Math 206 (Linear Algebra) Notes

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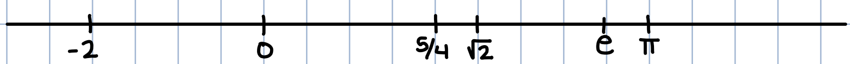
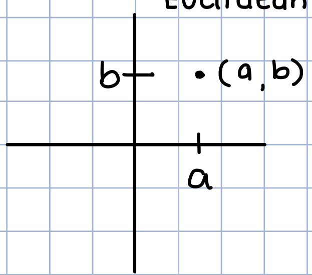
Section 1.1: Vectors in Euclidean Spaces

The set of all real numbers $\mathbb{R}$ is regarded geometrically as the Euclidean line or the Euclidean 1-space. The set of all ordered pairs of real numbers (a,b) is the Euclidean plane/ Euclidean 2-space or $\mathbb{R}^2$. The set of all ordered triples of real numbers (a,b,c) is the Euclidean 3-space, $\mathbb{R}^3$. Generally speaking, the set $\mathbb{R}^n$ of all ordered n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers is the Euclidean n-space.

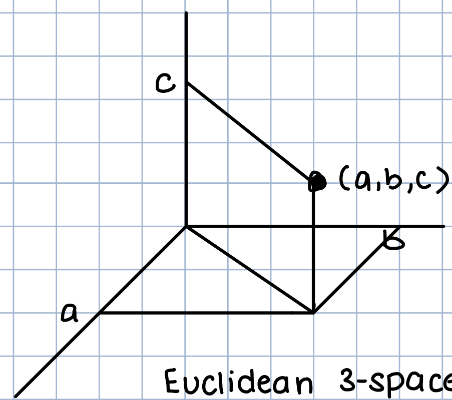
$\mathbb{R}$ = the set of real numbers
↳ Examples: $1, 0, -2, 5/4, \sqrt{2}, \pi, e$
The natural numbers: $0, 1, 2, \dots$
Integers: $\dots, -1, 0, 1, \dots$
Rational numbers can be written as fraction of integers
↳ Examples: $0, 1, -2, 5/4$
Algebraic numbers are roots of finite, non-zero polynomials in one variable with rational coefficients
↳ Example: $\sqrt{2}$ since it is the root of $x^2 - 2 = 0$
↳ Does not include imaginary numbers like $\sqrt{-9}$



Euclidean 1-space (line)



Euclidean 2-space (plane)



Euclidean 3-space (space)

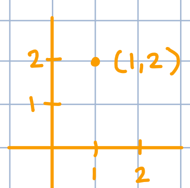
Vectors

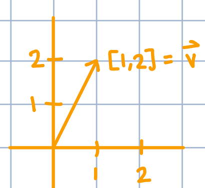
The motion in response to a force depends on the direction in which the force is applied and on the magnitude of the force.

Definition: An arrow pointed in the direction in which the force is acting with the length of the arrow representing the magnitude of the force is called a force vector

Vectors and points are both elements of $\mathbb{R}^N$ but just viewed differently. Mathematically there is no difference. We use parenthesis for points and brackets for vectors

Example:





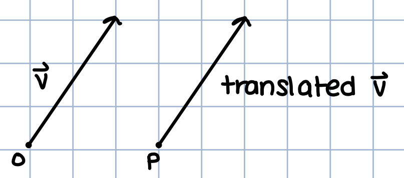
$\vec{v} = [v_1, v_2, \dots, v_n]$, v_i = the ith component

Definition: Two vectors $\vec{v} = [v_1, v_2, \dots, v_n]$ and $w = [w_1, w_2, \dots, w_n]$ are equal if $n=m$ and $v_i = w_i$ for each i

Definition: A vector containing only zeros as components is called a zero vector and is denoted $\vec{0}$

Example: In $\mathbb{R}^2$, $\vec{0} = [0,0]$. In $\mathbb{R}^4$, $\vec{0} = [0,0,0,0]$

If we draw an arrow having the same length and parallel to $\vec{v}$ but starting at a different point P, we refer to the arrows as $\vec{v}$ translated to P

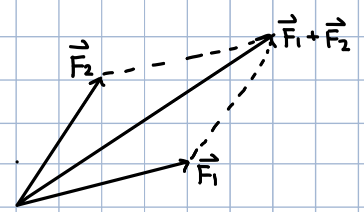


vector algebra

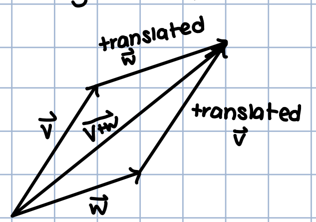
Physics tells us that if two force vectors/forces act on a body at the same time, then the two can be replaced by a single force, called the resultant force, which has the same effect.

