Rightarrow \underbrace{mg\sin\theta}_{\text{amplitude of the force}} = -ma$
The distance travelled by the pendulum is the arc of the circle which relates to θ using the formula $s=\text{arc length}=\ell\theta \Rightarrow a=\ddot{s}=\ell\ddot{\theta}$ where $\ddot{}$ denotes the second derivative
$\Rightarrow mg\sin\theta = -m\ell\ddot{\theta}$
This gives the differential equation: $\ddot{\theta}+\frac{g}{\ell}\sin\theta=0$

The idea of an evolution operator is given a start state, we should be able to find the state at time t but for this second order differential equation to have a unique solution, we need an initial position θ and velocity
so our state space X needs to be $S^1\times\mathbb{R}$ where S^1 is the circle for position θ and $\mathbb{R}$ is for the velocity
T is continuous so $T=\mathbb{R}$. Evolution is defined implicitly: $\dot{\theta}=v$, $\dot{v}=\frac{-g}{\ell}\sin\theta$, $(\theta, v)\in S^1\times\mathbb{R}$

A lot of real world situations can be described using differential equations

Let (X, T, φ^t) be a dynamical system. Assume it is invertible (so $T=\mathbb{R}$ or $T=\mathbb{Z}$)
Recall: $\varphi^0=\text{id}_X$ and $\varphi^{t+s}=\varphi^t\circ\varphi^s$. Notice that φ^{-t} is the inverse of φ^t so to each time t, we associate a bijective map $\varphi^t: X\to X$ such that 0 corresponds to id_X and $(s+t)$ corresponds to $\varphi^s\circ\varphi^t$
↳ This is a group action of T on X

A major question in dynamical systems is figuring out all possible outcomes of evolution

Let (X, T, φ^t) be a dynamical system:
Definitions:
i) A **trajectory** through $x_0\in X$ is a map from T to X given by $x(t, x_0)=\varphi^t(x_0)$
ii) An **orbit** through $x_0\in X$ is the range of the trajectory through x_0 i.e. $O(x_0)=\{x\in X: x=x(t, x_0), t\in T\}$
↳ trajectories and orbits are often used interchangeably but an orbit is the image of a trajectory
↳ note: if $x_1=x(t_1, x_0)$, then $O(x_1)=O(x_0)$
iii) A set $S\subseteq X$ is called an **invariant set** if $\varphi^t(x_0)\in S$ $\forall x_0\in S, t\in T$
↳ i.e. the evolution operator applied to a state in S will remain in that set

iv) An equilibrium state is an invariant set consisting of a single point, i.e. it is an $x^* \in X$ such that $\varphi^t(x_*) = x_* \ \forall t \in T$

↳ i.e. it is a fixed point of the evolution operator

↳ it is an orbit and also a trajectory

v) A trajectory is called periodic if $\exists t^* \in T$ such that $x(t+t^*, x_0) = x(t, x_0) \ \forall t \in T$

↳ idea no matter where you start, you'll always come back after t^*

↳ The smallest such t^* is called the period

vi) A cycle is the orbit of a periodic trajectory

Example: Let x denote the size of a population of bacteria and assume that the rate of change of the population is proportional to the size of the population with the coefficient of proportionality being a decreasing linear function of the population size. This process can be described using the following differential equation:

$\dot{x} = r(b-x)x$ ← can solve using separation of variables but we can make conclusions without solving

Notice that if $x(0)=0$ then $x(t)=0 \ \forall t \in \mathbb{R}$ ($\dot{x}=0$). Thus 0 is an equilibrium point

Also, if $x(0)=b$ then $\dot{x}=0 \Rightarrow x(t)=b \ \forall t \in \mathbb{R}$ so b is another equilibrium point

If you pick any other initial point there are three possibilities:

i) moves to $\pm\infty$ ← impossible since for x larger than b, $\dot{x}<0$ so decreases

ii) could go towards 0 $\Big\}$ depends on the notion of "stability"

iii) could go towards b

January 13th

Let (X, T, φ^t) be a dynamical system

Definition: An invariant set S_0 is called stable if for any neighborhood U of S_0 exists, there exists a neighborhood V of S_0 s.t. $\varphi^t(x) \in U \ \forall t>0 \ \forall x \in V$

↳ The idea is that for any neighborhood U, we can start a small enough distance away from S_0 s.t. you never leave U (it is essentially an ε-δ argument)

Example: If S_0 is an equilibrium state:

Definition: S_0 is called asymptotically stable if it is stable and if $\varphi^t(x) \to S_0$ as $t \to \infty \ \forall x \in V$ (for same V as above

↳ idea is we can choose V small enough such that the path converges to S_0

↳ Definition: $\varphi^t(x) \to S_0$ means for any neighborhood O of S_0, $\exists t^*>0$ s.t. $\varphi^t(x) \in O \ \forall t > t^*$

For many dynamical systems, the state space is an open subset of $\mathbb{R}^n$ (i.e. model the state of the system using tuples of $\mathbb{R}^n$) Also, the evolution is expressed using a system of differential equations. That is, $X = U \subseteq \mathbb{R}^n$, U open and $\dot{x} = f(x)$ where f is a continuous function on U

If evolution is described by a differential equation, is the flow (i.e. the family of evolution operators) well-defined?

The answer is given by the existence and uniqueness theorem for ODEs (we dont consider PDEs)

Theorem: Consider an initial value problem: $\dot{x}=f(x)$, $x(t_0)=x_0$, where $x\in U\subseteq\mathbb{R}^n$, U open, and $f\in C^r(U)$, $r\geq 1$. Then for $|t-t_0|$ small enough, $\exists!$ solution of the above IVP, $x(t,t_0,x_0)$, and the solution is a C^r function of (t,t_0,x_0) (usually just denoted as a function of t)

↳ Note: If $r=0$, you can prove existence but not uniqueness

Proof: Uses the contraction mapping principle

↳ The contraction mapping principle states that if $A:X\to X$ is a complete metric space and $\exists\,\alpha\in[0,1)$ such that $\rho(A(x),A(y))\leq\alpha\rho(x,y)$ where ρ is the metric, then $\exists!\,x^*$ s.t. $A(x^*)=x^*$

↳ A satisfying the hypothesis is called a contraction mapping

Proof: Form a sequence $x, Ax, A^2x, \ldots$ and show it converges to a fixed point

Definition: A Banach space is a complete normed vector space

Theorem: Let X be a Banach space. Let $A_y, y\in Y$ be a family of contractions such that $\exists\,\alpha\in[0,1]$, $\rho(A_y(x_1),A_y(x_2))\leq\alpha\rho(x_1,x_2)$ $\forall y\in Y$. If Y is a closed set of some Banach space (different), then for each $y\in Y$, $\exists!\,g(y)\in X$ s.t. $A_y(g(y))=g(y)$ where $g(y)$ depends continuously on y if A_y depends continuously on y

Consider $\dot{x}=f(x)$, $x(t_0)=x_0$

Notice that a function $\psi(t)$ is a solution iff $\psi(t)=x_0+\int_{t_0}^{t}f(\psi(\tau))d\tau$

So what if we consider a transformation $(A\psi)(t)=x_0+\int_{t_0}^{t}f(\psi(\tau))d\tau$

Notice that ψ is a solution of the IVP if it is a fixed point of A

January 18th

Initial value problem: $x\in\mathbb{R}^n$, $\dot{x}=f(x)$, $x(t_0)=x_0$ (1)

Theorem: Suppose $f\in C^r(U)$, $U\subseteq\mathbb{R}^n$ open for $r\geq 1$. Then for $|t-t_0|$ sufficiently small, (1) has a unique solution $x(t,t_0,x_0)$. Moreover, $x(t,t_0,x_0)$ is a C^r function of its arguments

↳ Note: the dependency on t is actually C^{r+1}

φ is a solution of (1) if $\varphi(t)=x_0+\int_{t_0}^{t}f(\varphi(\tau))d\tau$

So we'd like to consider $(A\varphi)(t)=x_0+\int_{t_0}^{t}f(\varphi(\tau))d\tau$

This A acts on functions φ such that $\varphi(t_0)=x_0$

It is more convenient to consider φ such that $\varphi(0)=0$ (by shifting)

↳ Notice that if $\varphi(0)=0$, then $\tilde{\varphi}(t)=\varphi(t-t_0)+x_0$ is such that $\tilde{\varphi}(t_0)=x_0$

So we want an operator where a fixed point is a solution

Notice that if φ is such that $\varphi(0)=0$, then $(A\varphi)(t)=\int_{t_0}^{t+t_0}f(\varphi(\tau-t_0)+x_0)d\tau$ is such that $(A\varphi)(0)=0$

Moreover, if $A\varphi=\varphi$, then $x(t)=\varphi(t-t_0)+x_0$ is a solution of (1). Note that we only need to consider φ such that $|\varphi(t)|\leq b$ where b is an appropriate constant

Notice that if f is bounded (by M), in a neighborhood of x_0, then for small enough $|t-t_0|$, $|\varphi(t)-x_0|\leq\int_{t_0}^{t}|f(\varphi(\tau))|dt\leq M|t-t_0|$, where φ is a solution. So if $|t-t_0|\leq c$, then $|\varphi(t)-x_0|\leq Mc=:b$

↳ The idea is that if the solution cannot go far away from x_0, then φ cannot go far away from 0

Recall:

i) A continuous function on a compact set is bounded

ii) A continuously differentiably function on a compact set is Lipschitz

↳ **Theorem:** Let $f: U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^n$ open. If $f \in C(U)$ then f is bounded on any compact set $K \subseteq U$. Moreover, if $f \in C^r(U), r \geq 1$, then f is Lipschitz on any compact $K \subseteq U$

↳ **Definition:** A function is Lipschitz if $\exists\ \lambda \geq 0$ such that $|f(x)-f(y)| \leq \lambda |x-y|\ \forall x,y \in K$. λ is called the Lipschitz constant

Proof of first theorem: Let $F = C(I_a(0), \mathbb{R}^n)$ i.e. F is a space of continuous functions $\varphi: I_a(0) \to \mathbb{R}^n$ where $I_a(0) = (-a,a)$.

Restrict F further to functions φ such that $\varphi(0)=0$ and $|\varphi(t)| \leq b\ \forall t \in I_a(0)$.

(we dont know what a and b are yet. we will choose them appropriately so that things work out)

Let $(A\varphi)(t) = \int_{t_0}^{t+t_0} f(\varphi(\tau - t_0)+x_0)\,d\tau$

Notice that $|f(x)| \leq M$ in some neighborhood of x_0. If a is small enough, then $\varphi(t-t_0)+x_0$ belongs to this same neighborhood $\forall t \in I_a(0)$ (since φ is a continuous function) $\Rightarrow |f(\varphi(t-t_0)+x_0)| \leq M$

Thus $|(A\varphi)(t)| \leq \int_{t_0}^{t+t_0} |f(\varphi(\tau-t_0)+x_0)|\,d\tau \leq M|t| \leq Ma$ for $t \in I_a(0)$

choosing a even smaller, we may assume $Ma \leq b$. Then $A: F \to F$.

Now $\|A\varphi_1 - A\varphi_2\| := \sup_{t \in I_a(0)} |(A\varphi_1)(t) - (A\varphi_2)(t)|$ ← this is the distance betwen $A\varphi_1$ and $A\varphi_2$

For any $t \in I_a(0)$, we have $|A\varphi_1(t) - A\varphi_2(t)| \leq \left|\int_{t_0}^{t+t_0} |f(\varphi_1(\tau-t_0)+x_0) - f(\varphi_2(\tau-t_0)+x_0)|\,d\tau\right| \leq \lambda \left|\int_{t_0}^{t+t_0} |\varphi_1(\tau-t_0) - \varphi_2(\tau-t_0)|\,d\tau\right|$

since we can make the neighborhood compact so f Lipschitz

where λ is a Lipschitz constant for f in the aforementioned neighborhood

Now $|\varphi_1(\tau-t_0) - \varphi_2(\tau-t_0)| \leq \sup_{t \in I_a(0)} |\varphi_1(t) - \varphi_2(t)| = \|\varphi_1 - \varphi_2\|$ = distance between φ_1 and φ_2 (since $\tau - t_0 \in I_a(0)$)

So $\forall t \in I_a(0)$, $|A\varphi_1(t) - A\varphi_2(t)| \leq \lambda \|\varphi_1 - \varphi_2\| \cdot |t| \leq \lambda a \|\varphi_1 - \varphi_2\| \Rightarrow \sup_{t \in I_a(0)} |A\varphi_1(t) - A\varphi_2(t)| = \|A\varphi_1 - A\varphi_2\| \leq \lambda a \|\varphi_1 - \varphi_2\|$

If we pick a such that $\lambda a < 1$, we can make A a contraction. Thus $\exists!\ \varphi \in F$ such that $A\varphi = \varphi$ (since F is complete)

Note that A depends on (t_0, x_0) and is a uniform contraction with respect to (t_0, x_0) so the fixed point $\varphi(t_0, x_0)$ depends on (t_0, x_0) continuously and differentiably (i.e. in a C^r way) $\boxtimes$

January 20th

Consider $\dot{x} = f(x)$ (1)

$x(t_0) = x_0$ (2)

where $f \in C^r(U), U \subseteq \mathbb{R}^n, U$-open, $r \geq 1$

Theorem: For any $x_0 \in U$, $\exists$ a maximal interval $J \ni t_0$ such that the IVP (1)-(2) has a unique solution on J. That is, if $\tilde{x}(t)$ is a solution of (1)-(2) on I then $I \subseteq J$ and $x(t) = \tilde{x}(t)$ for $t \in I$. Moreover, J is open (i.e. $J = (a,b)$) and if $b < \infty$ (respectively $a > -\infty$), then $\forall K \subseteq U$, K-compact, $\exists t$ such that $\exists t$ such that $x(t) \notin K$

Example: $\dot{x} = x^2$ $x \in \mathbb{R}$

$x(0) = x_0 > 0$

$\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x} = t - C \Rightarrow x = \frac{1}{C-t}$

$x(0) = x_0 \Rightarrow \frac{1}{C} = x_0 \Rightarrow x = \frac{1}{1/x_0 - t} = \frac{x_0}{1 - x_0 t}$ defined from $t=0$ up to $t = \frac{1}{x_0}$ $\Rightarrow$ maximal interval depends on x_0

$t \in C^r(U), r \geq 1, U$-open, $U \subseteq \mathbb{R}^n$

Definition: An autonomous differential does not explicitly depend on the independent variable (t)

Lemma: Suppose $x(t)$ satisfies (1), then $x(t+\tau)$ also satisfies (1) $\forall \tau \in \mathbb{R}$ (for autonomous systems)

Proof: $\left.\frac{dx(t+\tau)}{dt}\right|_{t=t_0} = \left.\frac{dx(t)}{dt}\right|_{t=t_0+\tau} = f(x(t_0+\tau)) = f(x(t+\tau))\big|_{t=t_0}$

Since t_0 is arbitrary, we are done $\boxtimes$

↳Note: this does not hold for non autonomous systems i.e. $\dot{x}=f(t_0,x)$

Example: $\dot{x}=e^t$

One solution is $x(t)=e^t$ but $x(t+\tau)=e^{t+\tau}$ but $\dot{x}(t+\tau)\neq e^t$

Theorem: For any $x_0 \in U$, there is a unique solution of (1) passing through x_0 (for autonomous systems)

Proof: Assume we have two solutions $x_1(t)$, $x_2(t)$ such that $x_1(t_1)=x_2(t_2)=x_0$

Consider $\tilde{x}_2(t)=x_2(t-(t_1-t_2))$. Then $\tilde{x}_2(t_1)=x_2(t_2)=x_0=x_1(t_1)$

$\Rightarrow \tilde{x}_2=x_1 \Rightarrow x_2=x_1$ (on maximal interval) $\boxtimes$

All of these results allow us to define the flow of a differential equation (i.e. the family of functions $\varphi_t: U \to U$ where $\varphi_t(x_0)=x(t,x_0)$ where x satisfies (1) and $x(0)=x_0$

(Here: $t\in J(x_0)$ where $J(x_0)$ is the maximal interval of existence)

Note: $\varphi^0(x_0)=x_0$ i.e. $\varphi^0=id$

$\varphi^{t+s}(x_0)=\varphi^s(\varphi^t(x_0))$ i.e. φ^t is an evolution operator

One more thing about maximal interval of existence: If $J=(a,\infty)$ and $\lim_{t\to\infty} x(t)$ exists and belongs to U, then $f(x^*)=0$, where $x^*=\lim_{t\to\infty} x(t)$ (a similar result holds for $J=(-\infty,b)$)

That is, x^* is an equilibrium point

January 23th

let $\dot{x}=f(x)$, $x\in\mathbb{R}^n$, $f\in C^r(\mathbb{R}^n)$, $r\geq 1$, $x(0)=x_0$

Definition: The corresponding flow is $\varphi_t(x_0)=\varphi(t,x_0)=x(t)$, where $x(t)$ is a solution of the above IVP.

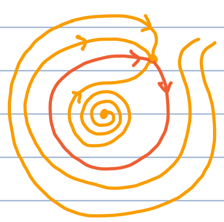
Since trajectories (or orbits) are invariant under φ_t, we can talk about their stability

Definition: A trajectory $\bar{x}(t)$ is called (Liapunov) stable if $\forall\varepsilon>0$ $\exists\delta>0$ such that if $|x(0)-\bar{x}(0)|<\delta$, then $|x(t)-\bar{x}(t)|<\varepsilon$ $\forall t>0$

Definition: $\bar{x}(t)$ is asymptotically stable if it is Liapunov stable and $|x(t)-\bar{x}(t)|\to 0$ as $t\to\infty$

Note: the main invariant sets we will look at are equilibrium points and periodic trajectories

Example:



periodic trajectory

Suppose we have a non-linear system: $\dot{x}=f(x)$
And suppose we are interested in the stability of $\bar{x}(t)$. We can ask what happens to $x(t)=\bar{x}(t)+y(t)$ where $y(t)$ is small.

We have $\dot{\bar{x}}(t)+\dot{y}(t)=f(\bar{x}(t)+y(t)) \overset{\text{taylor expansion}}{=} f(\bar{x}(t))+Df(\bar{x}(t))y+O(|y|^2)$

(if $g(y)=O(|y|^2)$ then $\exists\, c\geq 0$ s.t. $|g(y)|\leq c|y|^2$) (idea: if y is small, then y^2 is really small)

But $\dot{\bar{x}}(t) \underset{\text{identically equal}}{\equiv} f(\bar{x}(t))$

$\Rightarrow \dot{y}(t)=Df(\bar{x}(t))y(t)+O(|y|^2)$

so we might hope to figure out the stability by looking at the linear system $\dot{y}(t)=Df(\bar{x}(t))y(t)$
In general, analyzing solutions of a linear system with a time dependent constant. But, if $\bar{x}(t)$ is an equilibrium point, then $\bar{x}(t)=\bar{x} \Rightarrow Df(\bar{x})$ is time independent
So to understand, the stability of an equilibrium point, it might be enough to consider a linear system $\dot{y}=Ay$, where $A=Df(\bar{x})$, where $\bar{x}$ is an equalibrium point. Hence, lets focus more on linear systems

<u>Example</u>: $\dot{x}=ax$, $x(0)=x_0$, $x\in\mathbb{R}$,
$\Rightarrow x(t)=e^{at}\cdot x_0$

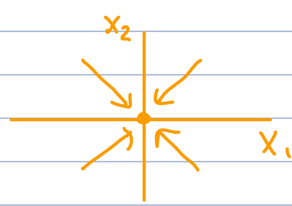
<u>Example</u>: $\dot{x}=Ax$, $x(0)=x_0$, $x(t)=e^{At}\cdot x_0$, $x\in\mathbb{R}^n$ but since A is a matrix,
$e^{At}=I+At+\frac{A^2t^2}{2!}\ t$

<u>Recall</u>: A is diagonalizable if $\exists P$ such that $A=P\,\text{diag}(\lambda_i)P^{-1}$ where $\text{diag}(\lambda_i)=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

In this case if A is diagonalizable, $\dot{x}=Ax$ can be transformed to $\dot{x}=P\,\text{diag}\,P^{-1}x \Rightarrow P^{-1}\dot{x}=\text{diag}(\lambda_i)P^{-1}x$
let $y=P^{-1}x$, then $\dot{y}=\text{diag}(\lambda_i)y$ i.e. $\dot{y}_1=\lambda_1 y_1$
$\vdots$
$\dot{y}_n=\lambda_n y_n$
$\Rightarrow y=e^{\text{diag}(\lambda_i)t}\cdot y_0=\text{diag}(e^{\lambda_i t})y_0$ where $y_0=P^{-1}x_0$
so $x=Py=P\,\text{diag}(e^{\lambda_i t})P^{-1}x_0$ so the behavior of $x(t)$ as $t\to\infty$ (or $-\infty$) is determined by the signs of λ_i
(if $\lambda_i>0$ for at least one i as $t\to\infty$ solution goes to ∞. If $\lambda_i<0$ $\forall i$, as $t\to\infty$, the solution goes to 0)

<u>Theorem</u>: consider a linear system $\dot{x}=Ax$, $x(0)=x_0$, $x\in\mathbb{R}^n$. Then each component of the solution is a linear combination of functions of the form $t^u e^{at}\cos(bt)$ and $t^u e^{at}\sin(bt)$, where $\lambda=a+ib$ is an eigenvalue of A and $0\leq u\leq n-1$. So if all eigen values have negative real parts, then $\bar{x}=0$ is an asymptotically stable equilibrium point. If at least one eigenvalue has a positive real part, then $\bar{x}=0$ is unstable

<u>Example</u>: $\dot{x}_1=\lambda_1 x_1$
$\dot{x}_2=\lambda_2 x_2$
Solution is $x(t)=\begin{pmatrix}c_1e^{\lambda_1 t}\\ c_2e^{\lambda_2 t}\end{pmatrix}$



January 25th

$\dot{x}=f(x)$
$x(0)=x_0$

<u>Recall</u>: If $\bar{x}$ is an equilibrium point and $Df(\bar{x})$ has eigen values with negative real parts, then $\bar{x}$ is asymptotically stable

Example: Love Affairs

Suppose we have two people whose love for each other changes with time. Lets call the first R(Romeo) and the second person J(Juliet). The more R loves J, the less J loves R. The more J loves R, the more R loves J. We want to understand how the amount of love for R and J changes with time.