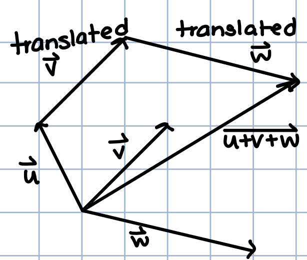
↳ The vector for the resultant force is the diagonal of the parallelogram having the two original vectors as edges.

↳ we consider the resultant vector to be the sum of the two original forces

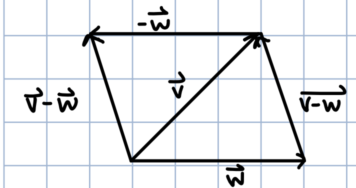


we can also think of connecting a translated vector to the end of the other. This is useful for adding multiple forces.

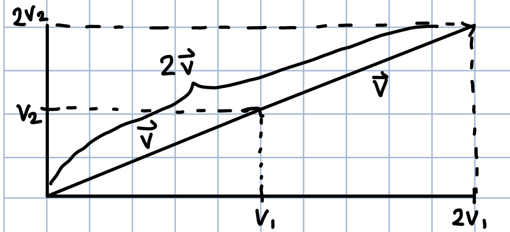




The difference between vectors is represented geometrically by the arrow from the tip of the second to the first or by adding a reversed arrow in the opposite direction.



Multiplying a vector by a scalar makes it that much longer or shorter



Vector algebra in $\mathbb{R}^n$

Let $\vec{v}=[v_1,v_2,\dots,v_n]$ and $w=[w_1,w_2,\dots,w_n]$ be vectors in $\mathbb{R}^n$. The vectors are added and subtracted as follows:

↳ vector addition: $\vec{v}+\vec{w}=[v_1+w_1, v_2+w_2,\dots,v_n+w_n]$

↳ vector subtraction: $\vec{v}-\vec{w}=[v_1-w_1, v_2-w_2,\dots,v_n-w_n]$

↳ If r is any scalar, the vector $\vec{v}$ is multiplied by r as follows:
scalar multiplication: $r\vec{v}=[rv_1, rv_2,\dots, rv_n]$

Example: Let $\vec{v}=[-3,5,-1]$ and $\vec{w}=[4,10,-7]$ in $\mathbb{R}^3$. Compute $5\vec{v}-3\vec{w}$

$5\vec{v}-3\vec{w}=5[-3,5,-1]-3[4,10,-7]=[-15,25,-5]-[12,30,-21]=[-27,-5,16]$

Properties of vector algebra in $\mathbb{R}^n$

Let $\vec{u},\vec{v}$, and $\vec{w}$ be any vectors in $\mathbb{R}^n$ and let r and s be any scalars in $\mathbb{R}$.

Properties of Vector Addition

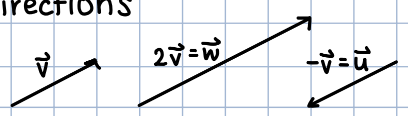
A1: $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$	Associative Law
A2: $\vec{v}+\vec{w}=\vec{w}+\vec{v}$	Commutative Law
A3: $\vec{0}+\vec{v}=\vec{v}$	Additive Identity
A4: $\vec{v}+(-\vec{v})=\vec{0}$	Additive Inverse

Properties Involving Scalar Multiplication

S1: $r(\vec{v}+\vec{w})=r\vec{v}+r\vec{w}$	Distributive Law
S2: $(r+s)\vec{v}=r\vec{v}+s\vec{v}$	Distributive Law
S3: $r(s\vec{v})=(rs)\vec{v}$	Associative Law
S4: $1\vec{v}=\vec{v}$	Preservation of Scale

Definition: Two non zero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are parallel, and we write $\vec{v}\parallel\vec{w}$, if one is a scalar multiple of the other. That is, for $c\in\mathbb{R}$, $\vec{v}=c\vec{w}$ or $\vec{w}=c\vec{v}$

↳ If $\vec{v}=r\vec{w}$ with $r>0$, then $\vec{v}$ and $\vec{w}$ have the same direction. If $r<0$, then they have opposite directions



Example: Are $\vec{v}=[2,1,3,-4]$ and $\vec{w}=[6,3,9,-12]$ parallel?

set $\vec{v}=r\vec{w}$ and try to solve for r.

$2=6r$, $1=3r$, $3=9r$, $-4=12r$

$r=1/3$ is common for all so they are parallel

Definition: Given vectors $\vec{v_1},\vec{v_2},\dots,\vec{v_k}$ in $\mathbb{R}^n$ and scalars $r_1,r_2,\dots,r_k$ in $\mathbb{R}$, the vector $r_1\vec{v_1}+r_2\vec{v_2}+\dots+r_k\vec{v_k}$ is a linear combination of vectors $v_1,v_2,\dots,v_k$ with scalar coefficients $r_1,r_2,\dots,r_k$

Every vector in $\mathbb{R}^2$ can be expressed uniquely as a linear combination of the vectors $[1,0]$ and $[0,1]$. That is, $\vec{v}=[v_1,v_2]=r_1[1,0]+r_2[0,1] \iff r_1=b_1$ and $r_2=b_2$

↳ Definition: we call these the standard basis vectors.

↳ For $\mathbb{R}^3$, these vectors are: $\vec{i}=[1,0,0]$, $\vec{j}=[0,1,0]$, $\vec{k}=[0,0,1]$

we denote the rth standard basis vector as $\vec{e_r}=[0,0,\dots,0,1,0,\dots,0]$. the rth component

Thus $\vec{v}=[v_1,v_2,\dots,v_n]=v_1\vec{e_1}+v_2\vec{e_2}+\dots+v_n\vec{e_n}$

Span of vectors

Definition: The span of a vector is the set of all linear combinations denoted $sp(v_1, v_2, \dots, v_k)$

↳ The span of a single vector are all the scalar multiples of it. If $\vec{v} \neq \vec{0}$ then this is a line. If $\vec{v} = 0$ then the span is $\vec{0}$.

↳ For 2 vectors $\vec{v}$ and $\vec{w}$:

- If $\vec{v} \parallel \vec{w}$ then $sp(\vec{v}, \vec{w})$ is a line
- If $\vec{v}$ is not parallel to $\vec{w}$, $sp(\vec{v}, \vec{w})$ is a plane
- If $\vec{v} = \vec{w}$ the span is $\vec{0}$

Example: Let $\vec{v} = [1,3]$ and $\vec{w} = [-2,5]$ in $\mathbb{R}^2$. Find scalars r and s such that $r\vec{v} + s\vec{w} = [-1,19]$

$r\vec{v} + s\vec{w} = r[1,3] + s[-2,5] = [r-2s, 3r+5s]$ so

$$\begin{matrix} r-2s=-1 \\ 3r+5s=19 \end{matrix} \Rightarrow \begin{matrix} 3r-6s=-3 \\ 3r+5s=19 \end{matrix} \Rightarrow 0+11s=22 \Rightarrow s=2 \Rightarrow r=3$$

we can also use column vectors and row vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \Rightarrow r\begin{bmatrix} 1 \\ 3 \end{bmatrix} + s\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 19 \end{bmatrix} \Rightarrow \begin{matrix} r-2s=-1 \\ 3r+5s=19 \end{matrix} \text{ as before}$$

The transpose of a row vector is the corresponding column vector. similarly the transpose of a column vector is the corresponding row vector, denoted $\vec{v}^T$

↳ $(\vec{v}^T)^T = \vec{v}$

Example: $[-1,4,15,-7]^T = \begin{bmatrix} -1 \\ 4 \\ 15 \\ -7 \end{bmatrix}$ $\qquad \begin{bmatrix} 2 \\ -30 \\ 45 \end{bmatrix}^T = [2,-30,45]$

Example: Is $[1,2,3]$ a linear combination of $[0,1,2]$ and $[2,2,2]$?

$[1,2,3] = r_1[0,1,2] + r_2[2,2,2]$?

$= [2r_2, r_1+2r_2, 2r_1+2r_2]$

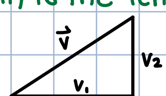
$2r_2 = 1$, $r_1 + 2r_2 = 2$, $2r_1 + 2r_2 = 3 \Rightarrow r_2 = 1/2$, $r_1 = 1$ so yes

Section 1.2: The Norm and Dot Product

The Magnitude of a vector

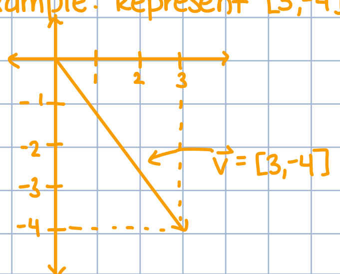
Definition: The magnitude of $\vec{v} = [v_1, v_2]$, denoted $\|\vec{v}\|$, is the length of the vector.

↳ By the pythagorean theorem: $\|v\| = \sqrt{v_1^2 + v_2^2}$



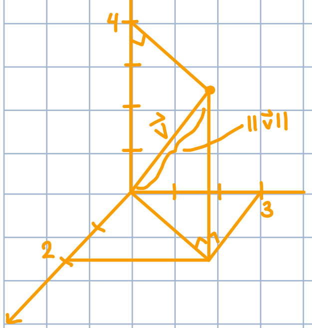
Let $\vec{v} = [v_1, v_2, \dots, v_n]$ be a vector in $\mathbb{R}^n$. The norm (or magnitude) of $\vec{v}$ is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

Example: Represent $[3,-4]$ geometrically and find its magnitude.



$\|v\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

Example: Represent [2,3,4] geometrically and find its magnitude.