Let $R(t)$ denote the amount of love R feels for J at time t and let $J(t)$ denote the amount of love J feels for R at time t.

We can assume that the rate of change of $R(t)$ is proportional to $J(t)$ and the rate of change of $J(t)$ is proportional to $R(t)$ i.e.

$\dot{R} = aJ$

$\dot{J} = -bR$

$(0,0)$ is the equilibrium point

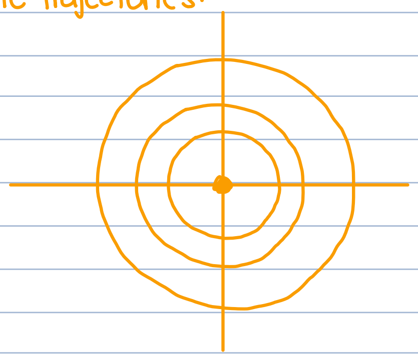
$$A = \begin{pmatrix} \partial\dot{R}/\partial R & \partial\dot{R}/\partial J \\ \partial\dot{J}/\partial R & \partial\dot{J}/\partial J \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}$$

eigenvalues of A: $\begin{vmatrix} 0-\lambda & a \\ -b & 0-\lambda \end{vmatrix} = \lambda^2 + ab$

$\lambda^2 + ab = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{ab}$ are purely imaginary, $Re(\lambda_i) = 0$ so the theorem gives no conclusion. But we know that the solutions are linear combinations of $\cos\sqrt{ab}\,t$ and $\sin\sqrt{ab}\,t$

Periodic trajectories:



This is stable but not asymptotically stable

(in general, if you have distinct eigenvalues with zero real parts, then it is stable but not asymptotically)

In general, the model can be: $\dot{R} = aR + bJ$

$\dot{J} = cR + dJ$

We want to interpret the model for different signs of a, b, c, d

Example: Population Dynamics

Assume we have a single species (e.g. bacteria)

Let $N(t)$ denote the size of the population at time t.

We could assume $\dot{N} = aN$

↳ Note: This assumes an unlimited amount of resources (food)

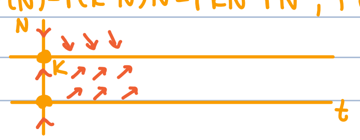
To take into account resources, assume $a = r(K-N)$, then we get $\dot{N} = r(K-N)N$

↳ K is called the carrying capacity

↳ This is called the logistic equation

Equilibrium points: $r(K-N)N = 0 \Rightarrow N = 0$ or $N = K$

$f(N) = r(K-N)N = rKN - rN^2$, $f'(N) = r(K-2N)$, $f'(0) = rK > 0 \Rightarrow$ unstable, $f'(K) = -rK < 0 \Rightarrow$ stable



For two species, populations can interact

January 27th

Example: Wolves and Rabbits

let $W(t)$ be the population of wolves and let $R(t)$ be the population of rabbits
we can model their populations with:

$\dot{R}(t) = a(b-R)R - cRW$

$\dot{W}(t) = -dW + eRW$

we reparametrize the system to reduce the number of parameters of the system

Let $R(t) = \alpha x(t)$, $W(t) = \beta y(t)$, and $t = \gamma\tau$

we want to figure out what α, β, and γ are

$$\frac{dR}{dt} = \frac{\alpha}{\gamma}\frac{dx}{d\tau} = a(b-\alpha x)\alpha x - c\alpha\beta xy$$

$$\frac{dW}{dt} = \frac{\beta}{\gamma}\frac{dy}{d\tau} = -d\beta y + e\alpha\beta xy$$

a has units $\frac{1}{\text{time size}}$

R, W has units size α has same units as R and b γ has units seconds, same as $1/ab$

$$\frac{dx}{d\tau} = \gamma ab\left(1-\frac{\alpha}{b}x\right)x - c\gamma\beta xy \xrightarrow{\alpha=b} = \gamma ab(1-x) - c\gamma\beta xy \xrightarrow{\gamma = 1/ab} (1-x)x - \frac{c\beta}{ab}xy \xrightarrow{\beta = ab/c}$$

← c has units $1/\text{size time}$, β is dimensionless

$$\frac{dy}{d\tau} = -\gamma dy + e\gamma\alpha xy \xrightarrow{\alpha=b} \gamma dy + e\gamma bxy \xrightarrow{\gamma=1/ab} \frac{d}{ab}y + \frac{e}{a}xy \xrightarrow{\beta=ab/c}$$

Denote $p = d/ab$, $q = e/a$, then

$\dot{x} = (1-x)x - xy = f(x,y) = x(1-x-y)$

$\dot{y} = qxy - py = g(x,y) = qy(x - p/q)$

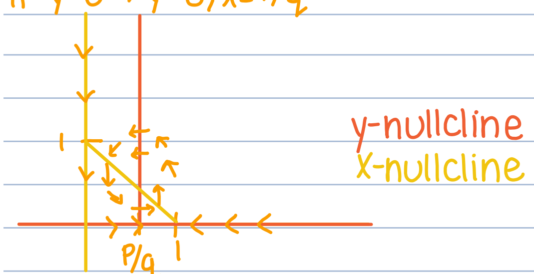
Definition: A planar system is a dynamical system with dimension 2

Definition: For a system $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

the x-nullcline is the set of points where $f(x,y)=0$ and y-nullcline is the set of points where $g(x,y)=0$

For the above example: If $\dot{x}=0 \Rightarrow x=0,\ y=1-x$

If $\dot{y}=0 \Rightarrow y=0,\ x=p/q$



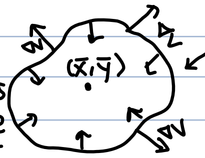
note: the first quadrant is an invariant set

Suppose $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

Consider an equilibrium $(\bar{x},\bar{y})$ and suppose we have a function $V: U \to \mathbb{R}$, $(\bar{x},\bar{y}) \in U$ such that
$V(\bar{x},\bar{y}) = 0$, $\dot{V}(x(t),y(t)) \leq 0$ in U

$$\dot{V}(x(t),y(t)) = \frac{d}{dt}(V(x(t)),y(t)) = \nabla V(x(t),y(t)) \cdot (f(x,y), g(x,y))$$

suppose f,g points inside the level set



positive level set of V
($\nabla V \perp$ level set)

so cannot leave the level set

↳ such a function is called a Liapunov function

Example (failure of linear stability analysis):

$\dot x = -y + x(x^2+y^2)$

$\dot y = x + y(x^2+y^2)$

equilibrium points: (0,0)

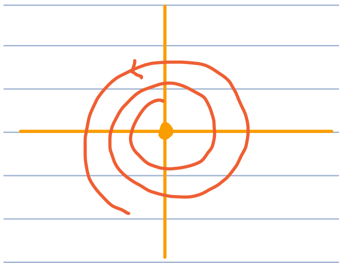
linearization at (0,0) gives

$\dot x = -y$, $\dot y = x$ $\quad Df(0,0)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ $\quad \lambda_{1,2}=\pm i$ ← for linear case, this is stable (i.e. a focus)

but our system is not linear

If we switch to polar coordinates ($x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$)

$\dot r = r^3$, $\dot\theta = 1$ ⇒ r is always increasing ⇒ origin is unstable (always moving away)



$$\theta = \arctan\frac{y}{x} \Rightarrow \dot\theta = \frac{1}{1+(y/x)^2}\cdot\frac{x\dot y - y\dot x}{x^2} = \frac{x\dot y - y\dot x}{x^2+y^2} = \frac{x\dot y - y\dot x}{r^2} = \frac{x^2+xy(x^2+y^2)+y^2-xy(x^2+y^2)}{r^2}$$

$$= \frac{r^2}{r^2} = 1$$

$$(r^2)^{\cdot} = 2x\dot x + 2y\dot y \Rightarrow 2r\dot r = 2x\dot x + 2y\dot y \Rightarrow \dot r = \frac{x\dot x + y\dot y}{r} = \frac{-xy+x^2r^2+xy+y^2r^2}{r} = (x^2+y^2)r = r^3$$

January 30th

Theorem: Consider a system $\dot x = f(x)$, $x\in\mathbb{R}^n$, $f\in C^r(\mathbb{R}^n)$, $r\geq 1$. Suppose that $\bar x$ is an equilibrium point and there exists a neighborhood $U\ni\bar x$ and a C^1 function $V: U\to\mathbb{R}$ such that:

1) $V(\bar x)=0$, $V(x)>0$ for $x\neq\bar x$

2) $\dot V(x) = \frac{d}{dt}(V(x(t))) = \nabla V\cdot\dot x = \nabla V\cdot f(x) \leq 0$ for $x\in U\setminus\{\bar x\}$

Then $\bar x$ is stable.

If in addition we have

3) $\dot V(x)<0$ for $x\in U\setminus\{\bar x\}$

Then $\bar x$ is asymptotically stable

Proof: Let $\overline{B_\delta(\bar x)} = \{x\in\mathbb{R}^n : |x-\bar x|\leq\delta\}$ and take δ sufficiently small such that $\overline{B_\delta(\bar x)}\subseteq U$

let $m = \inf_{x\in S_\delta(\bar x)} V(x)$, where $S_\delta(\bar x) = \{x\in\mathbb{R}^n : |x-\bar x| = \delta\}$

note that since $V(x)>0$ for $x\neq\bar x$, $m>0$.

let $U_1 = \{x\in\overline{B_\delta(\bar x)} : V(x)<m\}$ and note that $\bar x\in U_1$ and U_1 is open

Since $\dot V(x)\leq 0$, we know $V(x(t))$ is non-increasing so if $x(0)\in U_1$, then $V(x(t))<m$ $\forall t>0$. Hence $x(t)\in\overline{B_\delta(\bar x)}$ $\forall t>0$.

Now assume $\dot V(x)<0$ $\forall x\in U\setminus\{\bar x\}$

Take $x(0)\in U_1$ and consider $x(t)$. Since $\overline{B_\delta(\bar x)}$ is compact, every sequence has a convergent subsequence. So we can assume (passing to a subsequence if needed) that we have a sequence $t_n\to\infty$ such that $x(t_n)$ converges in $\overline{B_\delta(\bar x)}$

Let $\tilde x = \lim_{n\to\infty} x(t_n)$ (note that $\tilde x\in\overline{U_1}$ since $\overline{U_1}$ is compact)

Assume $\tilde x\neq\bar x$. Take $\varepsilon>0$ sufficiently small so that $\tilde x\notin\overline{B_\varepsilon(\bar x)}$

Repeating the earlier argument, we can find $\tilde U_1\subseteq\overline{B_\varepsilon(\bar x)}$ such that if $x(0)\in\tilde U_1$, then $x(t)\in\overline{B_\varepsilon(\bar x)}$.

Hence our original trajectory cannot intersect $\tilde U_1$. So $x(t)\in\overline{U_1}\setminus\tilde U_1$ which is compact.

In $\overline{U_1}\setminus\tilde U_1$, $\dot V(x)<0$. Let $K = \sup_{x\in\overline{U_1}\setminus\tilde U_1}\dot V(x)$ ($K<0$ so let $K=-L$ where $L>0$ i.e. $\dot V(x)\leq -L$ $\forall x\in\overline{U_1}\setminus\tilde U_1$)

Note that $V(x(t_n)) - V(x(0)) = \int_0^{t_n}\dot V(x(t))\,dt \leq -L\int_0^{t_n} dt = -Lt_n$. Hence $V(x(t_n))\leq V(x(0)) - L\cdot t_n$

so as $n\to\infty$, we must have $V(x(t_n))<0$ →← $\boxtimes$

Example: $\dot{x} = y$
$\dot{y} = -x + \varepsilon x^2 y$
(0,0) is the only equilibrium point
$J = \begin{pmatrix} 0 & 1 \\ -1+2\varepsilon xy & \varepsilon x^2 \end{pmatrix}$
$\lambda_{1,2}$ at (0,0) are $\pm i$ so linear stability analysis is not applicable
Let $V(x,y) = \frac{1}{2}(x^2+y^2)$

$\nabla V \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x,y) \cdot \begin{pmatrix} y \\ -x+\varepsilon x^2 y \end{pmatrix} = xy - xy + \varepsilon x^2 y^2 = \varepsilon x^2 y^2$
$\Rightarrow \dot{V} > 0$ for $\varepsilon > 0$ and $\dot{V} < 0$ for $\varepsilon < 0 \Rightarrow (0,0)$ is asymptotically stable if $\varepsilon < 0$

february 1st

Note: If V is a strict Liapunov function for $\dot{x} = f(x)$ (i.e. $\dot{V}(x) < 0$) defined on the whole $\mathbb{R}^n$, then the corresponding equilibrium point is globally asymptotically stable
↳ Definition: Globally asymptotically stable means for any trajectory $x(t)$, we have $x(t) \to \bar{x}$ as $t \to \infty$ (where $f(\bar{x}) = 0$)

Note: instead of $\mathbb{R}^n$, we can consider any open invariant set

Theorem: suppose the system $\dot{x} = f(x)$ has an equilibrium point, $\bar{x}$, such that $Df(\bar{x})$ has eigenvalues with only negative real parts. Then $\bar{x}$ is asymptotically stable

Lemma: Suppose $A: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with eigenvalues that all have negative real parts. Then there is an orthonormal basis in which $(x, Ax) = \langle x, Ax \rangle \leq -k|x|^2$, $k \geq 0$ (i.e. the angle between them is obtuse)

Proof: (we will do the Taylor expansion around $\bar{x}$)
$x = \bar{x} + y$, $\dot{x} = \dot{y} = f(\bar{x}+y) = Df(\bar{x})y + R(y)$, where $R(y) = O(|y|^2)$
Let $y = \varepsilon \cdot u$, then $\dot{u} = Df(\bar{x})u + \frac{R(\varepsilon u)}{\varepsilon} = Df(\bar{x})u + \bar{R}(u,\varepsilon)$, $\bar{R}(u,\varepsilon) = \frac{R(\varepsilon u)}{\varepsilon}$

Note: If we make any linear change of variables, $u = Tv$, we get $\dot{v} = T^{-1}Df(\bar{x})Tv + \dot{R}(Tv, \varepsilon)$
let $V(v) = \frac{1}{2}|v|^2$

Then $\nabla V \cdot f(v) = \nabla V \cdot (T^{-1}Df(\bar{x})T)v + \nabla V \cdot R(Tv,\varepsilon) = v(T^{-1}Df(\bar{x})T)v + v \cdot R(Tv_1, \varepsilon)$
$(V(v) = v_1^2 + v_2^2 + \ldots + v_n^2 \Rightarrow \nabla V = (2v_1, 2v_2, \ldots, 2v_n))$
since $Df(\bar{x})$ has eigen values with only negative real parts, we can choose T such that
$v \cdot (T^{-1}Df(\bar{x})T)v \leq -k|v|^2$, $k > 0$
Since $|R(Tv, \varepsilon)| \leq C_1 \varepsilon |v|$ for some C_1, then $|v \cdot R(Tu, \varepsilon)| \leq C_2 \varepsilon |v|^2$
Taking ε sufficiently small so that $C_2\varepsilon < K$, we get $\nabla V \cdot f(v) \leq -m|v|^2$, $m > 0$
$\Rightarrow \nabla V \cdot f(v) < 0 \ \forall v \neq 0$ so V is a strict Liapunov function ▨ ← proof sketch i dont get it

Definition: consider two systems $\dot{x} = f(x)$, $\dot{x} = g(x)$ in $\mathbb{R}^n$. Let the corresponding flows be $\varphi(t,x)$, $\psi(t,x)$. The two systems are C^r-conjugate if $\exists h: \mathbb{R}^n \to \mathbb{R}^n$, $h \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, h a bijection, $h^{-1} \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, with $\varphi(t, h(x)) = h(\psi(t,x))$
↳ C^0-conjugacy is called topological conjugacy
↳ If h is linear, we have linear conjugacy

february 3rd

Definition: Two systems $\dot{x}=f(x)$, $\dot{x}=g(x)$ with flows $\varphi(t,x)$, $\psi(t,x)$ are conjugate if $\exists\, h: \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism, such that $\varphi(t,h(x))=h(\psi(t,x))$. If h is linear, we have linear conjugacy

Lets investigate the conjugacy of linear systems $\dot{x}=Ax$, $\dot{x}=Bx$
It is easy to show that if A and B have only simple eigenvalues, then the two systems are linearly conjugate iff A and B have the same eigenvalues
↳ Eigenvalues are simple if they have multiplicity 1
Proof sketch: ($\Leftarrow$) use h to do a change of variables to give a matrix with the same eigenvalues
($\Rightarrow$) create a basis to diagonalize the systems ▨

To consider general C^r-conjugacy, it is helpful to introduce the following notation:
Definition: Let $m_+(A)$, $m_-(A)$, and $m_0(A)$ denote the number of eigenvalues of A with positive, negative, and zero real parts respectively

Theorem: consider $\dot{x}=Ax$, $x\in\mathbb{R}^n$ and suppose that $m_+(A)=n$, then the system C^0-conjugate to $\dot{x}=x$

Theorem: consider $\dot{x}=Ax$, $x\in\mathbb{R}^n$ and suppose that $m_0(A)=0$, then this system is C^0-conjugate to
$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}x_1\\ -x_2\end{pmatrix}$ $(x_1,x_2)\in\mathbb{R}^{m_+}\times\mathbb{R}^{m_-}$, where $m_+=m_+(A)$, $m_-=m_-(A)$

Corollary: Two systems $\dot{x}=Ax$, $\dot{x}=Bx$, $x\in\mathbb{R}^n$ with $m_0(A)=m_0(B)$ are C^0-conjugate if $m_+(A)=m_+(B)$ (or $m_-(A)=m_-(B)$)

Theorem (Hartman-Grobman): Consider a C^r, $r\geq 1$ system $\dot{x}=f(x)$, $x\in\mathbb{R}^n$. Suppose that $f(\bar{x})=0$ and $m_0(Df(\bar{x}))=0$. Then $\dot{x}=f(x)$ is locally C^0-conjugate to the linear system $\dot{\xi}=Df(\bar{x})\xi$
↳**Definition**: Such $\bar{x}$ as defined above is called hyperbolic

Definition: An invariant subspace of a linear map $A:\mathbb{R}^n\to\mathbb{R}^n$ is a linear subspace $E\subseteq\mathbb{R}^n$ such that $Ax\in E$ $\forall x\in E$