$\|\vec{v}\| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$

Example: Find the magnitude of $\vec{v} = [-2,1,3,-1,4,2,1]$

$\|\vec{v}\| = \sqrt{(-2)^2+1^2+3^2+(-1)^2+4^2+2^2+1^2} = \sqrt{36} = 6$

Properties of the norm in $\mathbb{R}^n$

For all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and for all scalars r,

↳ $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$ — Positivity

↳ $\|r\vec{v}\| = |r|\,\|\vec{v}\|$ — Homogeneity

↳ $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ — Triangle inequality

Unit vectors

Definition: A vector in $\mathbb{R}^n$ is a unit vector if it has magnitude 1

↳ Given any nonzero vector $\vec{v}$ in $\mathbb{R}^{2n}$, a unit vector having the same direction as $\vec{v}$ is given by $\frac{\vec{v}}{\|\vec{v}\|}$

The set of all unit vectors in $\mathbb{R}^1$ is a line, $\mathbb{R}^2$ is a circle, $\mathbb{R}^3$ is a sphere

Example: Find a unit vector having the same direction as $\vec{v} = [2,1,-3]$ and find a vector of magnitude 3 having direction opposite to $\vec{v}$.

$\|\vec{v}\| = \sqrt{2^2+1^2+(-3)^2} = \sqrt{14}$

$\vec{u} = \frac{1}{\sqrt{14}}[2,1,-3]$ is a unit vector with the same direction

$-3\vec{u} = \left(\frac{-3}{\sqrt{14}}\right)[2,1,-3]$ is the other required vector

All standard basis vectors are unit vectors so they are also called unit coordinate vectors.

The Dot Product

Definition: The dot product of vectors $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ in $\mathbb{R}^n$ is the scalar given by $\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n$

↳ The dot product is sometimes called the linear product or the scalar product. Geometrically, the dot product of two vectors is equal to the product of their magnitudes with the cosine of the angle between them i.e. $\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$

Note: $\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 + \ldots + v_n^2 = \|\vec{v}\|^2$

Definition: The angle between nonzero vectors $\vec{v}$ and $\vec{w}$ is: $\arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$

The Schwarz Inequality: $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

Example: Find the angle θ between $[1,2,0,2]$ and $[-3,1,1,5]$

$$\cos\theta = \frac{[1,2,0,2]\cdot[-3,1,1,5]}{\sqrt{1^2+2^2+0^2+2^2}\sqrt{(-3)^2+1^2+1^2+5^2}} = \frac{9}{(3)(6)} = \frac{1}{2} \Rightarrow \theta = 60°$$

Example: $[1,2,3]\cdot[-1,0,4] = -1+0+12 = 11$

Example: Find the angle between $[\sqrt{3},1,0]$ and $[0,\sqrt{2},0]$

$$\cos\theta = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Properties of the Dot Product in $\mathbb{R}^n$

Let $\vec{u},\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let r be any scalar in $\mathbb{R}$. The following properties hold:

D1: $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$	Commutative Law
D2: $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$	Distributive Law
D3: $r(\vec{v}\cdot\vec{w}) = (r\vec{v})\cdot\vec{w} = \vec{v}\cdot(r\vec{w})$	Homogeneity
D4: $\vec{v}\cdot\vec{v}\geq 0$ and $\vec{v}\cdot\vec{v}=0$ if and only if $\vec{v}=\vec{0}$	Positivity

Definition: Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are perpendicular or orthogonal, and we write $\vec{v}\perp\vec{w}$, if $\vec{v}\cdot\vec{w}=0$

↳ Note: If $\vec{v}=\vec{0}$, $\vec{v}\cdot\vec{w}=0$. If $\vec{v}\neq\vec{0}, \vec{w}\neq\vec{0} \Rightarrow \theta = \pi/2$

Example: Determine if $\vec{v}=[4,1,-2,1]$ and $\vec{w}=[3,-4,2,-4]$ are perpendicular.

$\vec{v}\cdot\vec{w} = (4)(3)+(1)(-4)+(-2)(2)+(1)(-4) = 0$ so yes

Proof of Schwarz Inequality: If $\vec{v}=\vec{0}$ or $\vec{w}=\vec{0}$ then both sides are 0. If $\vec{v}\neq 0, \vec{w}\neq\vec{0}$, let

$\vec{z} = \vec{v} - \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\cdot\vec{w}$ (note $\vec{z}\cdot\vec{w}=0$ since $\left(\vec{v}-\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\cdot\vec{w}\right)\vec{w} = \vec{v}\cdot\vec{w} - \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}\cdot\vec{w} = 0$)

$$\|\vec{v}\|^2 = \|z\|^2 + \left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)^2\|\vec{w}\|^2 = \|z\|^2 + \frac{(\vec{v}\cdot\vec{w})^2}{\|w\|^2} \geq \frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{w}\|^2}$$

$$\|v\|^2\|w\|^2 \geq (\vec{v}\cdot\vec{w})^2 \Rightarrow \|\vec{v}\cdot\vec{w}\| \leq \|v\|\|w\|$$

The Triangle Inequality: $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$

Proof: $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = (\vec{v}\cdot\vec{v})+2(\vec{v}\cdot\vec{w})+(\vec{w}\cdot\vec{w}) \leq (\vec{v}\cdot\vec{v})+2\|\vec{v}\|\|\vec{w}\|+(\vec{w}\cdot\vec{w})$
$= \|v\|^2+2\|\vec{v}\|\|\vec{w}\|+\|\vec{w}\|^2 = (\|\vec{v}\|+\|\vec{w}\|)^2 \Rightarrow \|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$

Section 1.3: Matrices and their Algebra

As seen in section 1.1, we can write the linear system: $\begin{aligned} x_1+2x_2 &= -1 \\ 3x_1+5x_2 &= 19 \end{aligned}$

in the unknowns x_1, x_2 as a column vector equation: $x_1\begin{bmatrix}1\\3\end{bmatrix}+x_2\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}-1\\19\end{bmatrix}$

we can abbreviate this as: $\underset{A}{\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix}}\underset{X}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}=\underset{B}{\begin{bmatrix}-1\\19\end{bmatrix}}$

Ax is equal to a linear combination of the column vectors of A: $\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\3\end{bmatrix}+x_2\begin{bmatrix}-2\\5\end{bmatrix}$

The Notion of a Matrix

Definition: A matrix is an ordered rectangular array of numbers, usually enclosed in parenthesis or square brackets. An $m \times n$ matrix has m rows and n columns.

Examples:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \; 2\times2 \qquad B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & -6 \\ -3 & -1 & -1 \end{bmatrix} \; 4\times3 \qquad C = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix} \; 2\times3$$

Definition: An $n \times n$ matrix is called a square matrix

An $1 \times n$ matrix is a row vector with n components and an $m \times 1$ matrix is a column vector with m components. The rows of a matrix are its row vectors and the columns are its column vectors.

a_{ij} = the matrix entry in row i and column j

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Matrix Multiplication

The product of the matrix A and column vector x is the linear combination of the column vectors of A having the components of x as coefficients.

For $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $A = [a_{ij}]$, $Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$

Examples:

$$\begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & -7 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 21 \\ -34 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\cdot0 + 0\cdot1 + \sqrt{2}\cdot4 \\ 2\cdot0 + (-1)\cdot1 + 3\cdot4 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ 11 \end{bmatrix}$$

Notes:

1) components of Ax are dot products of rows of A with x: $Ax = \begin{bmatrix} (1\text{st row})\cdot x \\ \vdots \\ (m\text{th row})\cdot x \end{bmatrix}$

2) Ax = linear combination of columns of A with coefficients x_i:

$Ax = x_1(1\text{st column}) + \dots + x_m(n\text{th column})$

3) Systems of linear equations can be written as $Ax = B$

Let $A = [a_{ik}]$ be an $m \times n$ matrix and let $B = [b_{kj}]$ be an $n \times s$ matrix. The matrix product AB is the $m \times s$ matrix $C = [c_{ij}]$ where c_{ij} is the dot product of the ith row vector of A and jth column vector of B: $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Example: Let A be a 2×3 matrix and B be a 3×5 matrix. What are the sizes of AB and BA?
AB is a 2×5 matrix, BA is not defined

Example: $\begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3+2\sqrt{2} \\ 4 & 12 \end{bmatrix}$

Example: $\begin{bmatrix} -2 & 3 & 2 \\ 4 & 6 & -2 \end{bmatrix}\begin{bmatrix} 4 & -1 & 2 & 5 \\ 3 & 0 & 1 & 1 \\ -2 & 3 & 5 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 8 & 9 & -13 \\ 38 & -10 & 4 & 32 \end{bmatrix}$

Note: Matrix multiplication is not commutative

Example: $A = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 4 & 10 \\ 10 & 28 \end{bmatrix}$ $BA = \begin{bmatrix} 3 & 5 \\ 15 & 29 \end{bmatrix}$

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

BUT, it is associative: $A(BC) = (AB)C$

Definition: A diagonal matrix is a square matrix where all off-diagonal entries are zero. That is, all entries are zero except possibly on the diagonal $a_{11}, a_{22}, \dots a_{nn}$

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The nxn Identity Matrix

Definition: The identity matrix is an nxn matrix $[a_{ij}]$ such that $a_{ii} = 1$ for $i = 1, \dots, n$ and $a_{ij} = 0$ for $i \neq j$. That is:

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}$$

Note: $AI = A$ and $IB = B$

Example: $\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$

Other Matrix Operations

Matrix Properties:

For matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$

↳ $A + B = C$ if A and B are the same size and $c_{ij} = a_{ij} + b_{ij}$

↳ $rA = B$ if $b_{ij} = ra_{ij}$

↳ $A^T = B$ if $a_{ij} = b_{ji}$ and $a_{ji} = b_{ij}$

Definition: If $A = A^T$ then the matrix A is symmetric

Example: $\begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 1 & -2 & 2 \end{bmatrix}$