Lemma: If E is an invariant subspace of a linear map $A:\mathbb{R}^n\to\mathbb{R}^n$, then E is an invariant set of the system $\dot{x}=Ax$ (i.e. invariant under the flow)
Proof: we need to show that $\forall x\in E$, $\varphi(t,x)\in E$ $\forall t\in\mathbb{R}$ where $\varphi(t,x)=e^{At}x$
$e^{At}=\lim_{k\to\infty}\sum_{i=0}^{k}\frac{(At)^i}{i!}$

For any fixed k, $\left(\sum_{i=0}^{k}\frac{(At)^i}{i!}\right)x\in E$ if $x\in E$ (because $A^ix\in E$ for $x\in E$ an t^i is a constant)

Since E is a linear subspace, it is complete so $\lim_{k\to\infty}\left(\sum_{i=0}^{k}\frac{(At)^i}{i!}\right)x\in E$

Using Jordan canonical form, we can prove that any linear map $A:\mathbb{R}^n\to\mathbb{R}^n$ has three invariant subspaces (E_s, E_u, E_c) whose sum is $\mathbb{R}^n$ (i.e. $E_s+E_u+E_c=\mathbb{R}^n$) and whose bases are formed by generalized eigenvectors corresponding to the eigenvalues with negative, positive, and zero real parts respectively.
Moreover, in this basis of generalized eigenvectors A has the structure

$$\begin{pmatrix} A_s & 0 & 0\\ 0 & A_u & 0\\ 0 & 0 & A_c\end{pmatrix} \leftarrow \text{dimensions depend on } m_+, m_-, m_0$$

Recall: Given a linear system $\dot{x}=Ax$, we can find a basis in which we have

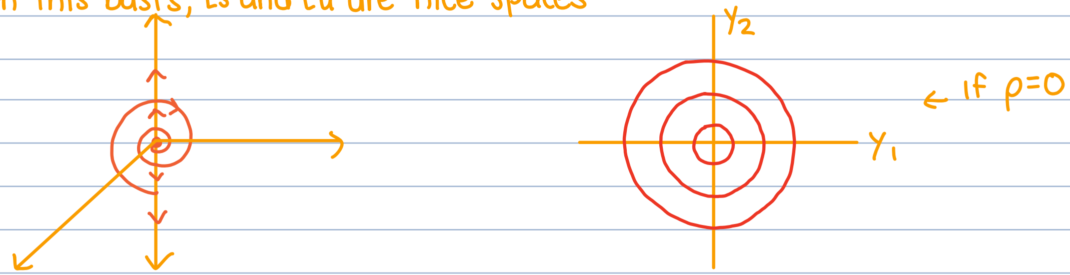
$$\dot{x}=\begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix}$$

where A_s, A_u, A_c have negative, positive, and zero real parts respectively

Example: suppose A has eigenvalues $\rho=\pm i\omega$ and λ where A is 3×3, $\rho<0$, and $\lambda>0$
we can find a basis such that the above holds so after setting $y=Tx$, where the colums of T are generalized eigenvectors, we get

$$\dot{y}=\begin{pmatrix} \rho & \omega & 0 \\ -\omega & \rho & 0 \\ 0 & 0 & \lambda \end{pmatrix} y$$

with this basis, E_s and E_u are nice spaces



consider $\dot{x}=f(x), x\in\mathbb{R}^n$, f is C^r, $r\geq2$, and suppose $f(\bar{x})=0$ (non-linear case)
Linearizing, we get $\dot{y}=Df(\bar{x})y+R(y)$, $R(y)=O(|y|^2)$
Applying a proper linearization $z=Ty$, we get

$$\dot{z}=\begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} z+\tilde{R}(z),\quad \tilde{R}(z)=O(|z|^2)$$

It is convenient to write $z=(u,v,w)\in\mathbb{R}^{m_-}\times\mathbb{R}^{m_+}\times\mathbb{R}^{m_0}$
then we have (since linear parts are completely decoupled)
$\dot{u}=A_s u+R_u(u,v,w)$
$\dot{v}=A_s v+R_v(u,v,w)$
$\dot{w}=A_c w+R_w(u,v,w)$

Theorem (Local stable, unstable, and center manifolds): consider $\dot{x}=f(x), x\in\mathbb{R}^n$, f is C^r, $r\geq2$
Assume that $(u,v,w)=(0,0,0)$ is the equilibrium point of the transformed system:
$\dot{u}=A_s u+R_u(u,v,w)$
$\dot{v}=A_u v+R_v(u,v,w)$
$\dot{w}=A_c w+R_w(u,v,w)$
then $\exists$ sets
$W^s_{loc}=\{(u,v,w): v=h_v^s(u), w=h_w^s(u), Dh^s_v(0)=Dh^s_w(0), |u|\text{-small}\}$
$W^u_{loc}=\{(u,v,w): u=h_u^c(v), w=h_w^u(v), Dh^u_u(0)=Dh^u_w(0), |v|\text{-small}\}$ $(h\in C^r)$
$W^c_{loc}=\{(u,v,w): u=h_u^c(w), v=h_v^c(w), Dh^c_w(0)=Dh^c_v(0)=0, |w|\text{-small}\}$
which are invariant under the flow and tangent to stable, unstable, and center subspaces of the linearized system
Moreover, trajectories in $W^s_{loc}(0)$ and $W^u_{loc}(0)$ have the same asymptotic properties as trajectories in the stable and unstable subspaces. That is, if $x(0)\in W^s_{loc}(0)$, then $x(t)\to0$ as $t\to\infty$. If $x(0)\in W^u_{loc}(0)$, $x(t)\to0$ as $t\to-\infty$

february 8th

Examples: $\dot{x}=x$, $\dot{y}=-y$, $\dot{x}=-x-y^2$, $\dot{y}=y+x^2$, $\dot{x}=x$, $\dot{y}=-y+x^2$

Looking at linearized system, the eigenvalues tell you which manifolds are present

consider a system without $m_0=0$

$\dot{u}=A_s u+R_u(u,v)$

$\dot{v}=A_u v+R_v(u,v)$

For stable manifold: $v=h_v^s(u)=h(u)$

Taking the derivative with respect to time we get $\dot{v}=Dh(u)\dot{u} \Rightarrow A_u v+R_v(u,v)=Dh(u)(A_s u+R_u(u,v))$

Then we consider an expansion of h into a taylor polynomial with unknown coefficients, plug it in, and equate the coefficients

Example: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ x^2\end{pmatrix}$ equilibrium point: $(0,0)$

$\lambda=\pm 1$ so we have an unstable and stable subspace for the linearized system

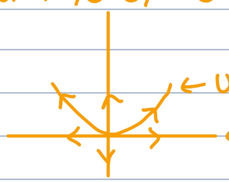
$E_u=$ x-axis so for local unstable manifold, we'll have $y=h(x)$ where $h(0)=0$ and $h'(0)=0$

let $h(x)=ax^2+bx^3+cx^4+O(x^5)$

$\dot{y}=h'(x)\dot{x} \Rightarrow -y+x^2=h'(x)x \Rightarrow -ax^2-bx^3-cx^4+O(x^5)+x^2\equiv(2ax+3bx^2+4cx^3+O(x^4))x$

$\Rightarrow (1-3a)x^2-4bx^2-5cx^4+O(x^5)\equiv 0$

$\Rightarrow a=1/3, b=0, c=0 \Rightarrow h(x)=x^2/3+O(x^5)$



Example: $\dot{x}=x^2$, $\dot{y}=-y$ $\quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}x^2\\ 0\end{pmatrix}$ equilibrium point is $(0,0)$

Stable manifold is 0 (since if $x=0$ then it stays 0) and also $y=0$ is a center manifold

Lets solve this explicitely: Since these are uncoupled, we can solve to get $y(t)=C_y e^{-t}$ and

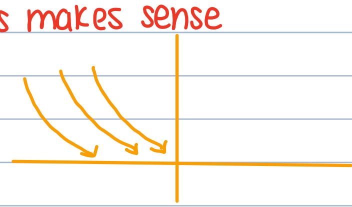
$\frac{dx}{dt}=x^2 \Rightarrow -\frac{1}{x}=t-C_x \Rightarrow x(t)=\frac{1}{C_x-t}$, $t=C_x-\frac{1}{x}$

looking for $x(0)=y(0)=0$

$y(x)=C_y e^{C_x-1/x}$ ← he got confused and none of this makes sense

as $x\to 0$, $y\to 0$

$y(x)=\begin{cases} C_y e^{1/x-C_x}, & x<0 \\ 0, & x\geq 0\end{cases}$ $\quad \begin{cases} e^{-1/x}, & x<0 \\ 0, & x\geq 0\end{cases}$

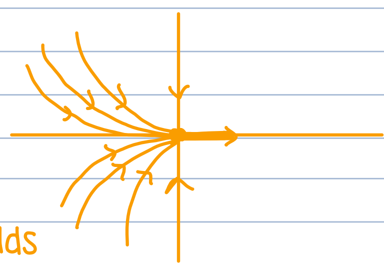


february 10th

Example: $\dot{x}=x^2$, $\dot{y}=-y$

$x=\frac{1}{C-t}$, $y=De^{-t}$

$y=\begin{cases} e^{1/x}, & x<0 \\ 0, & x\geq 0\end{cases}$ ← these are all center manifolds



Recall that $\dot{x}=f(x)$, where f is C^r on $\mathbb{R}^n$, $r\geq 2$ is linear conjugate to

$$\left.\begin{aligned}\dot{u}&=A_s u+R_u(u,v,w)\\ \dot{v}&=A_u v+R_v(u,s,w)\\ \dot{w}&=A_c w+R_w(u,v,w)\end{aligned}\right\}(1)\quad R_u, R_v, R_w \text{ have quadratic order}$$

in the neighborhood of $\bar{x}$ s.t. $f(\bar{x})=0$ where A_s, A_u, A_c have eigen values with negative, positive, and zero real parts

Lets focus on a system of the form (1)

Theorem: In a neighborhood of $(0,0,0)$, system (1) is conjugate to
$\dot{u}=A_s u$
$\dot{v}=A_u v$
$\dot{w}=A_c w+R(h_1(w),h_2(w),w)$, where h_1, h_2 are C^r functions, $h_i(0)=0$, $Dh_i(0)=0$

Now lets consider the case with $m_+=0$ (no eigen values with positive real parts), then

$$\left.\begin{aligned}\dot{u}&=A_s u+R_u(u,w)\\ \dot{w}&=A_c w+R_w(u,w)\end{aligned}\right\}(2)$$

Theorem: If $u=h_w^c(w)$ is a local representation of the center manifold, then the dynamics on the center manifold is described by $\dot{w}=A_c w+R_w(h_w^c(w),w)$ (3)

Theorem: suppose that the zero solution of (3) is asymptotically stable (unstable), then the zero solution of (2) are asymptotically stable (unstable). Moreover, if $u(t), w(t)$, is a solution of (2) with $u(0), w(0)$ small enough, then there is a solution $\bar{w}(t)$ of (3) s.t. $u(t)=h_w^c(\bar{w}(t))+O(e^{-\gamma t})$, $w(t)=\bar{w}(t)+O(e^{-\gamma t})$

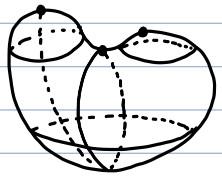
Proof (?): let $h=h_w^c$, then $u=h(w) \Rightarrow \dot{u}=Dh(w)\cdot\dot{w}$
$\Rightarrow A_s h(w)+R_u(h(w),w)=Dh(w)[A_c w+R_w(h(w),w)]$ or $N(h(w)) :\equiv Dh(w)[A_c w+R_w(h(w),w)]-A_s h(w)-R_u(h(w),w)\equiv 0$
use Taylor expansion and plug it in $\boxed{?}$

Theorem: suppose $\varphi:\mathbb{R}^{m_0}\to\mathbb{R}^{m_-}$ is C^1, $\varphi(0)=0$, $D\varphi(0)=0$ and $N(\varphi(w))=O(|w|^b)$, then $|h(w)-\varphi(w)|=O(|w|^b)$ as $w\to 0$

Example: $\dot{x}=x^2y-x^5$ $\quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}x^2y-x^5\\x^2\end{pmatrix}$
$\dot{y}=-y+x^2$
equilibrium: $(0,0)$
center manifold: $y=h(x)=ax^2+bx^3+O(x^4)$
$\dot{y}=(2ax+3bx^2+O(x^3))\dot{x}$
$\Rightarrow -(ax^2+bx^3+O(x^4))+x^2\equiv(2ax+3bx^2+O(x^3))(x^2(ax^2+bx^3+O(x^4))-x^5)$
$\Rightarrow (1-a)x^2+bx^2+O(x^4)\equiv 0$
$\Rightarrow a=1, b=0 \Rightarrow y=x^2+O(x^4)$
$\Rightarrow \dot{x}=x^2(x^2+O(x^4))-x^5=x^4+O(x^5)$

february 13th

Consider $\dot{x}=f(x)$
let $\bar{x}$ be an equilibrium (i.e. $f(\bar{x})=0$) and let $W_{loc}^s(\bar{x})$ and $W_{loc}^u(\bar{x})$ be the corresponding local stable and unstable manifolds. Then $W^s(\bar{x})=\bigcup_{t<0}\varphi(t,W_{loc}^s(\bar{x}))$ and $W^u(\bar{x})=\bigcup_{t\geq 0}\varphi(t,W_{loc}^u(\bar{x}))$ are global stable and unstable manifolds of $\bar{x}$, where φ is the flow

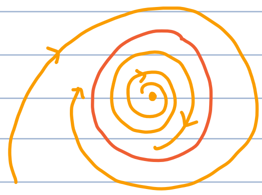


Recall that a continuous dynamical system is given by a vector field $\dot{x}=f(x), x\in\mathbb{R}^n$ and a discrete dynamical system is given by a map $g:\mathbb{R}^n\to\mathbb{R}^n$

↳ in the continuous case, $\varphi(t,x), t\in\mathbb{R}$ and in the discrete case $\varphi(t,x), t\in\mathbb{Z}$ and $\varphi(n,x)=f^n(x)$

Definition: Given a dynamical system with a flow $\varphi(t,x)$, a point $p\in\mathbb{R}^n$ is called an **ω-limit point** of (a trajectory starting at) a point x if $\exists \{t_i\}_{i=1}^{\infty}$ $t_i\to\infty$ s.t. $\varphi(t_i,x)\to p$ as $i\to\infty$. Similarly, an **α-limit point** is defined by taking $t_i\to-\infty$

Example:



any point on the periodic trajectory is an ω-limit point

If p is an ω limit point of x, then p is also an ω-limit point for any other point on that trajectory

Definition: we denote by **$\omega(x)$** (respectively **$\alpha(x)$**), the set of all ω-limit points of x (respectively α-limit points of x)

Theorem: Let $\varphi(t,x)$ be the flow of a continuous dynamical system and let M be a positively invariant set. Then $\forall x\in M$, $\omega(x)\cap M$ is closed in M. Moreover, if M is compact then:

i) $\omega(x)\neq\emptyset$

ii) $\omega(x)$ is compact

iii) $\omega(x)$ is invariant under $\varphi(t,x)$ (i.e. it is a union of orbits)

iv) $\omega(x)$ is connected

Proof: We will show that the complement of $\omega(x)$ is open in M. (let $\omega(x)\cap M=\omega(x)$)

Note that the case of $\omega(x)=\emptyset$ and $\omega(x)=M$ are trivial. Thus assume $\omega(x)\neq 0$ and let $q\notin\omega(x), q\in M$.

By definition of $\omega(x)$, $\exists\,\varepsilon>0\;\exists\,T>0$ s.t. $\forall t>T$, $\varphi(t,x)\notin B_\varepsilon(q)$ where $B_\varepsilon(q)=\{y:|y-q|<\varepsilon\}$. Hence, there is a neighborhood of q, U, such that $U\cap\omega(x)=\emptyset$. Since q was arbitrary, $\omega(x)^c$ is open.

Now assume M is compact.

i) let $x\in M$ and note that $\varphi(t,x)\in M$ $\forall t\geq 0$ since M is positively invariant. Take any sequence $t_i\to\infty$ and consider $\varphi(t_i,x)$. Since $\varphi(t_i,x)\in M$ and M is compact, $\exists\, i_k$ such that $\varphi(i_k,x)\to p\in M$ as $k\to\infty$

Hence $p\in\omega(x)$

ii) Since $\omega(x)$ is a closed subset of a compact set, it is compact (to be continued...)

february 15th

(... proof continued)

iii) Lets show that if $q\in\omega(x)$, then the whole orbit through q belongs to $\omega(x)$

First, note that $\varphi(s,q)$ is defined for all $s\in[0,\infty)$

Lets show that $\varphi(s,q)$ is also defined for $s\in(-\infty,0)$

Let t_i be a sequence such that $\varphi(t_i,x)\to q$ (such a sequence exists since $q\in\omega(x)$)

WLOG assume $t_i<t_{i+1}$. Note that $\varphi(s,\varphi(t_i,x))$ is defined for $s\in[-t_i,\infty)$, since $\varphi(s,\varphi(t_i,x))=\varphi(s+t_i,x)$
Take any $s\in(-\infty,0)$. $\exists N$ s.t. $\forall i>N$, $s>-t_i$ (since $t_i\to\infty$). For $i>N$, $\varphi(s,\varphi(t_i,x))$ is well-defined.
Now take the limit as $i\to\infty$ and use the fact that φ is continuous.
$\lim\limits_{i\to\infty}\varphi(s,\varphi(t_i,x))=\varphi(s,\lim\limits_{i\to\infty}\varphi(t_i,x))=\varphi(s,q)$

Now take any $q\in\omega(x)$ and let $p\in O(q):=$ orbit through q
Note that $\exists s\in\mathbb{R}$ such that $p=\varphi(s,q)$ by definition of $O(q)$
let t_i be such that $\varphi(t_i,x)\to q$ and consider the sequence t_i+s, then $\varphi(s+t_i,x)=\varphi(s,\varphi(t_i,x))$, so by continuity of φ, $\lim\limits_{i\to\infty}\varphi(s+t_i,x)=\varphi(s,\lim\limits_{i\to\infty}\varphi(t_i,x))=\varphi(s,q)=p$

iv) (Recall that a set $A\subseteq\mathbb{R}^n$ is disconnected if $\exists U,V$ open such that $A\subseteq U\cup V$, $U\cap\bar{V}=\emptyset$, $\bar{U}\cap V=\emptyset$, $A\cap U\neq\emptyset$, $A\cap V\neq\emptyset$. Also if $A\in\mathbb{R}^n$ is connected and f continuous, then $f(A)$ is connected)
Assume $\omega(x)$ is disconnected, then $\exists U,V\subseteq\mathbb{R}^n$, U,V open such that $\omega(x)\subseteq U\cup V$, $U\cap\bar{V}=\emptyset$, $\bar{U}\cap V=\emptyset$, $\omega(x)\cap U\neq\emptyset$, $\omega(x)\cap V\neq\emptyset$
let t_i,s_i be such that $\varphi(t_i,x)\to p\in U$ and $\varphi(s_i,x)\to q\in V$ as $t_i\to\infty$, $s_i\to\infty$.
Take large i such that $\varphi(t_i,x)\in U$. Then $\exists i_2>i_1$ such that $s_{i_2}>t_{i_1}$ and $\varphi(s_{i_2},x)\in V$.
Now, consider $\varphi([t_{i_1},s_{i_2}],x)$ (the image of the interval $[t_{i_1},s_{i_2}]$ under the map $\varphi(\cdot,x)$) since $[t_{i_1},s_{i_2}]$ is connected and $\varphi(\cdot,x)$ is continuous, $\exists\bar{t}_1\in(t_{i_1},s_{i_2})$ such that $\varphi(\bar{t},x)\in M\setminus(U\cup V)=K$-compact and corresponding $\bar{t}_2,\bar{t}_3,\dots$ such that $\varphi(\bar{t}_i,x)\in K\ \forall i$.
Since K is compact, we can find a subsequence $\bar{t}_{i_1},\bar{t}_{i_2},\dots$ such that $\varphi(\bar{t}_{i_k},x)\to\bar{p}\in K$ as $k\to\infty$.
But then $\bar{p}\in\omega(x)$ by definition →← since $\omega(x)\subseteq U\cup V$ ▨

Consider the evolution operator $\varphi(t,x)$ for a continuous or discrete dynamical system on $\mathbb{R}^n$