Example: $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix}$ is undefined

Note: $A - B = A + (-1)B$

Examples:

1) $2\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{bmatrix} - 3\begin{bmatrix} -1 & 0 & 5 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 & -7 \\ -12 & 10 & 13 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ 5 & 7 \end{bmatrix}$

3) $\begin{bmatrix} 5 & -6 & -2 & 8 \\ -6 & 3 & 1 & 11 \\ -2 & 1 & 0 & 4 \\ 8 & 11 & 4 & -1 \end{bmatrix}$ is symmetric

Properties of Transpose Operation

↳ $(A^T)^T = A$ transpose of a transpose

↳ $(A+B)^T = A^T + B^T$ transpose of a sum

↳ $(AB)^T = B^T A^T$ transpose of a product

Properties of Matrix Algebra

↳ $A + B = B + A$ commutative law of addition

↳ $(A+B) + C = A + (B+C)$ Associative law of addition

↳ $A + 0 = 0 + A = A$ Identity for addition

↳ $r(A+B) = rA + rB$ Left distributive law

↳ $(r+s)A = rA + sA$ Right distributive law

↳ $(rs)A = r(sA)$ Associative law of scalar multiplication

↳ $(rA)B = A(rB) = r(AB)$ Scalars pull through

↳ $A(BC) = (AB)C$ Associative law of matrix multiplication

↳ $IA = A$ and $BI = B$ Identity for matrix multiplication

↳ $A(B+C) = AB + AC$ Left distributive law

↳ $(A+B)C = AC + BC$ Right distributive law

Section 1.4: Solving Systems of Linear Equations

The most general type of linear system has m equations in n unknowns and can be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

In two variables where $a_{i1} \neq 0$ or $a_{i2} \neq 0$, $a_{i1}x_1 + a_{i2}x_2 = b_i$ determines a line and the solution to the system is the intersection of these m lines

m	line solution	single solution	no solution
1		not possible	not possible
2	lines coincide		
3			OR

we can rewrite this system as a single matrix equation $A\vec{x}=\vec{b}$ with $A=[a_{ij}]$ coefficient matrix

Definition: The augmented or partitioned matrix is a shorthand of the system above written

$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = [A|b]$$

Example: $2x_1+x_3=1$
$x_1+2x_2-x_3=0$ $\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right]$

Elementary Row Operations

we determine the solutions to $Ax=b$ by manipulating the augmented matrix using elementary row operations. The row operations change the matrix but do not affect the solution set.

Elementary Row Operations

- (Row Interchange) Interchange the ith and jth row vectors in the matrix $R_i \longleftrightarrow R_j$
- (Row Scaling) Multiply the ith row vector in a matrix by a nonzero scalar $R_i \longrightarrow sR_i$
- (Row Addition) Add to the ith row vector of a matrix s times the jth row $R_i \longrightarrow R_i+sR_j$

Definition: If a matrix B can be obtained from a matrix A through elementary row operations, then A is row equivalent to B

Example: $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & 0 & 3 & 3 \\ 1 & 2 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -4 & 3 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$

$R_1 \to 3R_1$

R_1-2R_2

Theorem: If $[A|b]$ and $[H|c]$ are row equivalent augmented matrices, then the linear systems $Ax=b$ and $Hx=c$ have the same solution sets.

Example: Solve

$$\begin{bmatrix} 0 & 2 & -4 \\ 3 & 6 & 12 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & -4 & 0 \\ 3 & 6 & 12 & -9 \\ 2 & 0 & 4 & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 6 & 12 & -9 \\ 0 & 2 & -4 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \xrightarrow{R_1\to\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & -4 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2\to\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \xrightarrow{R_3\to -2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & -4 & -4 & 6 \end{array}\right] \xrightarrow{R_3\to 4R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -12 & 6 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1+2x_2+4x_3=-3 \\ x_2-2x_3=0 \\ -12x_3=6 \end{array} \Rightarrow \begin{array}{l} x_1=1 \\ x_2=-1 \\ x_3=-1/2 \end{array} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ -1/2 \end{bmatrix}$$

Row Echelon Form

Definition: A matrix is in Row Echelon form if:
1) All rows containing only zeros appear below rows with nonzero entries
2) The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

↳ The first nonzero entry in a row of a row echelon form matrix is the pivot for that row

Examples:

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not row echelon

$\begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is not row echelon

$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in row echelon

$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon

*circles are pivots

If $[A|b]$ is in row echelon form then $Ax=b$ is easily solved by back substitution

Example: Find all solutions to

$$[H|c] = \left[\begin{array}{ccc|c} -5 & -1 & 3 & 3 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 2 & -4 \end{array}\right] \Rightarrow \begin{array}{r} -5x_1 - x_2 + 3x_3 = 3 \\ 3x_2 + 5x_3 = 8 \\ 2x_3 = -4 \end{array} \Rightarrow x_3 = -4/2 = -2 \Rightarrow 3x_2 + 5(-2) = 8 \Rightarrow x_2 = 6 \Rightarrow -5x_1 - 6 + 3(-2) = 3$$
$$\Rightarrow x_1 = -3$$

Example: $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right] \Rightarrow \begin{array}{r} x_2 + x_3 = 3 \\ 2x_3 = 1 \end{array} \Rightarrow \begin{array}{l} x_3 = 1/2 \\ x_2 = 5/2 \end{array}$ x_1 is free

Example: $\left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$ Since the last row means $0x_1 + 0x_2 + 0x_3 - 1$, there are no solutions

Definition: A linear system having no solutions is inconsistent. If A linear system has one or more solutions, the system is consistent

Example:

$$[H|c] = \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} x_1 - 3x_2 + 5x_4 = 4 \\ x_3 + 2x_4 = -7 \\ x_5 = 1 \end{array} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 5s + 4 \\ r \\ -2s - 7 \\ s \\ 1 \end{bmatrix}$$

← general solution

for scalars r, s

↳ x_2 and x_4 are called free variables

Theorem: We can always reduce a matrix to row echelon form using elementary row operations.

↳ The algorithm to do this is Gaussian elimination/reduction

Gaussian Elimination

Steps

1) If the first column of A contains only zeros, cross it off mentally and continue until the left column of the remaining matrix is nonzero or until columns are gone
2) Use row interchange if necessary to obtain a pivot, p, in the top row. For each row below with a nonzero entry, r, in the first column, add $-r/p$ times the top row to it to create zeros
3) Go back to step 1 and repeat with the next columns
   ↳ Note: You can multiply the pivots row by $1/p$ to make calculations simpler

Example:

$$\begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example:

Solve
$$x_2 - 3x_3 = -5$$
$$2x_1 + 3x_2 - x_3 = 7$$
$$4x_1 + 5x_2 - 2x_3 = 10$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 0 & -4 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$$\Rightarrow \vec{x} = [-1, 4, 3]$$

Example: Determine whether $\vec{b} = [1, -7, -4]$ is in the span of $\vec{v} = [2,1,1]$ and $\vec{w} = [1,3,2]$

$\vec{b}$ is $sp(\vec{v}, \vec{w})$ if and only if $\vec{b} = x_1\vec{v} + x_2\vec{w}$ for some scalars x_1 and x_2. Thus this means

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -4 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -5 & 15 \\ 0 & -1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2$, $x_2 = -3$ so $\vec{b} = 2\vec{v} - 3\vec{w}$ which is in $sp(\vec{v}, \vec{w})$

Theorem: Let A be an $m \times n$ matrix. $Ax = b$ is consistent if and only if the vector $\vec{b}$ in $\mathbb{R}^m$ is in the span of the column vectors of A

Definition: A matrix in row echelon form with all pivots equal to 1 and with zeros above an below each is in reduced row echelon form

↳ We use the Gauss-Jordan method to reduce a matrix into reduced row echelon form
   ↳ Theorem: The reduced row echelon form of a matrix A is unique

Theorem: Let $Ax = b$ be a linear system and let $[A|b] \sim [H|c]$, where H is in row echelon form.

1) The system $Ax = b$ is inconsistent if and only if the augmented matrix $[H|c]$ has a row with all entries 0 to the left of the partition and a nonzero entry to the right of the partition. That is, a row $[0, 0, \dots, 0 \mid c]$, $c \neq 0$
2) If $Ax = b$ is consistent and every column of H contains a pivot, the system has a unique solution
3) If $Ax = b$ is consistent and some column of H has no pivot, the system has infinitely many solutions, with as many free variables as there are pivot-free columns in H

Example: $\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 3 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 0 & 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right] \xrightarrow{R_2 \to -3R_1+R_2} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right]$

$\xrightarrow{R_2 \leftrightarrow -R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -3 & 1 \end{array}\right] \xrightarrow{R_3 \to 3R_2+R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right] \Rightarrow$ inconsistent

Example: $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Consistent and 3 columns with no pivots

$2x_4+3x_5=2$, $x_4=1-\frac{3}{2}x_5$, $x_2=1$

x_1, x_3, and x_5 are free variables

$\begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 1-\frac{3}{2}x_5 \\ x_5 \end{bmatrix} \leftarrow$ solution set (general solution)

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is an example of a particular solution

Definition: Any matrix that can be obtained from an identity matrix by means of one elementary row operation is an elementary matrix

Theorem: Let A be an $m \times n$ matrix and let E be an $m \times n$ elementary matrix. Multiplication of A on the left by E effects the same elementary row operation on A that was performed on the identity matrix to obtain E.