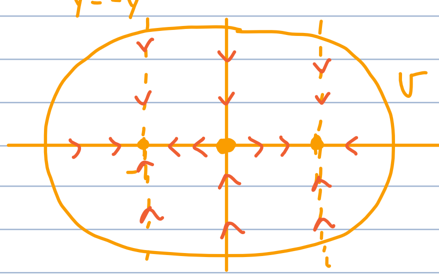
Definition: A point $x_0\in\mathbb{R}^n$ is called a **non-wandering point** (of φ) if $\forall U\ni x_0$, $\forall T>0$ $\exists\,|t|>T$ such that $\varphi(t,U)\cap U\neq\emptyset$. The set of all non-wandering points is called a **non-wandering set**.

february 17th

By $\varphi(t,x)$ we mean an evolution operator of a continuous or discrete dynamical system

Definition: A closed invariant set A of $\varphi(t)$ is called an **attracting set** if $\exists U\supseteq A$, U-open set such that $\varphi(t,U)\subseteq U\ \forall t>0$ and $\bigcap\limits_{t>0}\varphi(t,U)=A$. The set U is called a trapping region

Example: $\dot{x}=x-x^3$, $\dot{y}=-y$ equilibrium points: $(0,0),(0,-1),(0,1)$



$\bigcap\limits_{t>0}\varphi(t,U)=[-1,1]$ ← so $[-1,1]$ is an attracting set

Definition: A closed invariant set A is called **topologically transitive** if for any two (relatively) open sets $U,V\subseteq A$, $\exists t>0$ s.t. $\varphi(t,U)\cap V\neq\emptyset$

Definition: An **attractor** is a topologically transitive attracting set

Definition: The basin of attraction of an attracting set A is $\bigcup_{t\leq 0} \varphi(t,U)$, U - trapping region

Example: $\dot{x} = -y + x(1-z^2-x^2-y^2)$
$\dot{y} = x + y(1-z^2-x^2-y^2)$
$\dot{z} = 0$

For fixed z, converting to polar coordinates:
$\dot{\theta} = 1$
$\dot{r} = r(\sqrt{1-z^2} - r)$

Theorem: Suppose G is an invariant set of a continuous flow $\varphi(t,x)$. Also assume that there is a C^r, $r\geq 1$ function $V: G \to \mathbb{R}$ such that $\dot{V}(x) \leq 0$ $(\dot{V} = \nabla V \cdot f)$. Then $\forall x \in G$, $\omega(x) \cap G \subseteq E$, where $E = \{x \in G : \dot{V}(x) = 0\}$
Also, $\alpha(x) \cap G \subseteq E$

Proof: let $p \in \omega(x) \cap G$. Note that $V(p) = \inf_{t>0} V(\varphi(t,x))$. Indeed consider $V(\varphi(t,x))$ for some time t. Since $t_i \to \infty$ such that $\varphi(t_i,x) \to p$. Moreover we can assume $t_i < t_{i+1}$
$\exists K > 0$ such that $t_i > t$ $\forall i > K$ so $\dot{V}(\varphi(t_i,x)) \leq V(\varphi(t,x))$
Passing to the limit, $V(p) \leq V(\varphi(t,x))$ so $V(p)$ is a lower bound.
Since $\forall \varepsilon > 0$ $\exists t_i$ such that $V(\varphi(t_i,x)) - V(p) < \varepsilon$, it is the greatest lower bound (to be continued...)

february 22nd

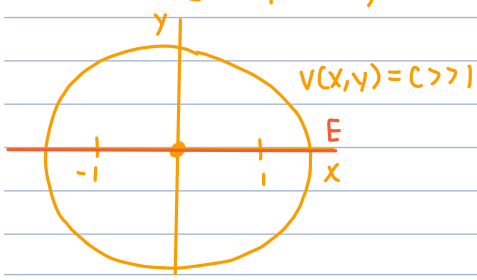
(...proof continued/rewritten) let $p \in \omega(x) \cap G$. We showed that $V(p) = \inf_{t>0} V(\varphi(t,x))$. Now consider the trajectory through p, $\varphi(t,p)$. $\varphi(t,p)$ exists for small t and $\varphi(t,p) \in G$ since G is invariant.
Fix some small t and consider $t_k + t$ where $\varphi(t_k, x) \to p$. Then $\varphi(t_k+t, x) = \varphi(t, \varphi(t_k, x)) \to \varphi(t,p) = q$
Using a similar argument, we can show that $V(q) = \inf_{t>0} V(\varphi(t,x))$. But then $V(q) = V(p)$, that is $V(\varphi(t,p)) = V(p)$ $\forall$ small t. Hence $\dot{V}(p) = 0$ ▨
↳ Note: a similar argument works for $\dot{V} \geq 0$

Corollary (Lasalle Invariance Principle): Consider a continuous flow $\varphi(t,x)$. Let G be a compact positively invariant set (with nonempty interior and $C^r, r\geq 1$ boundary) Suppose that $V: G \to \mathbb{R}$ is a C^r, $r \geq 1$ function such that $\dot{V}(x) \leq 0$ for $x \in G$, then $\forall x \in G$, $\varphi(t,x) \to M$, where M is defined as follows:
If $E = \{x \in G : \dot{V}(x) = 0\}$, then $M = \{x \in E : O^+(x) \subseteq E\}$ ← the positively invariant part of E

Proof: Take any $x \in G$. Clearly $\varphi(t,x) \to \omega(x)$ as $t \to \infty$ (by definition of $\omega(x)$) and since $\omega(x) \subseteq E$ is positively invariant, $\omega(x) \subseteq M$. Thus $\varphi(x,t) \to M$ as $t \to \infty$ ▨

Example: $\dot{x} = y$
$\dot{y} = x - x^3 - \delta y$
equilibrium points: $(0,0)$, $(\pm 1, 0)$

let $V(x,y) = \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$, $\dot{V}(x,y) = -\delta y^2$



on $\{y=0\}$ and not an equilibrium point, $\dot{y} \neq 0$ so the positively invariant part of E is $\{(0,0), (\pm 1, 0)\}$

Definition: For a flow $\varphi(t,x)$ (continuous or discrete), a periodic trajectory through x_0 is a function $\varphi(t,x_0)$ such that $\exists T>0$ $\varphi(t+T,x_0)=\varphi(t,x_0)$ $\forall t\in\mathbb{R}$

Lets focus on planar continuous systems:

$$\left.\begin{aligned}\dot{x}&=f(x,y)\\ \dot{y}&=g(x,y)\end{aligned}\right\}(1)$$

Definition: For $f:\mathbb{R}^n\to\mathbb{R}^n$, the divergence of f is $\underline{\operatorname{div}} f(x)=\underline{\nabla} f(x)=\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)$ (trace of the Jacobian)

Theorem: suppose that the system (1) is such that $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\neq 0$ in a simply connected domain D.

Then there are no periodic orbits in D.

↳ simply connected essentially means there are no holes

february 24th

$\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$ (1)

Theorem (Bendixon's criterion): suppose $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\neq 0$ in a simply connected region $D\subseteq\mathbb{R}^2$, then system (1) does not have periodic trajectories in D

↙line integral

Proof: Assume there is a periodic orbit $C\subseteq D$. Consider the following integral: $\oint_C f\,dy-g\,dx$

(let $\gamma:[0,1]\to\mathbb{R}^2$ be C^r, $r\geq 0$, and such that $\gamma(0)=\gamma(1)$. let $f,g:\mathbb{R}^2\to\mathbb{R}^2$, then

$\oint_C f\,dx+g\,dy=\int_0^1(f(\gamma(t))\cdot\gamma_x'(t)+g(\gamma(t))\cdot\gamma_y'(t))\,dt$ where $\gamma=\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}$, $C=\operatorname{im}\gamma$)

$\oint_C f\,dy-g\,dx=\int_0^T(f(\gamma(t))\cdot\dot{y}(t)-g(\gamma(t))\dot{x}(t))\,dt=\int_0^T(f\cdot g-g\cdot f)\,dt=0$ where γ is the periodic trajectory (of period T). (recall: $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$)

By Green's theorem: $\oint_C f\,dy-g\,dx=\int_{D_C}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)dx\,dy\neq 0$ since $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\neq 0\to\leftarrow$ ▨

Theorem (Bendixon's criterion pt 2): suppose there is a C^1 function $B:\mathbb{R}^2\to\mathbb{R}^1$ such that $\frac{\partial}{\partial x}(Bf)+\frac{\partial}{\partial y}(Bg)\neq 0$ is a simply connected region $D\subseteq\mathbb{R}^2$. Then there are no periodic trajectories in D.

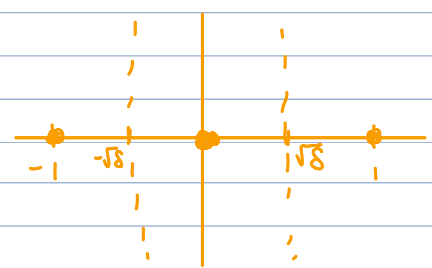
Example: $\dot{x}=y$

$\dot{y}=x-x^3-\delta y$, $\delta>0$

$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=-\delta<0$ so no periodic orbits in $\mathbb{R}^2$

Example: $\dot{x}=y$

$\dot{y}=x-x^3-\delta y+x^2y$, $\delta>0$



$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=-\delta+x^2$ so divergence is 0 when $x^2=\delta$ i.e. $x=\pm\sqrt{\delta}$

so any periodic trajector must cross at least one of those lines

Definition: Two vectors $u, v \in \mathbb{R}^n$ are transversal if they are linearly independent, i.e. $u \neq \alpha v$ and $v \neq \alpha u$ for some $\alpha \in \mathbb{R}$. Two vector spaces $U, V \subseteq \mathbb{R}^n$ with $\dim U + \dim V = n$ are transversal if for any $u \in U$, $v \in V$, u and v are transversal

Definition: Recall that a regular curve in $\mathbb{R}^n$ is a $C^r, r \geq 1$ map $\gamma: I \to \mathbb{R}^n$, where I-open interval, and $\gamma'(t) \neq 0$. An arc is the image of a regular curve

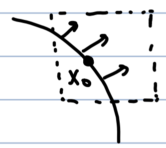
Definition: A planar arc Σ is transversal to the vector field (f,g) if $\gamma'(t)$ is transversal to (f,g) at every point $\gamma(t)$, where $\Sigma = \operatorname{im}\gamma$. In otherwords, (f,g) is nowhere tangent to Σ

Definition: A local section of the vector field (f,g) at (x_0,y_0) is an arc Σ containing (x_0,y_0) and transverse to (f,g)

If (x_0,y_0) is not an equilibrium point, we can always construct a local section through (x_0,y_0)

february 27th

$$\dot{x} = f(x,y) \quad (1)$$
$$\dot{y} = g(x,y)$$



vector field is never tangent to the curve in the box

Let Σ be a local section at $x_0 \in \mathbb{R}^2$

↳ Note by definition of local section, x_0 is not an equilibrium point

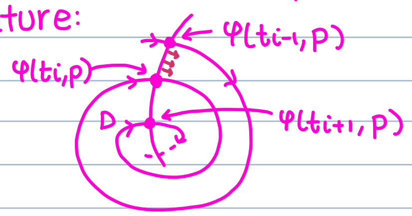
Theorem: Suppose $z_0 \in \mathbb{R}^2$, $\varphi(t_0, z_0) = x_0$, and Σ is a local section at x_0. Then $\exists\, U \ni z_0$, U-open, and a differentiable function $\tau: U \to \mathbb{R}$ s.t. $\tau(z_0) = t_0$ and $\varphi(\tau(z), z) \in \Sigma, z \in U$

↳ τ gives time the time that it intersects the local section

↳ This tells you that you know the behaviour of the vector field around the local section

Theorem: Let M be a compact, positively invariant set for system (1) and let Σ be an arc transverse to the vector field. Then any positive orbit $O^+(p)$, for $p \in M$, intersects Σ in a monotone sequence. That is, if $\varphi(t_{i-1}, p), \varphi(t_i, p), \varphi(t_{i+1}, p) \in \Sigma$, then $\varphi(t_i, p)$ lies between $\varphi(t_{i-1}, p)$ and $\varphi(t_{i+1}, p)$

picture:



↳ idea is that the vector field is transverse to the arc

Proof: Clearly, if $O^+(p)$ does not intersect Σ or intersects at one or two points, the statement is vacuously true. In the other cases, it is enough to consider the points $x_{i-1} = \varphi(t_{i-1}, p)$, $x_i = \varphi(t_i, p)$, $x_{i+1} = \varphi(t_{i+1}, p)$, $t_{i-1} < t_i < t_{i+1}$

Consider the region bounded by the arc segment $[x_{i-1}, x_i]$ and the part of the orbit from x_{i-1} to x_i, call it D

Assume that the vector field is pointing inside D. (If not, consider the closure of the complement of D). Then D is positively invariant (if youre on the local section you must move strictly inside D). Then x_{i+1} belongs to the interior of D. Hence $x_i \in [x_{i-1}, x_{i+1}]$ ▨

Theorem: Let M be a compact positively invariant set for (1), and let Σ be an arc transverse to (1). Then for any $p \in M$, $\omega(p)$ intersects Σ in at most one point

Proof: Suppose $q_1, q_2 \in \omega(p) \cap \Sigma$, $q_1 \neq q_2$. Then $\exists \{t_i\}, t_i \to \infty, t_i < t_{i+1}$ s.t. $\varphi(t_i, p) \to q_1$. Also $\exists \{s_i\}$, $s_i \to \infty$, $s_i < s_{i+1}$ s.t. $\varphi(s_i, p) \to q_2$. Since Σ is a transverse arc, we can construct sequences $\bar{t}_i$ and $\bar{s}_i$ such that $\varphi(\bar{t}_i, p) \in \Sigma$, $\varphi(\bar{t}_i, p) \to q_1$ and $\varphi(\bar{s}_i, p) \in \Sigma$, $\varphi(\bar{s}_i, p) \to q_2$. But then $\varphi(t, p)$ intersects Σ in a non-monotonic sequence ▨

March 1st

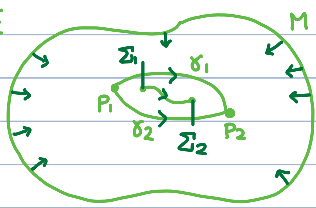
$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \quad (1)$$

Theorem: Let M be a compact, positively invariant set for (1). If $\omega(p)$ does not contain equilibrium points, $p \in M$, then $\omega(p)$ is a closed orbit (i.e. the image of a periodic trajectory)

Proof: let $q \in \omega(p)$. We'll show that $O^+(q)$ is a closed orbit.
let $x \in \omega(q)$ (note $x \in \omega(p)$) and let Σ be a local section through x.
Note that $\exists \{t_n\} \nearrow \infty$ s.t. $\varphi(t_n, q) \to x$. Since Σ is transverse to the vector field, we can find $\{\bar{t}_n\} \nearrow \infty$ s.t. $q_n = \varphi(\bar{t}_n, q) \to x$ and $q_n \in \Sigma$ (same argument as previous theorem).
Since $\omega(p)$ cannot intersect Σ at more than one point (previous theorem), $q_n = x \; \forall n$. Hence $O^+(q)$ is a closed orbit.
Now we'll show that $\omega(p)$ coincides with $O^+(q)$.
let Σ_1 be a local section through q.
$\omega(p)$ intersects Σ_1 only at q. Also, $\omega(p)$ does not contain any equilibrium points and is connected. Thus $\omega(p)$ must coincide with $O^+(q)$ since otherwise $\omega(p)$ would intersect Σ at more than just one point (by connectedness) ▨

Theorem: Let M be a compact positively invariant set for (1). Suppose that $\omega(p)$, $p \in M$, contains two distinct equilibrium points: p_1, p_2. Then $\exists!$ orbit $\gamma \subseteq \omega(p)$ s.t. $\omega(\gamma) = p_2$, $\alpha(\gamma) = p_1$

Proof:



Assume $\exists \gamma_1, \gamma_2$ with $\alpha(\gamma_i) = p_1$, $\omega(\gamma_i) = p_2$, $i = 1, 2$.
let $q_1 \in \gamma_1$, $q_2 \in \gamma_2$ and let Σ_1 and Σ_2 be local sections through q_1 and q_2
$\exists t_1 > 0$ such that $\varphi(t_1, p)$ intersects Σ_1 and $\exists t_2 > t_1$ such that $\varphi(t_2, p)$ intersects Σ_2. Then the region bounded by the parts of γ_i from q_i to p_2, parts of Σ_i from q_i to $\varphi(t_i, p)$, and the part of $O^+(p)$ between $\varphi(t_1, p)$ and $\varphi(t_2, p)$ is positively invariant.
But then points of γ_i outside of this region cannot be ω-limit points of p $\rightarrow\leftarrow$ ▨

Theorem: Let M be compact, positively invariant set for (1). Then for any $p \in M$, $\omega(p)$ is one of the following:
i) a single orbit
ii) a closed orbit
iii) A finite number of equilibrium points p_i, $i = 1, \dots, k$, and orbits γ such that $\alpha(\gamma), \omega(\gamma) \in \{p_1, \dots, p_k\}$

↳ This is called the Poincare-Bendixon Theorem (one of them)

Proof: If $\omega(p)$ contains only equilibrium points, there can be at most one since $\omega(p)$ is connected.
If $\omega(p)$ does not contain equilibrium points, then it is a closed orbit (proved previously)

If $\omega(p)$ contains $p_1,\dots,p_n$ equilibrium points, then it must contain orbits connecting these points and there can be at most one orbit γ with distinct $\omega(\gamma)$ and $\alpha(\gamma)$ by the previous theorem $\boxtimes$

March 3rd

consider the system:

$\dot{x}=-x+ay+x^2y$
$\dot{y}=b-ay-x^2y$ $\quad a,b>0$ } describes the process of breaking down sugar

we are dealing with concentrations in this system so we're only interested in the first quadrant

Nuclines:

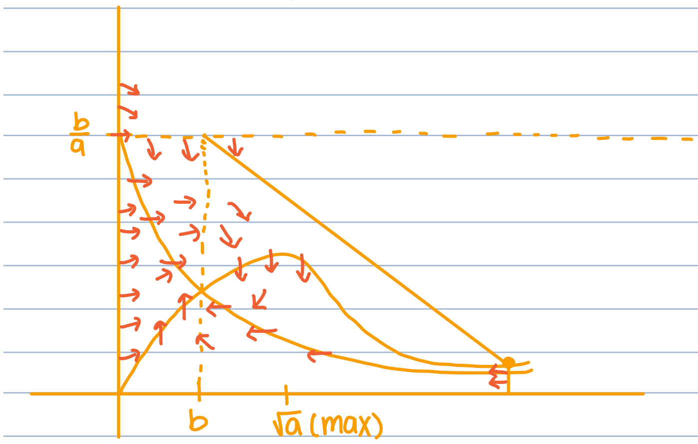
x: $0=-x+ay+x^2y \Rightarrow y=\frac{x}{a+x^2}$

y: $b-ay-x^2y=0 \Rightarrow y=\frac{b}{a+x^2}$

If $x=0 \Rightarrow \dot{x}=ay>0$, $\dot{y}=b-ay$ ($\dot{y}>0$ for $y<b/a$, $\dot{y}<0$ for $y>b/a$)

If $y=0$, $\dot{x}=-x$, $\dot{y}=b>0$

plugging in $y=\frac{b}{a+x^2}$ to $\dot{x}$ gives $\dot{x}=-x+\frac{ab}{a+x^2}+\frac{x^2b}{a+x^2}=\frac{-x(a+x^2)+(a+x^2)b}{a+x^2}=b-x$



$\frac{dy}{dx}\approx -1$ around $(b,\frac{b}{a})$

$\frac{dy}{dx}$ needs to be ≤ -1 on $y=-x+b+\frac{b}{a}$ (to point inside)

on this line we have:

$\dot{x}=-x+a(-x+b+b/a)+x^2y=(1+a)(b-x)+x^2y$

$\dot{y}=b-a(-x+b+b/a)-x^2y=-a(b-x)-x^2y$

$$\left|\frac{\dot{y}}{\dot{x}}\right|=\left|\frac{-a(b-x)-x^2y}{(1+a)(b-x)+x^2y}\right|=\left|\frac{a+x^2y/x-b}{-(1+a)+x^2y/x-b}\right|=\left|\frac{x^2y/x-b\ -a}{x^2y/x-b-(1-a)}\right|$$

consider $\frac{x^2(b-x+b/a)}{-(1+a)(x-b)}$

$\frac{b-ay-x^2y}{-x+ay+x^2y}<-1 \Rightarrow b-ay-x^2<x-ay-x^2y$

$b-x<0$

} Mileyko got confused for this entire part (continued next class)

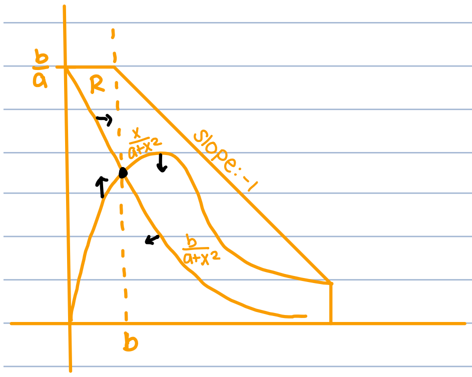
March 6th

consider the system

$\dot{x} = -x + ay + x^2 y = f(x,y)$

$\dot{y} = b - ay - x^2 y = g(x,y)$

with picture:



The region R is compact and positively invariant.
If we figure out if/when the equilibrium point $\bar{p}$ is unstable, then after cutting out a small disk around $\bar{p}$ (i.e. $R \setminus B_\varepsilon(\bar{p})$, $\varepsilon \ll 1$) we get a compact positively invariant set without equilibrium points so $R \setminus B_\varepsilon(x)$ contains a closed orbit (by Poincare-Bendixon)

First lets find $\bar{p}$:

$$\begin{aligned} -x+ay+x^2y&=0 \\ b-ay-x^2y&=0 \end{aligned} \Rightarrow \bar{p} = \left(b, \frac{b}{a+b^2}\right)$$

At p we have

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{pmatrix} = \begin{pmatrix} b^2-a & b^2+a \\ -2b^2/(b^2+a) & -(b^2+a) \end{pmatrix}$$

note: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A-\lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - tr(A)\lambda + \det A \Rightarrow \lambda_{1,2} = \frac{tr(A) \pm \sqrt{(tr A)^2 - 4\det(A)}}{2}$

$tr(A) = \frac{b^2-a}{b^2+a} - (b^2+a) = \frac{b^2-a^2-(b^2+a)^2}{b^2+a}$, $\det(A) = -(b^2-a) + 2b^2 = b^2+a > 0$

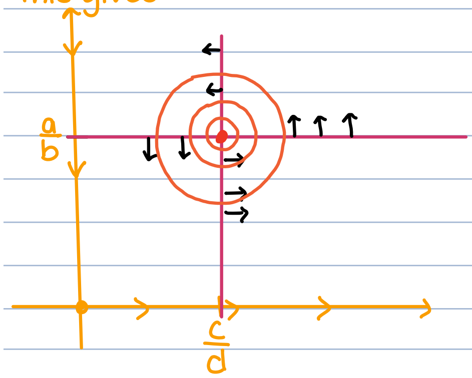
so if $tr(A) < 0$, then $Re(\lambda_{1,2}) < 0$ so stable

for instability we need $b^2 - a - (b^2+a)^2 > 0$

Example: Predator and prey (fish)

Assume that in the absense of predators, the population of the prey fish increases at a rate proportional to the size of the population with the constant of proportionality $a > 0$. Assume that in the absense of prey, the predators die off at a rate proportional to the size of the population with coefficient of proportionality $-c$, $c > 0$. Lastly, if the presence of predators decreases a by a quantity proportional to the population of predators and $-c$ increases by a quantity proportional to the population of prey if prey are present.

If x is the population of prey and y is the population of predators then we have
$\dot{x}=x(a-by)$
$\dot{y}=y(-c+dx)$
This gives:



Notice that: $\dot{x}\frac{1}{x}(c-dx)=(a-by)(c-dx)$

and $\dot{y}\frac{1}{y}(a-by)=-(a-by)(c-dx)$

$\Rightarrow \frac{1}{x}(c-dx)\dot{x}+\frac{1}{y}(a-by)\dot{y}=0 \Rightarrow \left\langle \begin{pmatrix} \frac{1}{x}(c-dx) \\ \frac{1}{y}(a-by) \end{pmatrix}, \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right\rangle = 0$

Let $F(x)=c\ln x-dx$ and $G(y)=a\ln y-by$.
Then if $H(x,y)=F(x)+G(y)$ we get $\dot{H}(x,y)=F'(x)\dot{x}+G'(y)\dot{y}=0$
So every trajectory belongs to a level set $H(x,y)=$ constant. These are closed curves.

March 8th

Assume we have a population and a disease.
Let S be the people who are susceptible, I be the people who are infected, and R be the people who have recovered.
Then if N is the population size, $N=S+I+R$ (SIR)
Make the following assumptions:
- rate of S is proportional to N
- Susceptible people can become infected after meeting an infected person (assuming people meet continuously
- people die at a rate of death proportional to the population
- Assume for infected people, they die at a faster rate
- Infected people recover at a rate proportional to I

Some possible additional assumptions: (SIRS)
- recovered people can become susceptible again at a rate proportional to I
- rate of birth of infected is βI

This gives the following table of terms contributing to the rate of change:

Susceptible $=S$	Infected $=I$	Recovered $=R$	
$+bN$			
$-\beta S I/N$	$+\beta S I/N$		SIR
	$-\gamma I$	$+\gamma I$	SIRS
$-\omega S$	$-(\omega_i+\omega)I$	$-\omega R$	
$+\alpha R$		$-\alpha R$	
$-p\beta I$	$+p\beta I$		

we have the following differential equations:

SIR

$\dot{S}=bN-\beta S\frac{I}{N}-\omega S$ Assuming N is constant $(N=S+I+R)$

$\dot{I}=\beta S\frac{I}{N}-(\gamma+\omega_i+\omega)I \Rightarrow \dot{I}=\left(\frac{\beta}{N}(N-I-R)-(\gamma+\omega_i+\omega)\right)I$

$\dot{R}=\gamma I-\omega R$ $\qquad \dot{R}=\gamma I-\omega R$

SIRS

$\dot{S}=bN-pbI-\beta S\frac{I}{N}-\omega S+\alpha R$

Assuming N is constant

$\dot{I}=\beta S\frac{I}{N}-(\gamma+\omega_i+\omega-pb)I \Rightarrow \dot{I}=\left(\frac{\beta}{N}(N-I-R)-(\gamma-\omega_i+\omega-pb)\right)I$

$\dot{R}=\gamma I-(\omega+\alpha)R$ $\qquad \dot{R}=\gamma I-(\omega+\alpha)R$

In both cases we can write the system as

$\dot{I}=(r-aI-aR)I$

$\dot{R}=\gamma I-cR$

where $r=\beta+pb-\gamma-\omega_i-\omega$, $a=\beta/N$, and $c=\omega+\alpha$

The possible equilibrium points are $(0,0)$ and (I_*,R_*) where $r-aI_*-aR_*=0$, $\gamma I_*-cR_*=0$

Note that (I_*,R_*) is relevant only when $I_*,R_*>0$ (no negative population)

suppose it is relevant, then $r=aI_*+aR_*$ and $-\gamma I_*+cR_*=0$ so

$\dot{I}=(a(I_*-I)+a(R_*-R))I$

$\dot{R}=-\gamma(I_*-I)+c(R_*-R)$

let $V(I,R)=I-I_*\ln I+d(R_*-R)^2$ (Liapunov function)

$\nabla V=\left(\frac{1}{I}(I-I^*),-2d(R_*-R)\right)$

$\Rightarrow \dot{V}=\frac{1}{I}(I-I_*)\cdot\dot{I}-2d(R_*-R)\cdot\dot{R}=[a(I_*-I)+a(R_*-R)](I-I_*)-2d(R_*-R)[c(R_*-R)-\gamma(I_*-I)]$

$=-a(I_*-I)^2-a(R_*-R)(I_*-I)-2dc(R_*-R)^2+2d\gamma(I_*-I)(R_*-R)$

Thus if $d=\frac{a}{2\gamma}$, $\dot{V}=-a(I_*-I)^2-2dc(R_*-R)^2<0$

This is true everywhere so (I_*,R_*) is globally asymptotically stable (so no closed orbits)

March 10th

Example: Van der Pol Equation

$\ddot{x}+\mu(x^2-1)\dot{x}+x=0$

↳ Describes the voltage of a point

Example: Lienard System (A more general system than Van der Pol)

$\ddot{x}+f(x)\dot{x}+g(x)=0$

Since $\frac{d}{dt}(\underbrace{\dot{x}+F(x)}_{y})=-g(x)$, where $F(x)=\int_0^x f(s)ds$,

we can rewrite the system as

$\dot{x}=y-F(x)$

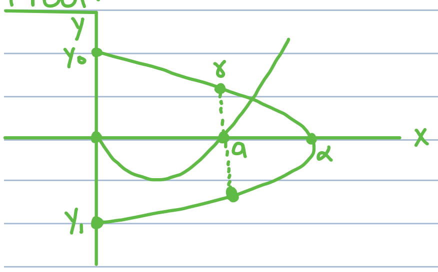
$\dot{y}=-g(x)$

Let $G(x)=\int_0^x g(s)ds$ and $u(x,y)=\frac{y^2}{2}+G(x)$

Theorem (Lienard's Theorem): Suppose that $g, F\in C'$, f even, g odd, $xg(x)>0$ for $x\neq 0$, $F'(0)=f(0)<0$, $\exists!\, a>0$ s.t. $F(a)=0$, and $F(x)$ monotonically increasing to ∞ for $x\geq a$. Then the Lienard system has a unique limit cycle and it is stable.

↳ Note: since f is even $\Rightarrow$ F is odd

Proof:



If $\exists\, y_0$ such that $y_1=-y_0$, we get a closed orbit (because if $(x,y)\in X$, then $(-x,-y)\in X$) trajectory

$\varphi(\alpha)=\int_\gamma d(u(x,y))=u(0,y_1)-u(0,y_0)=\frac{y_1^2}{2}-\frac{y_0}{2}$

$du=ydy+g(x)dx=ydy-(y-F(x))dy=F(x)dy$

$\frac{dx}{dy}=\frac{y-F(x)}{-g(x)} \Rightarrow dx=\frac{y-F(x)}{-g(x)}dy$

at a, $\varphi(a)=\int_\gamma F(x)dy=\int_0^T F(x(t))(-g(x(t)))dt$

It can be shown that $\varphi(\alpha)$ is monotonically decreases to $-\infty$ as $\alpha\to-\infty$ (rigorous proof in Perko) ▨

For the van der Pol equation, $f(x)=\mu(x^2-1)$, $g(x)=x$, $\mu>0$

To use the theorem we need to show $\exists!\, a>0$ such that $F(a)=0$, $F(x)=\int_0^x f(s)ds$, $F(x)\nearrow\infty$, $x\geq a$, $x\to\infty$

$F(x)=\mu\int_0^x(s^2-1)ds=\mu\left(\frac{x^3}{3}-x\right)=\frac{\mu x}{3}(x-\sqrt{3})(x+\sqrt{3})$

$\Rightarrow F(x)=0$ for $a=\sqrt{3}>0$ and $F(x)\nearrow\infty$ for $x\geq\sqrt{3}$, $x\to\infty$

Thus the system has a unique limit cycle which is stable

Other examples:

$\dot{x}=y$, $\dot{y}=-f(x)y-g(x)$ and $\dot{x}=y-F(x)$, $\dot{y}=-g(x)$

March 13th

In classical mechanics, systems are described by generalized coordinates: (q,p), where $q=(q_1,\dots,q_n)$ is the position vector, and $p=(p_1,\dots,p_n)$ is the momentum vector.

The phase space is a set $U\subseteq\mathbb{R}^{2n}$.

If the function $H(q,p)$ is the total energy, then the dynamics can be described by:

$$\dot{q}=\frac{\partial H}{\partial p}, \quad \dot{p}=\frac{\partial H}{\partial q}$$

where $\frac{\partial H}{\partial p}=\left(\frac{\partial H}{\partial p_1},\dots,\frac{\partial H}{\partial p_n}\right)$ and $\frac{\partial H}{\partial q}=\left(\frac{\partial H}{\partial q_1},\dots,\frac{\partial H}{\partial q_n}\right)$

↳ H is called the Hamiltonian

Note that if $J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ where $I_n=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

then the equations can be written as

$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = JDH$, where $DH=\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$

Definition: A symplectic form on $\mathbb{R}^{2n}$ is a non-degenerate skew-symmetric bilinear form. A canonical symplectic form is given by $\Omega(u,v)=(u,Jv)$, $u,v \in \mathbb{R}^{2n}$

↳ More generally, $\Omega(u,v)=(u,Av)$ where A is a non-singular skew symmetric matrix

Given a function $H:\mathbb{R}^{2n}\to\mathbb{R}$ and a canonical symplectic form Ω, we can define a vector field, X_H, by:

I think this is inner product

$\Omega(X_H,v)=(DH,Jv) \quad \forall v\in\mathbb{R}^{2n}$ (1)

If (1) holds for $\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}=X_H$, then $\Omega((\dot{q},\dot{p}),v)=(DH,v)$

$\Rightarrow ((\dot{q},\dot{p}),Jv)=(DH,v)$

$\Rightarrow (J^T(\dot{q},\dot{p}),v)=(DH,v)$

$\Rightarrow -(J(\dot{q},\dot{p}),v)=(DH,v)$

$\Rightarrow (DH+J(\dot{q},\dot{p}),v)=0 \ \forall v\in\mathbb{R}^{2n}$

$\Rightarrow DH=-J(\dot{q},\dot{p})$

$\Rightarrow \frac{\partial H}{\partial q}=-\dot{p}, \quad \frac{\partial H}{\partial p}=\dot{q}$

$\Rightarrow \dot{q}=\frac{\partial H}{\partial p}, \quad \dot{p}=\frac{-\partial H}{\partial q}$

Definition: Given two functions, $F,G:\mathbb{R}^{2n}\to\mathbb{R}$ (smooth). We define the Poisson bracket of F and G by $\{F,G\}=\Omega(X_F,X_G)=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial q_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_i}\right)$

↳ we can show that given a Hamiltonian H, and a function F, we have $\dot{F}=\{F,H\}$

Notice that $\dot{H}=\{H,H\}=0$ so all orbits belong to level sets of H

Definition: A Hamiltonian system is completely integrable if there exists first integrals $I_1,\dots,I_{n-1},I_n=H$ such that the I_k are functionally independent (except maybe on a set of measure zero) and $\{I_k,I_e\}=0$, $1\le k,e\le n$

↳ Note that a set $M_f=\{(q,p)\in\mathbb{R}^{2n}: I_k(q,p)=f_k\}$, $f=(f_1,\dots,f_n)$ is an invariant set

↳ M_f is looking at level sets of each of the first integrals

Theorem: M_f is a manifold that is as differentiable as the least differentiable integral, and it is invariant. Also, M_f is diffeomorphic to the n-dimensional torus $T^n=\underbrace{S^1\times S^1\times\dots\times S^1}_{n}$ (S^1 a circle)

Moreover, the flow of the system gives rise to quasiperiodic motion on T^n, that is, if $\varphi\in T^n$ then $d\varphi/dt=\omega$, where $\omega(f)=(\omega_1(f),\dots,\omega_n(f))$

$\dot{x}=f(x),\ x\in\mathbb{R}^n$

Definition: An $(n-1)$ dimensional surface $S\subseteq\mathbb{R}^n$ is transverse to the vector field f if f is never tangent to S, i.e. $n(x)\cdot f(x)\neq 0\ \forall x\in S$ where $n(x)$ is the unit normal vector

Definition: A local section of f at some $x_0\in\mathbb{R}^n$ is a surface $S\subseteq B_\varepsilon(x_0)$ transverse to f, $x_0\in S$

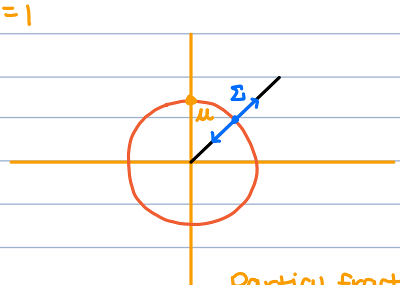
Theorem: let $\gamma(t)$ be a periodic orbit with period T
$\dot{x}=f(x),\ x\in\mathbb{R}^n$
and let $x_0=\gamma(0)$ (or any point on γ). let Σ be a local section at x_0.
Then $\exists$ a neighborhood U of x_0 and a function $\tau: U\to\mathbb{R}$ such that τ is as smooth as f, $\tau(x_0)=T$, and $\varphi(\tau(x),x)\in\Sigma\ \forall x\in\Sigma\cap U$ ($\varphi(t,x)$ is the flow of the system)
↳ $\tau(x)$ is a return function i.e. plug in a point around x_0 and it will output the time at which you come back (so if you plug in x_0, $\tau(x_0)=T$)

Proof: Implicit function theorem. Consider $F(t,x)=(\varphi_t(x)-x_0)\cdot f(x_0)$ if S is a hyperplane perpendicular to $f(x_0)$. Show $\frac{dF}{dt}(T,x_0)\neq 0$, then by implicit function theorem, we are done $\boxtimes$

Definition: The function $P(x)=\varphi(\tau(x_0+x),x_0+x)-x_0$ is called a Poincare function.
↳ $P(0)=0\Rightarrow 0$ is a fixed point for P

Theorem: If $\gamma(t)$ is a periodic orbit of $\dot{x}=f(x),\ x\in\mathbb{R}^n$, then $\gamma(t)$ is asymptotically stable iff 0 is an asymptotically stable fixed point of a Poincare map

Example:
$\dot{x}=-y+x(\mu^2+x^2-y^2)$
$\dot{y}=x+y(\mu^2-x^2-y^2)$ $\quad \mu>0$
let $x=r\cos\theta,\ y=r\sin\theta,\ r^2=x^2+y^2$
$\Rightarrow r\dot{r}=x\dot{x}+y\dot{y}=x^2(\mu-x^2-y^2)+y^2(\mu^2-x^2-y^2)=(\mu^2-r^2)r^2$
$\dot{x}=\dot{r}\cos\theta-r\sin\theta\,\dot{\theta}$
$\dot{y}=\dot{r}\sin\theta+r\sin\theta\,\dot{\theta}$ $\quad\Rightarrow x\dot{y}-y\dot{x}=\dot{\theta}(r^2\cos^2\theta+r^2\sin^2\theta)=\dot{\theta}\cdot r^2$
$\dot{r}=r(\mu^2-r^2)$
$x^2+y^2=\dot{\theta}\cdot r^2\Rightarrow r^2=\dot{\theta}\cdot r^2\Rightarrow\dot{\theta}=1$
Thus we have:
$\dot{r}=r(\mu-r^2)$
$\dot{\theta}=1$



$\theta(t)=\theta_0+t$

$$\int_{r_0}^{r}\frac{d\rho}{\rho(\mu^2-\rho^2)}=t\Rightarrow\overbrace{\int_{r_0}^{r}\frac{d\rho}{\rho}+\frac{1}{2\mu}\left[\int_{r_0}^{r}\frac{d\rho}{\mu-\rho}-\int_{r_0}^{r}\frac{d\rho}{\mu+\rho}\right]}^{\text{Partial fraction decomposition}}=\frac{1}{\mu^2}\ln\frac{r}{r_0}-\frac{1}{2\mu^2}\ln\frac{\mu^2-r^2}{\mu^2-r_0^2}$$

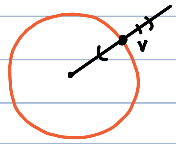
$\Rightarrow \left(\frac{r}{r_0}\right)^2\left(\frac{\mu^2-r_0^2}{\mu^2-r^2}\right)=e^{2\mu^2 t}$

$\Rightarrow r^2(\mu^2-r_0^2)=(\mu^2-r^2)r_0e^{2\mu^2 t}$

$\Rightarrow r^2(\mu^2-r_0^2+r_0e^{2\mu^2 t})=\mu r_0^2 e^{2\mu^2 t}$

$\Rightarrow r(t)=\left(\frac{\mu r_0^2 e^{2\mu^2 t}}{\mu^2-r_0^2+r_0e^{2\mu^2 t}}\right)^{1/2}$ $\tau(x)=2\pi$ since polar coordinates

$\Rightarrow P(r)=\left(\frac{\mu r e^{4\mu^2\pi}}{\mu^2-r^2+r^2e^{4\mu^2\pi}}\right)$



March 17th

$\dot{x}=f(x),\ f\in C^r,\ r\geq 1$

let $\varphi(t,x)$ be the corresponding flow, $\frac{d\varphi}{dt}(t,x)=f(\varphi(t,x))$

Suppose we have periodic orbit $\gamma(t)=\varphi(t,x_0)$ with period T and with Σ as a local section at x_0 (assume it is defined by $\langle x-x_0, f(x_0)\rangle=0$)

The poincare map $P:\Sigma\to\Sigma$ is given by $P(\xi)=\varphi(\tau(x_0+\xi),x_0+\xi)-x_0$.

We are interested in eigenvalues of $DP(0)$ since 0 corresponds to the fixed point x_0

Note that the map $\varphi(\tau(x_0+\xi),x_0+\xi)-x_0$ is defined not just on Σ but in a neighborhood of 0 so $DP(0)$ is the restriction of the derivative of $\varphi(\tau(x_0+\xi),x_0+\xi)-x_0$ to Σ

$D\varphi\big|_{\xi=0}=\frac{d\varphi}{dt}\big|_{\xi=0}+D\varphi\big|_{\xi=0}$, where $D\varphi$ denotes derivative of φ with respect to x

$=f(\varphi(\tau(x_0),x_0))\cdot D\tau(x_0)+D\varphi(\tau(x_0),x_0)=f(x_0)\cdot D\tau(x_0)+D\varphi(T,x_0)$ since $\tau(x_0)=T$, $\varphi(T,x_0)=x_0$

Notice that this linear map acts on a vector $\eta\in\mathbb{R}^n$ by

$\eta\to f(x_0)D\tau(x_0)\cdot\eta+\gamma(T)\cdot\eta=\langle D\tau(x_0),\eta\rangle f(x_0)+Y(T)\eta$ where $Y(T)=D\varphi(T,x_0)$

Notice that any vector can be written as a sum $c\cdot f(x_0)+\xi,\ \xi\in\Sigma$

If $\eta=c\cdot f(x_0)+\xi$, then $Y(T)\eta=cf(x_0)+Y(T)\xi=DP(0)\xi-\langle D\tau,\xi\rangle f(x_0)+cf(x_0)$

Theorem: $\lambda\neq 1$ is an eigenvalue of $DP(0)$ $\Leftrightarrow$ it is an eigenvalue of $Y(T)$
$\lambda=1$ is an eigenvalue of $D(P)$ $\Leftrightarrow$ 1 is an eigen value of multiplicity >1 of $Y(T)$

Consider $x(t)=y(t)+\gamma(t)$, $|y(t)|\ll 1$, then $\dot{y}+\dot{\gamma}=f(y(t)+\gamma(t))=f(\gamma(t))+Df(\gamma(t))\cdot y(t)+O(|y|^2)$

$\Rightarrow \dot{y}=Df(\gamma(t))y+O(|y|^2)$

Thus the linearized system around $\gamma(t)$ is $\dot{y}=A(t)y$, where $A(t)=Df(\gamma(t))$ is a periodic matrix

(Definition: A matrix $\Phi(t)$ is a fundamental matrix for the system $\dot{y}=A(t)y$, if it satisfies:
i) $\dot{\Phi}=A(t)\Phi$
ii) $\det\Phi\neq 0$)

↳ Then the solution to the IVP $\dot{y}=A(t)y,\ y(0)=y_0$ is given by $y(t)=\Phi(t)\Phi^{-1}(0)y_0$
Moreover, $\det\Phi(t)=\det\Phi(0)\cdot\exp\left(\int_0^t tr(A(t))dt\right)\leftarrow$ liouville's formula

It turns out that $Y(t)=D\varphi(t,x_0)$ is a fundamental matrix of this linearized system and $Y(0)=I$

↳ Proof: $\varphi(0)=id\Rightarrow D\varphi(0,x_0)=id$

$$D\varphi(t,x)=\begin{pmatrix}\partial\varphi_1/\partial x_1 & \cdots & \partial\varphi_1/\partial x_n\\ \vdots & & \vdots\\ \partial\varphi_n/\partial x_1 & \cdots & \partial\varphi_n/\partial x_n\end{pmatrix}$$

$\frac{d\varphi}{dt}(t,x)=f(\varphi(t,x))$

$\frac{\partial}{\partial t}\left(\frac{\partial \varphi_i}{\partial x_j}(t,x)\right)=\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}\varphi_i(t,x)\right)=\frac{\partial}{\partial x_j}\left(f_i(\varphi(t,x))\right)=\sum_{k=1}^{n}\frac{\partial f_i}{\partial x_k}\frac{\partial \varphi_k}{\partial x_j}$

$\frac{\partial}{\partial t}D\varphi(t,x)=Df(\varphi(t,x))\,Df(t,x)$, at x_0 we have $\frac{d}{dt}Y(t)=A(t)Y(t)$ (Proof concludes with the following lemma)

Lemma: If $y_0=f(x_0)$ and $y_1=f(\varphi(t_1,x_0))$, then $y_1=Y(t_1)y_0$

Proof: $y(t)=f(\varphi(t,x_0))$

$\frac{dy}{dt}=Df(\varphi(t_0,x_0))\cdot\frac{\partial\varphi}{\partial t}(t,x_0)=Df(\varphi(t,x_0))\cdot f(\varphi(t,x_0))=\overbrace{Df(\varphi(t,x_0))}^{A(t)}\cdot y$

$Y(t)$ is a fundamental matrix for $\dot{y}=A(t)y \Rightarrow y(t)=Y(t)Y^{-1}(0)\cdot y_0=Y(t)y_0$ (since $Y(0)=id$) ▨

↳ so $f(x_0)=Y(T)\cdot f(\varphi(T,x_0))=Y(T)\cdot f(x_0)$ so $f(x_0)$ is an eigenvector of $Y(T)$ with eigenvalue 1

Notice $\det Y(T)=\lambda_1\cdots\lambda_n=\exp\left(\int_0^t tr(Df(\gamma(s)))ds\right)=\exp\left(\int_0^t div\, f(\gamma(s))ds\right)$

↳ For $\mathbb{R}^2$ we get $\lambda_2=\exp\left(\int_0^t div\, f(\gamma(s))ds\right)$

You can represent $Y(t)=Z(t)e^{Dt}$

using the "logarithm" of $Y(T)$ gives $Y(T)=e^{DT}$

so define $Z(t)=Y(t)e^{-Dt}$

March 20th

Liouville's Theorem

consider the dynamical system

$\dot{x}=f(x), x\in\mathbb{R}^n$ with flow φ_t

let D_0 = domain in $\mathbb{R}^n$ and $D_t=\varphi_t(D_0)$

let $v(t)$ = volume of D_t

Lemma: $\left.\frac{dv}{dt}\right|_{t=0}=\int_{D_0}\nabla\cdot f\,dx$, where $\nabla\cdot f=\frac{\partial f_1}{\partial x_1}+\ldots+\frac{\partial f_n}{\partial x_n}$ (i.e. the divergence of f)

Proof: $v(t)=\int_{D_0}\det\frac{\partial\varphi_t(x)}{\partial x}dx$, where $\varphi_t(x)=x+f(x)t+O(t^2)$ ← taylor expansion

$\frac{\partial\varphi_t}{\partial x}=id+\frac{\partial f}{\partial x}t+O(t^2)$

$\det(I+\varepsilon A)=\underbrace{f_0(A)}_{1}+\varepsilon\underbrace{f_1(A)}_{tr(A)}+\varepsilon^2 f_2(A)+\ldots$

$\Rightarrow \det\frac{\partial\varphi}{\partial x}(x)=1+tr\left(\frac{\partial f}{\partial x}\right)\cdot t+O(t^2)$

$\Rightarrow v(t)=v(0)+\int_{D_0}t\,\nabla\cdot f\,dx+O(t^2)$ ▨

Liouville's Theorem: suppose $\nabla\cdot f=0$, then $\forall D_0$, $\underset{\text{volume of } D_t=\varphi_t(D_0)}{v(t)}=v(0)$ ← volume of D_0

↳ i.e. the volume is preserved

Poincare Recurrence Theorem: Assume $g:\mathbb{R}^n\to\mathbb{R}^n$ is continuous an one-to-one, and assume $D\subseteq\mathbb{R}^n$ is compact, $g(D)=D$, (i.e. D is invariant under g). Let $\bar{x}\in D$ and let U be a neighborood of $\bar{x}$.
Then $\exists x\in U$ such that $g^n(x)\in U$ for some $n>0$

Proof: consider the sequence $U, g(U), g^2(U), \dots, g^n(U), \dots$
Since g is volume preserving $\Rightarrow$ some of the $g^i(U)$ must intersect since otherwise D would be infinite $\rightarrow\leftarrow$
Assume $g^k(U)\cap g^l(U)\neq\emptyset$, $k>l$
$\Rightarrow g^{k-l}(U)\cap U\neq\emptyset$
Thus if we let $y=g^{k-l}(x)$, then $x\in U$ and $g^n(x)\in U$ where $n=k-l$

Example: Lorenz system (predicting climate)
$\dot{x}=\sigma(x-y)$
$\dot{y}=rx-y-xz$
$\dot{z}=xy-bz$
equilibrium points: $(0,0,0)$, $Q_\pm=(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$

March 22nd

Example:
$\dot{x}=\sigma(y-x)$
$\dot{y}=rx-y-xz$
$\dot{z}=xy-bz$

$$J=\begin{pmatrix}-\sigma & \sigma & 0\\ r-z & -1 & -x\\ y & x & -b\end{pmatrix} \qquad J\big|_{(0,0,0)}=\begin{pmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & b\end{pmatrix}$$

$$\begin{vmatrix}-\sigma-\lambda & \sigma\\ r & -1-\lambda\end{vmatrix}=\lambda^2+(\sigma+1)\lambda+\sigma-\sigma r \Rightarrow \lambda=\frac{1}{2}\left(-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma(1-r)}\right) \qquad 0\le r<1\Rightarrow\lambda<0$$

Proposition: If $r<1$, then all solutions tend to the origin
Proof: construct a Liapunov function:
$L(x,y,z)=x^2+\sigma y^2+\sigma z^2$
$(2x, 2\sigma y, 2\sigma z)(\dot{x}\dot{y}\dot{z})=2x\sigma(y-x)+2\sigma rxy-2\sigma y^2-2\sigma xyz+2\sigma zxy-2\sigma z^2=-2\sigma(x^2+y^2+z^2)+2\sigma xy(1+r)$ ▨
↳ when $r>1$, this is no longer true:
$v(x,y,z)=rx^2+\sigma y^2+\sigma(z-2r)^2$ (ellipsoid centered at $(0,0,2r)$)
$v(x,y,z)=v>0$

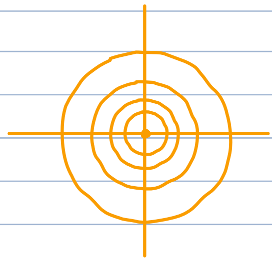
Proposition: $\exists v^*$ such that at any solution that starts outside $v=v^*$, eventually it enters it and stays trapped
Proof: $\dot{v}=(2rx, 2y\sigma, 2\sigma(z-2r))(\dot{x}\dot{y}\dot{z})=2r\sigma(xy-x^2)+2\sigma rxy-2\sigma y^2-2\sigma xyz+2\sigma xyz-2\sigma bz^2$
$-4\sigma rxy+4\sigma rbz=-2\sigma(rx^2+y^2+b(z-r)^2-br^2)$
$rx^2+y^2+b(z-r)^2=\mu$ defines an ellipsoid, $\mu>0$
when $\mu>br^2\Rightarrow\dot{v}<0$
choose v^* such that $rx^2+y^2+b(z-r)^2=br^2$ with $\dot{u}$ in its interior
divergence: $-\sigma-1-b<0$, $\dot{v}=\int \operatorname{div} V\,dxdydz=-(\sigma+1+b)V$
$\Rightarrow v(t)=e^{-(\sigma+1+b)t}v_0$
which shrinks exponentially to 0

Consequence: the volume of the set of points whose solution remains for all time (backwards and forward) in the ellipsoid $v=v^*$ is 0

April 3rd

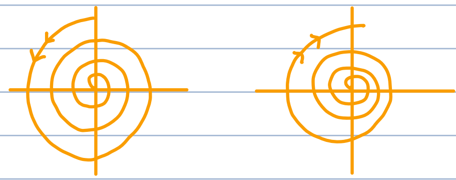
Example: Simple Harmonic oscillator

$\dot{x} = y$
$\dot{y} = -\omega_0^2 x$



Consider the perturbation:

$\dot{x} = y$
$\dot{y} = -\omega_0^2 x - \varepsilon y, \ |\varepsilon| \ll 1$



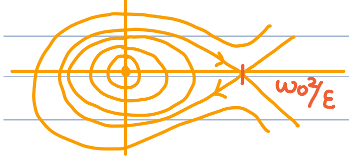
orientation of arrows depends on ε

Another Pertubation:

$\dot{x} = y$
$\dot{y} = -\omega_0^2 x + \varepsilon x^2$

Notice that $h(x,y) = \frac{y^2}{2} + \frac{\omega_0^2 x^2}{2} - \varepsilon \frac{x^3}{3}$ is constant along trajectories.

We have two equilibrium points: $(0,0), (\omega_0^2/\varepsilon, 0)$ $\varepsilon > 0$



↳ This is an example of a structurally unstable dynamical system since one small change affected it greatly

Recall that two vector fields $f, g \in C^r(\mathbb{R}^n)$ are topologically conjugate if $\exists$ homeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $h(\varphi(t,x)) = \psi(t, h(x))$ where φ and ψ are flows of f and g respectively

↳ instead of preserving time parametrization, we can allow it to change so $\forall x \ \exists \ \tau(x,t)$ s.t. $\frac{\partial \tau}{\partial t} > 0$ and $h(\varphi(t,x)) = \psi(\tau(x,t), h(x))$

↳ such systems/vector fields are called topologically equivalent

Note that the space of dynamical systems on $\mathbb{R}^n$ coincides with the space of vector fields i.e. it is a space of C^r functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, $C^r(\mathbb{R}^n, \mathbb{R}^n)$.

↳ One of the possible norms on $C^r(\mathbb{R}^n, \mathbb{R}^n)$ would be $\sup_{x \in \mathbb{R}^n} \|f(x)\|$ but this doesnt work unless we restrict ourselves to the subspace of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ consisting of bounded functions.

Moreover we need to control $Df(x)$

To eliminate the problem with unboundedness we can consider either compact invariant subsets of $\mathbb{R}^n$ or compact manifolds without a boundary

so suppose $f \in C^r(E, \mathbb{R}^n)$, $\bar{E}$-compact, E-open. Assume $\varphi(t,x) \in E \ \forall x \in E, t \in \mathbb{R}$.

Define $\|f\|_1 = \sup_{x\in E}\|f(x)\| + \sup_{x\in E}\underbrace{\|Df(x)\|}_{\substack{\shortparallel \\ \sup_{\substack{\|y\|\neq 0 \\ y\in\mathbb{R}^n}} \frac{\|Df(x)y\|}{\|y\|}}}$ ← standard euclidean norm

The distance between $f, g \in C^r(\bar{E}, \mathbb{R}^n)$ is simply $\|f-g\|_1$

Also works for compact manifolds

Definition: The vector field f on E is structurally stable if for any vector field g on f s.t. $\|f-g\|_1 < \varepsilon$, f and g are topologically equivalent if ε is sufficiently small

April 5th

Definition: If X is a topological space, then $U \subseteq X$ is called residual if $U = \bigcap_{i=0}^{\infty} U_i$, U_i-open, dense.

If X is s.t. every residual set is dense, then it is called a Baire space

Theorem (Peixoto): Let f be a C^r, $r\geq 1$, vector fixed on a 2-dimensional compact, differentiable, manifold (without boundary), then f is structurally stable iff

1) The number of equilibria and closed orbits is finite and each is hyperbolic

↳ Definition: An equilibrium is hyperbolic if the eigenvalues of linearization dont have 0 real part.

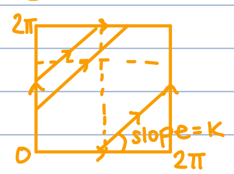
2) There are no orbits connecting equilibrium points

3) The non-wandering set consists of equilibrium points and limit cycles only

Moreover, the set of structurally stable vector fields is open and dense subset of $C^r(M, \mathbb{R}^2)$, where M is our manifold

Example: A torus

$\dot{x} = \omega_1$
$\dot{y} = \omega_2$ defines a flow on a torus

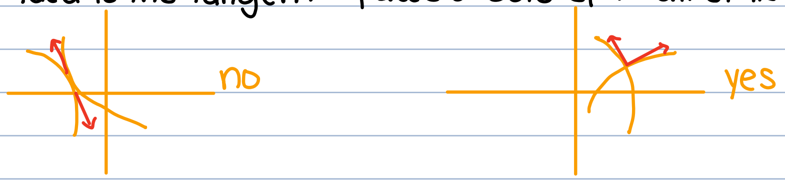


$K = \frac{\omega_2}{\omega_1}$. The whole torus becomes the nonwandering set but this is not structurally stable

Unfortunately, the set of structurally stable dynamical systems on 3-dimensional manifolds is not residual

Definition: Let $M, N \subseteq \mathbb{R}^n$ be submanifolds. We say that M and N intersect transversally if $M \cap N = \emptyset$ or $\forall x \in M \cap N$, $T_xM + T_xN = \mathbb{R}^n$ (T_xM is the tangent space of M at x. Similarly for T_xN)

↳ idea is the tangent spaces should span all of $\mathbb{R}^n$



Definition: A Morse-Smale system is one for which:

1) The number of equilibrium points and closed orbits is finite and each is hyperbolic
2) All stable and unstable manifolds intersect transversally
3) The non-wandering set consists of equilibrium points and closed orbits only

↳ Note: A Morse-Smale system on a compact manifold is structurally stable

Many dynamical systems that model real world systems depend on parameters:

$\dot{x} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$

μ - parameters

$f(x, \mu) = 0$

$f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$

April 7th

$\dot{x} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$

Assume $f(x_0, \mu_0) = 0$

We know that if $D_x f(x_0, \mu_0)$ doesn't have eigenvalues with zero real part, then small perturbations should not change the behavior of the system in the neighborhood of (x_0, μ_0).
To make this slightly more rigorous, let's define the notion of local structural stability

Definition: Consider a vector field f on an open set $U \subseteq \mathbb{R}^n$. We say that f is structurally stable on U if $\exists V \supseteq U$ such that f is topologically equivalent to any g on V with $\|f - g\|_1$ sufficiently small.

↳ So if (x_0, μ_0) is hyperbolic, then $f(x, \mu_0)$ is structurally stable on a sufficiently small neighborhood of x_0. However, if (x_0, μ_0) is not hyperbolic, then $f(x_0, \mu_0)$ may not be structurally stable

Definition: We say that an equilibrium point (x_0, μ_0) of $\dot{x} = f(x, \mu)$ undergoes a bifurcation at $\mu = \mu_0$ if $f(x, \mu_0)$ is not locally structurally stable

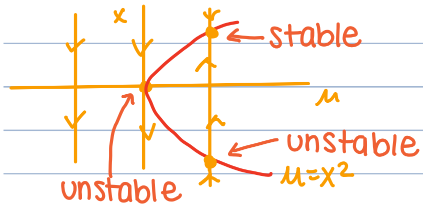
Let's focus on the case where μ is a scalar

Example: $\dot{x} = \mu - x^2$, $x \in \mathbb{R}^1$, $\mu \in \mathbb{R}^1$

We have two equilibrium points if $\mu > 0$: $x = \pm\sqrt{\mu}$, one equilibrium point if $\mu = 0$: $x = 0$, and no equilibrium points if $\mu < 0$.

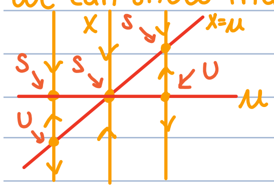
It is easy to see that $\frac{\partial f}{\partial x} = -2x$ ($f(x, \mu) = \mu - x^2$) is not 0 at $x = \pm\sqrt{\mu}$ and is 0 at $x = 0$

Lets consider the set of equilibrium points in the (μ, x)-plane



← This is called saddle-node bifurcation

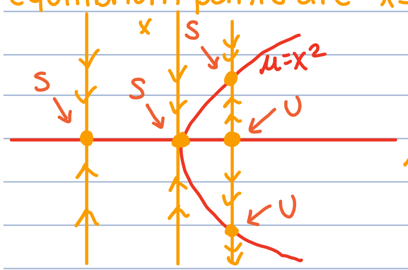
Example: $\dot{x} = \mu x - x^2$
equilibrium points are $x=0$ and $x=\mu$
we can show that for $\mu=0$, $x=0$ is not hyperbolic



(two equilibrium points merge to one then split to two)

Transcritical bifurcation

Example: $\dot{x} = \mu x - x^3$
equilibrium points are $x \equiv 0$ and $x = \pm\sqrt{\mu}$ (if $\mu > 0$)



← pitchfork bifurcation

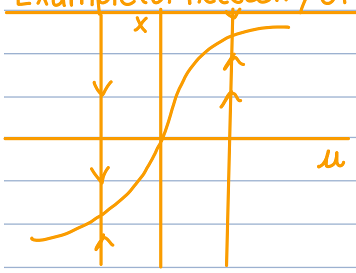
Question: when does one of these bifurcations happen in a general system?

$\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^1$, $\mu \in \mathbb{R}^1$

Note that if $Df = \left(\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial \mu}\right) \neq 0 \;\; \forall x, \mu$, then $f(x,\mu)=0$ defines a smooth curve in the (μ, x)-plane

so we'll have a saddle-node bifurcation if $\exists (x_0, \mu_0)$ such that $f(x_0,\mu_0)=0$ and the above curve is tangent to the vertical line through (x_0,μ_0) and (locally) lies to one side of this line

Example (of necessity of lying to one side): $\dot{x} = \mu - x^3$



$\dot{x} = \mu - x^3$

April 10th

$\dot{x}=f(x,\mu),\ x\in\mathbb{R}^1,\ \mu\in\mathbb{R}^1$
Assume $f(0,0)=0,\ \frac{\partial f}{\partial x}(0,0)=0$

For saddle-node, we need a unique curve (in (μ,x)-plane) passing through $(0,0)$ and this curve should lie on one side of $\mu=0$
From Implicit Function Theorem, we need $\frac{\partial f}{\partial \mu}(0,0)\neq 0$ to have a unique curve, $\mu(x)$, passing through $(0,0)$. For this curve to be tangent to $\mu=0$, we need $\frac{d\mu}{dx}(0)=0$ and for it to be on one side of $\mu=0$, it's enough to have $\frac{d^2\mu}{dx^2}(0)\neq 0$

consider $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$
$F(x,y)=0,\ F(0,0)=0,\ (x,y)\in\mathbb{R}^n\times\mathbb{R}^m$
If $D_yF(0,0)$ is non-singular then $\exists g:\mathbb{R}^n\to\mathbb{R}^m$ s.t. $F(x,g(x))\equiv 0$
Then since $f(x,\mu(x))\equiv 0$,

$$\frac{\partial f}{\partial x}(x,\mu(x))+\frac{\partial f}{\partial \mu}(x,\mu(x))\cdot\frac{d\mu}{dx}(x)\equiv 0$$

$$\frac{d\mu}{dx}(0)=\frac{-\partial f/\partial x(0,0)}{\partial f/\partial \mu(0,0)}=0 \text{ if } \frac{\partial f}{\partial x}(0,0)=0$$

$$\frac{d^2\mu}{dx^2}(0)=\frac{-\partial^2 f/\partial x^2(0,0)}{\partial f/\partial \mu(0,0)}\neq 0 \text{ if } \frac{\partial^2 f}{\partial x^2}(0,0)\neq 0$$

so to have a S-N bifurcation of $\dot{x}=f(x,\mu)$ at a $\overbrace{\text{non-hyperbolic}}^{\frac{\partial f}{\partial x}(0,0)=0}$ equil point $(0,0)$ we need

i) $\frac{\partial f}{\partial \mu}(0,0)\neq 0$

ii) $\frac{\partial^2 f}{\partial x^2}(0,0)\neq 0$

Assume $f(0,0)=0,\ \frac{\partial f}{\partial x}(0,0)=0$

For a transcritical bifurcation we need two curves of equilibria passing through $(0,0)$.
Implicit function theorem tells us that we need $\frac{\partial f}{\partial \mu}(0,0)=0$

Recall that we need to have $x=0$ equil for all μ. so, $f(x,\mu)=xF(x,\mu)$, where

$$F(x,\mu)=\begin{cases} f(x,\mu)/x, & x\neq 0\\ \partial f/\partial x(0,\mu), & x=0\end{cases}$$

Note that:

$$F(0,0)=0,\ \frac{\partial F}{\partial x}(0,0)=\frac{\partial^2 f}{\partial x^2}(0,0),\ \frac{\partial^2 F}{\partial x^2}(0,0)=\frac{\partial^3 f}{\partial x^3}(0,0),\ \frac{\partial F}{\partial \mu}(0,0)=\frac{\partial^2 f}{\partial \mu\partial x}(0,0)$$

we need a curve of equilibria different from $x=0$ passing through $(0,0)$ in the (μ,x)-plane. so we need $\frac{\partial F}{\partial \mu}(0,0)\neq 0$. For the resulting curve $\mu(x)$ (s.t. $F(x,\mu(x))\equiv 0$) to be different from $x=0$ we require $0<\left|\frac{\partial \mu}{\partial x}(0)\right|<\infty$

$$\frac{d\mu}{dx}(0)=\frac{-\partial F/\partial x(0,0)}{\partial F/\partial \mu(0,0)}=-\frac{\partial^2 f/\partial x^2(0,0)}{\partial^2 f/\partial x\partial \mu(0,0)}\neq 0 \text{ if } \frac{\partial^2 f}{\partial x^2}(0,0)\neq 0$$

so, for transcritical bifurcation, we need:

i) $\frac{\partial f}{\partial \mu}(0,0)=0$ iii) $\frac{\partial^2 f}{\partial x^2}(0,0)\neq 0$

ii) $\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\neq 0$

For pitchfork bifurcation we need:

1) $\frac{\partial f}{\partial \mu}(0,0)=0$

2) $\frac{\partial^2 f}{\partial x^2}(0,0)=0$

3) $\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\neq 0$

4) $\frac{\partial^3 f}{\partial x^3}(0,0)\neq 0$

April 12th

$\dot{x}=F(x,\mu)$, $x\in\mathbb{R}^n$, $\mu\in\mathbb{R}^k$

Suppose $F(0,0)=0$. If $D_xF(0,0)$ has eigenvalues with zero real part, we have a non-hyperbolic equilibrium point.
Suppose there are c eigenvalues with zero real part and assume the rest of the eigenvalues have negative real part.
To figure out what happens for small μ, we treat it as a variable:
$$\dot{x}=F(x,\mu)=D_xF(0,0)x+D_\mu F(0,0)\mu+\underbrace{F_2(x,\mu)}_{=O(|x|^2,|\mu|^2)}$$

For $\dot{\mu}=0$:
Let T be matrix s.t. $T^{-1}D_xF(0,0)T=J$, Jordan can. form where $J=\begin{pmatrix} J_1 & \\ & J_2\end{pmatrix}$,

J_1 has eigenvalues with zero real part (it's $c\times c$)
$x=T\begin{pmatrix}u\\v\end{pmatrix}$, u is a c-dim vector

Then we get $T\begin{pmatrix}\dot{u}\\\dot{v}\end{pmatrix}=D_xF(0,0)T\begin{pmatrix}u\\v\end{pmatrix}+D_\mu F(0,0)\mu+F_2(u,v,\mu)$

$\dot{u}=J_1u+\Lambda_1\mu+f(u,v,\mu)$
$\dot{\mu}=0$
$\dot{v}=J_2v+\Lambda_2\mu+g(u,v,\mu)$
$T^{-1}D_\mu F(0,0)=\begin{pmatrix}\Lambda_1\\\Lambda_2\end{pmatrix}$, $\begin{pmatrix}f\\g\end{pmatrix}=T^{-1}F_2(T\begin{pmatrix}u\\v\end{pmatrix},\mu)$

so, the center manifold can be expressed as $v=h(u,\mu)$, $Dh(0,0)=0$, $h(0,0)=0$
Then we can approximate: $h(u,\mu)=$ second order terms + third order term + ...
Note: $\dot{v}=D_xh(u,\mu)\cdot\dot{u}+D_\mu h(u,\mu)\cdot\underbrace{\dot{\mu}}_{=0}=J_2h(u,\mu)+\Lambda_2\mu+g(u,h(u,\mu),\mu)$

$D_xh(u,\mu)[J_1u+\Lambda_1\mu+f(u,h(u,\mu),\mu)]-J_2h(u,\mu)-\Lambda_2\mu-g(u,h(u,\mu),\mu)=0$

Example:

$\dot{x} = \frac{x}{2} + y + x^2 y$

$\dot{y} = x + 2y + \varepsilon y + y^2$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 y \\ \varepsilon y + y^2 \end{pmatrix}$$

$\det(A - \lambda I) = \lambda^2 - 5/2\,\lambda = 0 \Rightarrow \lambda = 0, \lambda = 5/2$

$\lambda = 0: \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \lambda = \frac{5}{2}, \begin{pmatrix} -2 & 1 \\ 1 & -1/2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$T = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad T^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u+v \\ 2v-u \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 5/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} (2u+v)^2(2v-u) \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 5/2\, v \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 2(2u+v)^2(2v-u) - \varepsilon(2v-u) - (2v-u)^2 \\ (2u+v)^2(2v-u) + 2\varepsilon(2v-u) + 2(2v-u)^2 \end{pmatrix}$$

$v = h(u, \varepsilon) = a_1 u^2 + a_2 \varepsilon u + a_3 \varepsilon^2 + O(3)$

$(2a_1 u + a_2 \varepsilon + O(2))(O(2)) = 5/2\,(a_1 u^2 + a_2 \varepsilon u + a_3 \varepsilon^2 + O(3)) + 1/5\,[-2\varepsilon u + 2u^2 + O(3)]$

$$O(3) = \underbrace{\left(\frac{5}{2}a_1 + \frac{2}{5}\right)}_{=0} u^2 + \left(\frac{5}{2}a_2 - \frac{2}{5}\right)\varepsilon u + \underbrace{\frac{5}{2}a_3}_{=0} \varepsilon^2 + O(3)$$

$a_1 = \frac{-4}{25}, a_2 = \frac{4}{25}$

$$\dot{u} = \frac{1}{5}\left[-\varepsilon\left(\frac{8u}{25}(\varepsilon - u) - u\right) - u^2\right] + O(3) = \frac{u}{5}\left[-\varepsilon\left(\frac{8}{25}(\varepsilon - u) - 1\right) - u\right] = \frac{u}{5}\left[\varepsilon - u - \frac{8\varepsilon}{25}(\varepsilon - u)\right] = \frac{u}{5}(\varepsilon - u)\left(1 - \frac{8\varepsilon}{25}\right) + O(3)$$

transcritical bifurcation

April 17th

$\dot{x} = f(x), f(x_0) = 0$

First assume $x = x_0 + y$, then $\dot{y} = \tilde{f}(y), \tilde{f}(0) = f(x_0) = 0$

$\tilde{f}(y) = D\tilde{f}(0)y + G(y)$, where $G(y) = O(|y|^2)$ so $\dot{y} = D\tilde{f}(0)y + G(y)$

let $J = T^{-1} D\tilde{f}(0) T$, where J is Jordan canonical form of $D\tilde{f}(0)$

let $y = Tu$, then $\dot{u} = Ju + \tilde{G}(u)$

Taylor expand $\tilde{G}(u)$ further: $\dot{u} = Ju + F_2(u) + F_3(u) + \ldots + F_{r-1}(u) + O(|u|^r)$

Note that each component of the vector $F_k(u)$ is a linear combination of monomials of degree k.

That is, if $F_k(u) = \begin{pmatrix} F_u^1(u) \\ \vdots \\ F_u^k(u) \end{pmatrix}$, then $F_u^i(u) = \sum C_{i_1 i_2 \cdots i_n} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n}$, $\sum_{i=1}^{n} \alpha_i = k$

How can we simplify $F_2(u)$?

let $u = v + h_2(v)$, then $\dot{v} = Dh_2(v) \cdot \dot{v} = Jv + Jh_2(v) + F_2(v + h_2(v)) + \ldots$

$(I + Dh_2(v))\dot{v} = Jv + Jh_2(v) + F_2(v) + \underbrace{DF_2(v) \cdot h_2(v)}_{O(|v|^3)} + \ldots$

Note that $(I+Dh_2(v))^{-1}=I-Dh_2(v)+O(|v|^2)$

$\Rightarrow \dot{v}=(I-Dh_2(v)+O(|v|^2))(Jv+Jh_2(v)+F_2(v)+O(|v|^3))$

$\quad = Jv+Jh_2(v)-Dh_2(v)\cdot Jv+F_2(v)+O(|v|^3)$

we want to pick $h_2(v)$ such that $Jh_2(v)-Dh_2(v)Jv+F_2(v)$ is as simple as possible

Note that $F_2(v)\in H_2$, where H_2 is the vector space of homogeneous polynomials of degree 2

Consider the operator $L_2: H_2 \to H_2$

$(L_2P)(v)=DP(v)JP(v)-Jv=[P(v),Jv]$

so $Jh_2(v)-Dh_2(v)Jv=-(L_2h_2)(v)$, if $h_2\in H_2$

L_2 is linear so $H_2=\operatorname{Im}L_2 \oplus G_2$, where G_2 is the complement of $\operatorname{Im}L_2$

so $F_2:=F_2^1+F_2^2$, where $F_2^1\in \operatorname{Im}L_2$, $F_2^2\in G_2$

Pick h_2 to cancel F_2^1

Note, if $v=(v_1,v_2)$ then the basis of H_2 is

$\begin{pmatrix} v_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1v_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2^2 \end{pmatrix}$

Example:

$J=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then

$$L_2=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

basis: $\overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}^{G_2}$

$\begin{pmatrix} v_1^2 \\ \frac{1}{2}v_1v_2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ v_1^2 \end{pmatrix}$

April 19th

$\dot{x}=f(x,\mu)$, $x\in\mathbb{R}^n$, $\mu\in\mathbb{R}$, f is C^r, $r\geq 5$

Assume $f(0,\mu)=0$ and $D_xf(0,0)$ has two complex conjugate eigenvalues with zero real part (and the other eigenvalues have negative real part)

as you change μ you will at some point cross the imaginary axis

using center manifold theory, we get the following:

$\dot{u}=\alpha(\mu)u-\beta(\mu)+f_1(u,v,\mu)$

$\dot{v}=\beta(\mu)u+\alpha(\mu)v+f_2(u,v,\mu)$,

where $\lambda(\mu)=\alpha(\mu)\pm i\beta(\mu)$ are the eigenvalues crossing the imaginary axis.

Note: $\alpha(0)=0$

using normal form we get

$\dot{u}=\alpha(\mu)u-\beta(\mu)v+(a(\mu)u-b(\mu)v)(u^2+v^2)+O(5)$

$\dot{v}=\beta(\mu)u+\alpha(\mu)v+(b(\mu)u+a(\mu)v)(u^2+v^2)+O(5)$

In polar coordinates:

$\dot{r} = \alpha(\mu) r + a(\mu) r^3 + O(r^5)$

$\dot{\theta} = \beta(\mu) + b\mu r^2 + O(r^4)$

Taylor expand α, β, a, b:

$\dot{r} = \alpha'(0)\mu r + a(0) r^3 + O(\mu^2, \mu r^3, r^5)$

$\dot{\theta} = \beta(0) + \beta'(0)\mu + b(0) r^2 + O(\mu^2, \mu r^2, r^4)$

Consider the truncated system:

$\dot{r} = d\mu r + a r^3$

$\dot{\theta} = \omega + c\mu + b r^2$

$d = \alpha'(0)$

$a = a(0)$

$\omega = \beta(0)$

$c = \beta'(0)$

$b = b(0)$

Note that if $\frac{d\mu}{a} < 0$, then $\dot{r} = 0$ at $r = \sqrt{\frac{-d\mu}{a}}$

In fact, $(r(t), \theta(t)) = \left(\sqrt{\frac{-d\mu}{a}}, (\omega + (c - \frac{bd}{a})\mu)t + \theta_0\right)$

is a periodic orbit if μ is small enough. It's asymptotically stable for $a<0$, unstable for $a>0$

There are 4 cases:

1) $d>0, a>0$
stable eq. point for $\mu<0$, unstable eq pt for $\mu>0$, unstable periodic orbit for $\mu<0$

2) $d>0, a<0$, stable equilibrium point $\mu<0$, unstable equilibrium point $\mu>0$
stable periodic orbit $\mu>0$

3) $d<0, a>0$, ...

4) $d<0, a<0$, ...

If $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f^1(u,v,0) \\ f^2(u,v,0) \end{pmatrix}$ at $\mu = 0$, then $a = \frac{1}{16}\left[f^1_{uuu} + f^1_{uvv} + f^2_{uuv} + f^2_{vvv}\right] +$

$\frac{1}{16\omega}\left[f^1_{uv}(f^1_{uu} + f^1_{vv}) - f^2_{uv}(f^2_{uu} + f^2_{vv}) - f^1_{uu} f^2_{uu} + f^1_{vv} f^2_{vv}\right]$

April 21st

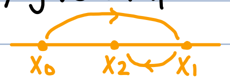
Chaos (in discrete maps)

discrete maps are known as difference equations (successive iterations)

Example: $x_{n+1} = \cos x_n$, a 1-dimensional map

↳ The sequence $x_0, x_1, \ldots$ is called the orbit starting at x_0

These systems come in several ways:
1) They help us analyze differential equations
↳ Example: Poincare map, Lorenz map
2) They model natural phenomena
↳ sometimes we want the time to be discrete
↳ Example: Animal population
3) They give simple examples of chaos since you can have wilder behavior
↳



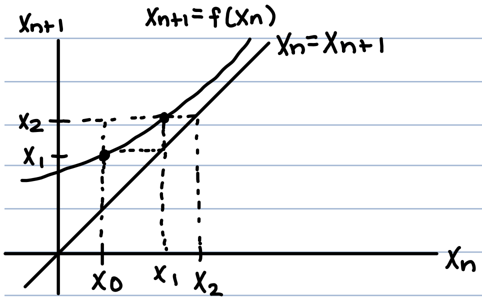
order one difference equation: $x_{n+1}=f(x_n)$, $f\in C^\infty$
order two: $x_{n+1}=f(x_n, x_{n-1})$

Definition: A fixed point is a point x^* such that $f(x^*)=x^*$ i.e. orbit remains at x^* for all future iterations

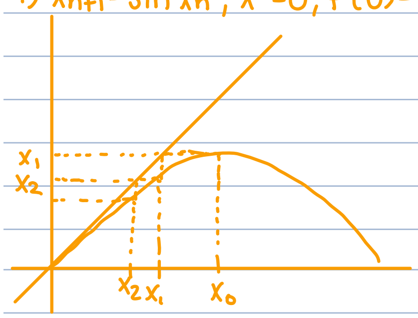
we want to study stability starting at $x_n=x^*+\varepsilon_n$ i.e. are points attracted or repelled
we look at the linearization using Taylor series expansion:
$x^*+\varepsilon_{n+1}=x_{n+1}=f(x+\varepsilon_n)=f(x^*)+f'(x^*)\varepsilon_n+O(\varepsilon_n^2)$
since $f(x^*)=x^* \Rightarrow \underbrace{\varepsilon_{n+1}=f'(x^*)\varepsilon_n}_{\text{linearized map (equivalent of the Jacobian)}}+O(\varepsilon_n^2)$

This gives: $\varepsilon_0 \longrightarrow \varepsilon_1=f'(x^*)\varepsilon_0$
$\varepsilon_2=(f'(x^*))^2\varepsilon_0$
$\vdots$
$\varepsilon_n=(f'(x^*))^n\varepsilon_0$
if $|f'(x^*)|<1 \Rightarrow \varepsilon_n\to 0$ so x^* is linearly stable
if $|f'(x^*)|>1 \Rightarrow \varepsilon_n\to\infty$ so unstable
if $|f'(x^*)|=1$, inconclusive so you need to study more terms

Cowebs
Definition: Cowebs are graphical representations of differential equations
↳ gives information about stability



Examples:
1) $x_{n+1}=\sin x_n$, $x^*=0$, $f'(0)=1$

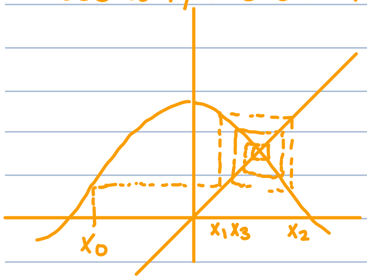


brings you to the origin so x^* is locally stable

2) $x_{n+1} = \cos x_n$, $x_{n+1} \xrightarrow{n\to\infty} ?$

a calculator gives $x_n \to 0.739...$

To see why there is a fixed point there, look at the coweb

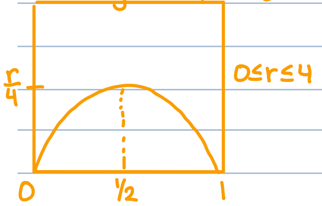


Example: The discrete logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

P = population, r = rate, K = maximum population

This gives the discrete model: $x_{n+1} = \overbrace{r x_n (1 - x_n)}^{f(x_n)}$ (we normalized $x_n \to x_n/K$ (dimensionless)), $x_n \geq 0$



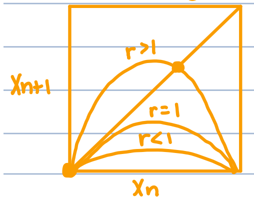
Fixed points: $r x^* (1 - x^*) = x^*$

$\Rightarrow x^* = 0$ or $r(1 - x^*) = 1 \Rightarrow x^* = 1 - 1/r$ (for $r \geq 1$)

stability: $f'(x^*) = r - 2rx^*$

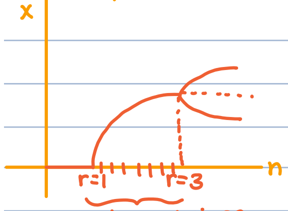
$f'(0) = r$ so stable if $r<1$, unstable if $r>1$ (transcritical bifurcation one fixed point became two and the original changed stability)

$f'(1 - 1/r) = 2 - r$ so stable if $-1 < 2 - r < 1 \Rightarrow 1 < r < 3$ and unstable at $r > 3$



(r=1 is tangent)

at r=3, there is "period-doubling", $f(f(x)) = f^2(x) = x$

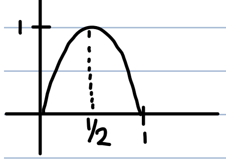


dashed lines indicate instability

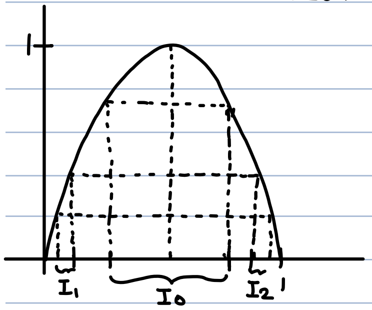
April 24th

$x_{n+1} = r x_n (1-x_n)$

let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = rx(1-x)$
we know that at $r=3$ the equilibrium point $x^* = 1 - 1/r$ loses stability.
we can consider the map $f^2 = f \circ f$
$f(f(x)) = f(rx(1-x)) = r^2 x(1-x)(1-rx(1-x))$
for $r>3$, we get two additional fixed points of f^2: p, q
we can show that for $1+\sqrt{6} < r < 3$, p, q are stable
$r^2(1-x)(1-rx(1-x)) = 1$
$r(1-x)(1-rx+rx^2) = 1$
$r(1-x-rx+rx^2+rx^2-rx^3) = 1$
$r(1-x-rx+2rx^2-rx^3) = 1$
$r^2x^3 - 2r^2x^2 + (r^2+r)x - r + 1 = 0$
$r^2(x - {}^{r-1}/_r)x^2 = 0$
when $r=4$, max $f = 1$



$x_{n+1} = r x_n (1-x_n)$
Note that there's an interval around $1/2$ such that $\forall x \in I_0$ $f^n(x) \xrightarrow{n\to\infty} -\infty$
Then consider $f^{-1}(I_0)$. We also have $f^n(x) \to -\infty$



<u>Note</u>: $f(I_0) = I_1 \sqcup I_2$
we can continue $f^{(-n)}(I_0) \leftarrow$ inverse image
<u>Note</u>: $f^{-n}(I_0)$ consists of 2^n disjoint open intervals
let $\Lambda = [0,1] \setminus \bigcup_{n=0}^{\infty} f^{-n}(I_0)$, this is invariant with respect to R

Divide $[0,1]$ into $L = [0, 1/2]$ and $R = [1/2, 1]$
consider any $x \in \Lambda$ then either $x \in R$ or $x \in L$
similarly $f(x) \in R$ or $f(x) \in L$, $f^{(n)}(x) \in R$ or $f^{(n)}(x) \in L$ $\forall n$
so to each $x \in \Lambda$ we can associate a sequence LRRRLRLLR...
To make things simpler, let's use 0 for L and 1 fo R
let Σ denote the space of infinite sequences of 0s and 1s i.e. $\Sigma = \{f : \mathbb{N} \to \{0,1\}\}$, $f = (s_0, s_1, \dots)$
we can endow Σ with a metric

$$d(s,t) = d((s_0, s_1, \dots, s_n, \dots), (t_0, t_1, \dots, t_n, \dots)) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Note that given $s = (s_0, \dots)$, $t = (t_0, \dots)$
$s_0 = t_0, \dots, s_n = t_n \Leftrightarrow d(s,t) < 1/2^n \Rightarrow d(s,t) \le 1/2^n$
Note that we have a map $h: \Omega \to \Sigma$

Theorem: If $r>4$, then h is a homeomorphism

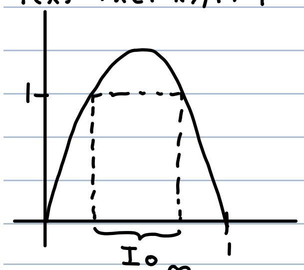
Recall: two maps are topologically conjugate if $\exists h: X\to X$, a homeomorphism such that $h\circ f = g\circ h$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

so we can define $g: \Sigma \to \Sigma$ so that the above h is a conjugacy

April 26th

$f(x) = rx(1-x),\ r>4$

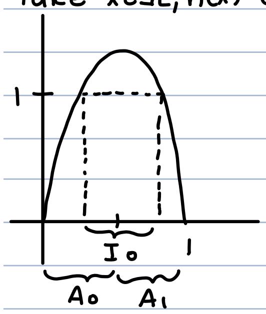


$\Lambda = [0,1] \setminus \bigcup_{n=0}^{\infty} f^{-n}(I_0)$

$h: \Lambda \to \Sigma$, $\Sigma = \{s: \mathbb{N} \to \{0,1\}\}$, $d(s,t) = \sum_{i=0}^{\infty} \frac{(s_i - t_i)}{2^i}$

Theorem: h is a homeomorphism for $r>4$

Take $x\in\Lambda$, $h(x) = (s_0, s_1, \dots)$, $x \in A_{s_0}$, $f(x) \in A_{s_1}$, $f^2(x) \in A_{s_2}, \dots$



note that then $h(f(x)) = (s_1, s_2, \dots)$

let $\sigma: \Sigma \to \Sigma$ be defined by $\sigma((s_0, s_1, \dots)) = (s_1, s_2, \dots)$

↳ This is a shift map

Theorem: h is a conjugacy between $f: \Lambda \to \Lambda$ and $\sigma: \Sigma \to \Sigma$

Proof: we need to show that $h\circ f = \sigma \circ h$

take $x \in \Lambda$, $h(x) = (s_0, s_1, \dots)$

$h(f(x)) = (s_1, s_2, \dots)$, $\sigma(h(x)) = \sigma((s_0, s_1, \dots)) = (s_1, s_2, \dots)$ $\boxtimes$

consider $\sigma: \Sigma \to \Sigma$

the fixed points are: $(0,0,\dots)$ and $(1,1,\dots)$

let $s_1, \dots, s_k$ be any finite sequence and let $\overline{s_1, \dots, s_k}$ denote the element of Σ obtained by repeating the sequence $s_1, \dots, s_k$ i.e. $\overline{s_1, \dots, s_k} = (s_1, \dots, s_k, s_1, \dots, s_k, s_1, \dots, s_k, \dots)$

Then $\overline{s_1, \dots, s_k}$ is a periodic point with period k. Thus we have infinitely many periodic points.

Claim: periodic points are dense

Proof: let $s \in \Sigma$. Take $\varepsilon > 0$ and let $\frac{1}{2^n} < \varepsilon$

if $s = (s_1, \dots, s_n, \dots)$, let $\bar{s} = \overline{s_1, \dots, s_n}$, then $d(s, \bar{s}) \le \frac{1}{2^n} < \varepsilon$ ▨

For $n \in \mathbb{N}$, let $s^{n,1}, s^{n,2}, \dots, s^{n,n}$ be all finite sequences of length n

let $\hat{s} = (s^{1,1}, s^{1,2}, s^{2,1}, s^{2,2}, s^{2,3}, s^{2,4}, \dots)$ then for any $s \in \Sigma$ and $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $d(\sigma^n(\hat{s}), s) < \varepsilon$

(Recall: A dynamical system $\varphi(t,x)$ is transitive on X if $\forall\, U, V \subseteq X$, U, V-open, $\exists\, t$ such that $\varphi(t,U) \cap V \neq \emptyset$)

↳ so dense ⇒ transitive

so $\forall\, s,t \in \Sigma$, $s \neq t$, $\exists n \in \mathbb{N}$ such that $d(\sigma^n(s), \sigma^n(t)) \ge 1$

Definition: A dynamical system $\varphi(t,x)$ exhibits sensitive dependence on initial conditions if $\exists \alpha > 0$ such that $\forall x, y \in X$ $\exists t > 0$ such that $d(\varphi(t,x), \varphi(t,y)) \ge \alpha$

Definition: A compact invariant set Λ is chaotic if:

i) the system exhibits sensitive dependence on initial conditions on Λ

ii) Λ is transitive

iii) (optional) $\exists$ infinitely many periodic orbits dense in Λ

↳ A dynamical system is chaotic if it contains a chaotic invariant set

Example: We just showed the logistic equation is chaotic

Theorem: conjugacy preserves chaos

i.e.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

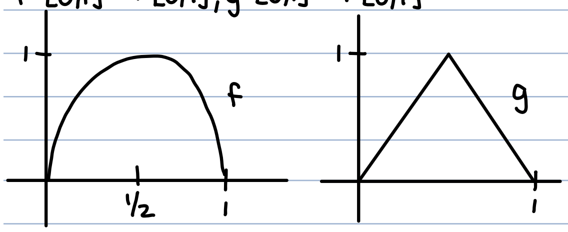
in fact even if h is n-to-1 (n finite), it preserves chaos

↳ with this h, we call it semi-conjugacy

what if $r = 4$ i.e. $f(x) = 4x(1-x)$?

let $g(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} < x \le 1 \end{cases}$

$f: [0,1] \to [0,1]$, $g: [0,1] \to [0,1]$



Theorem: $h(x) = \frac{1}{2}(1 - \cos(2\pi x))$ is a semi-conjugacy from g to f

April 28th

$f: \mathbb{R}^n \to \mathbb{R}^n$

$x_{n+1} = f(x_n) = f^{n+1}(x_0)$

we might want to look at $f^n(x_0 + \delta_n) - f^n(x_0) = s_n$

Suppose $f:\mathbb{R}\to\mathbb{R}$, then $f^n(x_0+\delta_0)-f^n(x_0)\approx (f^n)'(x_0)\delta_0+O(\delta_0^2)$

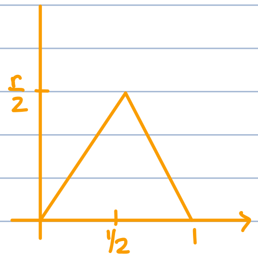
so $\frac{|\delta_n|}{|\delta_0|}\approx|(f^n)'(x_0)|=\prod_{i=0}^{n-1}|f'(x_i)|$ $\quad\left((f(f(x)))'=f'(f(x))\cdot f'(x),\ (f(f(f(x))))'=f'(\underbrace{f(f(x))}_{x_2})f'(\underbrace{f(x)}_{x_1})f'(x)\right)$

$\ln\left|\frac{\delta_n}{\delta_0}\right|=\sum_{i=0}^{n-1}\ln|f'(x_i)|$

on average, the expansion is $\frac{1}{n}\sum_{i=0}^{\infty}\ln|f'(x_i)|$

Taking the limit we get $\lambda(x_0)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|f'(x_i)|$ ← Liapunov exponent

Example: $g(x)=\begin{cases} rx, & 0\le x<1/2 \\ r-rx, & 1/2\le x\le 1\end{cases}$



note that $|g'(x)|=r$

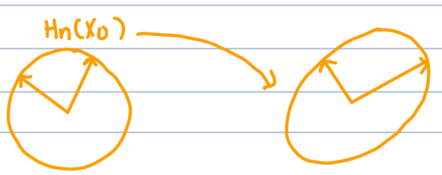
$\lambda=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln r=\ln r$

Given δ_0, we can estimate $\frac{|\delta_n|}{|\delta_0|}\approx\frac{|Df^n(x_0)\cdot\delta_0|}{|\delta_0|}=\frac{\left|\prod_{i=0}^{n-1}Df(x_i)\,\delta_0\right|}{|\delta_0|}$

$|x|=\sqrt{(x,x)}$, $|Ax|=\sqrt{(Ax,Ax)}=\sqrt{x^TA^TAx}$

take $|\delta_0|=1$

$\frac{1}{n}\ln|\delta_n|\approx\frac{1}{n}\ln|Df^n(x_0)\delta_0|=\frac{1}{2n}\ln|\overbrace{\delta_0^TH_n(x_0)\delta_0}^{(Df^n(x_0)\delta_0,\,Df^n(x_0)\delta_0)}|$, where $H_n(x_0)=(Df^n(x_0))^TDf^n(x_0)$



$\exists\, n$ directions $e_1,\dots,e_n$ s.t. $\exists\lim_{n\to\infty}\frac{1}{2n}\ln|e_i^TH_n(x_0)e_i|=\lambda_i$

$\dot{x}=f(x)$

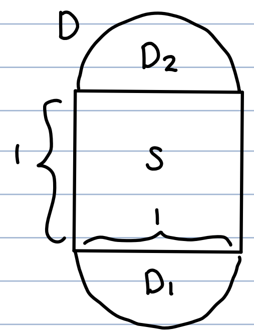
$\dot{\xi}=Df(x(t))\cdot\xi$

If $\varphi(t,x_0)$ is the fundamental matrix, we look at $\frac{|\varphi(t,x_0)e|}{|e|}$

Liapunov exponent → $\lambda(x_0,e)=\lim_{t\to\infty}\frac{1}{t}\frac{|\varphi(t,x_0)e|}{|e|}$

Example: Smale's horseshoe map

May 1st

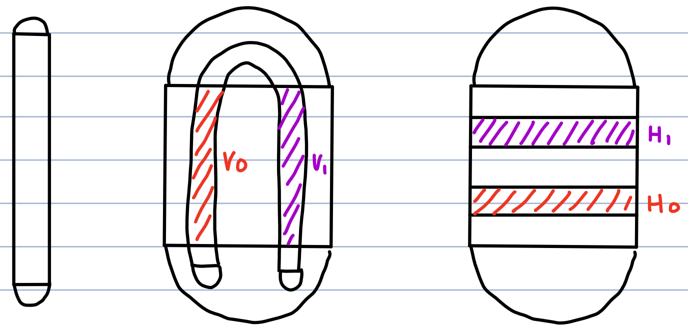


$F: D \to D$

shrink linearly in horizontal direction by $\delta < 1/2$

expand in vertical direction by $1/\delta$

$\exists!$ equilibrium point in D_1 attracting all orbits in D_1



Note that $F^{-1}(S)$ is $H_0 \cup H_1$

$F(H_0)=V_0$, $F(H_1)=V_1$

image of a horizontal line segment in $F^{-1}(S)$ is a horizontal line segment shrunk by δ

image of a vertical line segment in $F^{-1}(S)$ is a vertical line segment expanded by $1/\delta$

to understand the dynamics, note that $\forall x \in D_1 \cup D_2$, $F^n(x) \longrightarrow x^* \in D_1$, x^*-equilibrium point

so the positively invariant set in S consists of points that always stay in S under action of F, that is

$\Lambda_+ = \{x \in S : F^n(x) \in S \ \forall n \in \mathbb{N}\}$

note that if $F(x) \in S$, then $x \in H_0 \cup H_1$ so if $F^2(x) \in S$, then $F(x) \in H_0 \cup H_1 \Rightarrow x \in F^{-1}(H_0 \cup H_1)$

so if $F^{n+1}(x) \in S$, then $x \in F^{-n}(H_0 \cup H_1)$

Hence $\Lambda_+ = \bigcap_{n=0}^{\infty} F^{-n}(H_0 \cup H_1)$

we can also look at the negatively invariant set: $\Lambda_- = \{x \in S : F^{-n}(x) \in S \ \forall n \in \mathbb{N}\}$

if $F^{-1}(x) \in S \Rightarrow x \in V_0 \cup V_1$

similarly, $F^{-2}(x) \in S \Rightarrow F^{-1}(x) \in V_0 \cup V_1 \Rightarrow x \in F(V_0 \cup V_1)$

if $F^{-n-1}(x) \in S \Rightarrow x \in F^n(V_0 \cup V_1)$ so $\Lambda_- = \bigcap_{n=0}^{\infty} F^n(V_0 \cup V_1)$

Thus we get the invariant set $\Lambda = \Lambda_+ \cap \Lambda_-$

To each $x \in \Lambda$, we associate a bi-infinite sequence $(\dots s_{-2}, s_{-1}, s_0, s_1, s_2, \dots)$ of 0s and 1s

$s_k = i$ if $F^k(x) \in H_i$, $i=0,1$, $k \in \mathbb{Z}$.

This gives a map $h: \Lambda \longrightarrow \Sigma_2 \leftarrow$ space of bi-infinite sequences

$\Sigma_2 = \{s : \mathbb{Z} \to \{0,1\}\}$

$d(s,t) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$

The map $h: \Lambda \to \Sigma_2$ is a homeomorphism

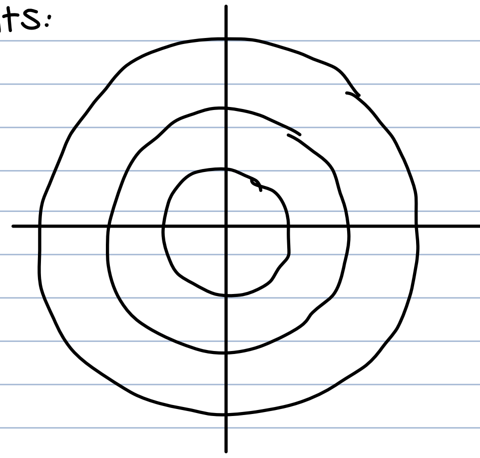
if $\sigma: \Sigma_2 \longrightarrow \Sigma_2$ is the left shift map: $\sigma((\dots, s_{-1}, s_0, s_1, \dots)) = ((\dots, s_0, s_1, s_2, \dots))$, then h is a conjugacy from $F|_\Lambda$ to σ

May 3rd

$\dot{r} = \sin(\pi/r)$
$\dot{\theta} = r$

1) periodic orbits: $\dot{r}=0 \Rightarrow \sin \pi/2 = 0 \Rightarrow \pi/r = \pi n,\ n \in \mathbb{N}\setminus\{0\}$
$r = 1/n,\ n=1,2,\dots$
so, periodic orbits are: $(1/n,\ \theta_0 + t/n)$
$\frac{d}{dr}(\sin\frac{\pi}{r}) = -\frac{\pi}{r^2}\cos\frac{\pi}{r}$

at $r = \frac{1}{2k}$ the sign<0, $r=\frac{1}{2k+1}$ sign>0
consider the annulus bounded by two stable periodic orbits:
$(r_1,\theta_1),(r_2,\theta_2)$
$\dot{\theta} = \sin \pi/r \geq \sin\frac{\pi}{\min(r_1,r_2)} > 0$



let $\theta_1(t) - \theta_2(t) = f(t)$
$\dot{f}(t) = \sin\left(\frac{\pi}{r_1(t)}\right) - \sin\left(\frac{\pi}{r_2(t)}\right)$

$r_1(0) < r_2(0)$
$\dot{r}_1 = \sin\left(\frac{\pi}{r_1}\right) > \sin\left(\frac{\pi}{r_2(t)}\right) = \dot{r}_2(t)$

$\Rightarrow \dot{f}(t) > 0$

$\forall\, U_1, U_2,\ \exists T>0$ s.t. $\varphi(T, U_1) \cap U_2 \neq \emptyset$