Example: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}$

Section 1.5 Inverses of Square Matrices

A system of n unknowns $x_1, x_2, \ldots, x_n$ can be expressed in matrix form as $Ax=b$ where A is the $n \times n$ coefficient matrix, x is the $n \times 1$ column vector with ith entry x_i and b is an $n \times 1$ column vector with constant entries. The analogous equation with scalars is $ax=b$.

If $a \neq 0$ we solve it by multiplying by $1/a$ i.e.

$$\underbrace{\left(\tfrac{1}{a}\right)ax=\left(\tfrac{1}{a}\right)b}_{\text{multiplication by } 1/a} \Rightarrow \underbrace{\left[\left(\tfrac{1}{a}\right)a\right]x=\left(\tfrac{1}{a}\right)b}_{\text{Associativity of multiplication}} \Rightarrow \underbrace{1x=\left(\tfrac{1}{a}\right)b}_{\text{property of } 1/a} \Rightarrow \underbrace{x=\left(\tfrac{1}{a}\right)b}_{\text{property of } 1}$$

Similarly, we must find a $n \times n$ matrix C such that $CA=I$ so that we have

$C(Ax)=Cb \Rightarrow (CA)x=Cb \Rightarrow Ix=Cb \Rightarrow x=Cb$

Using back substitution we have $x=Cb$ so $Ax=A(cb)=(AC)b \Rightarrow AC=I$

Theorem: Let A be an $n \times n$ matrix. If C and D are matrices such that $AC=DA=I$, then $C=D$. In particular if $AC=CA=I$ then C is unique

Proof: Let C and D be matrices such that $AC=DA=I$. We have $D(AC)=(DA)C \Rightarrow D(AC)=DI=D$ and $(DA)C=IC=C$ so $C=D$. Now let $AC=CA=I$ and suppose $AD=DA=I$. Then $AC=I=DA \Rightarrow D=C$ ▨

Definition: An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix C such that $CA = AC = I$ where I is the $n \times n$ identity matrix. The matrix C is the inverse of A, denoted A^{-1}.
↳ If A is not invertible then it is singular

Theorem: Every elementary matrix is invertible

Theorem: Let A and B be invertible $n \times n$ matrices. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Lemma: Let A be an $n \times n$ matrix. The linear system $Ax = b$ has a solution for every choice of column vector $b \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity.

Theorem: Let A and C be $n \times n$ matrices. Then $CA = I$ if and only if $AC = I$

Computation of A^{-1}

To find A^{-1}, if it exists, proceed as follows:
1) Form augmented matrix $[A | I]$
2) Apply Gauss-Jordan method to attempt to reduce $[A|I]$ to $[I|C]$. If the reduction can be carried out, then $A^{-1} = C$. Otherwise A^{-1} does not exist

Theorem: The following conditions for an $n \times n$ matrix A are equivalent:
i) A is invertible
ii) A is row equivalent to the identity matrix I
iii) The system $Ax = b$ has a solution for each n-component column vector b
iv) A can be expressed as a product of elementary matrices
v) The span of the column vectors of A is $\mathbb{R}^n$

Example: Solve $\begin{matrix} 2x + 9y = -5 \\ x + 4y = 7 \end{matrix}$ and compute the inverse of the coefficient matrix

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 9 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -4 & 9 \\ 0 & 1 & 1 & -2 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 83 \\ -19 \end{bmatrix}$$

Example: Express $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ as a product of elementary matrices

we apply the steps above to the identity matrix

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

To express an invertible matrix as a product of elementary matrices, write in left to right the inverses of the elementary matrices corresponding to the row operations that reduce A to I

Section 1.6 Homogeneous Systems, Subspaces, and Bases

Definition: A linear system $Ax = b$ is homogeneous if $b = 0$
↳ A homogeneous linear system $Ax = 0$ is always consistent because $x = \vec{0}$ is a solution
↳ The zero vector is called the trivial solution. All other solutions are nontrivial

Theorem: Let $Ax=0$ be a homogeneous linear system. If h_1 and h_2 are solutions of $Ax=0$, then so is the linear combination rh_1+sh_2 for any scalars r and s

Proof: Let h_1 and h_2 be solutions of $Ax=0$, so $Ah_1=0$ and $Ah_2=0$
$A(rh_1+sh_2)=A(rh_1)+A(sh_2)=r(Ah_1)+s(Ah_2)=r0+s0=0$ as needed.

In fact every linear combination of solutions of a homogeneous system $Ax=0$ is again a solution of the system

Definition: A subset W of $\mathbb{R}^n$ is closed under vector addition if for all $\vec{u},\vec{v}\in W$ the sum $\vec{u}$ is in W

Definition: If $r\vec{v}\in W$ for all $\vec{v}\in W$ and scalars r, then W is closed under scalar multiplication.

Definition: A nonempty subset $W\subseteq\mathbb{R}^n$ that is closed under both vector addition and scalar multiplication is a subspace of $\mathbb{R}^n$

↳ The solution set of every homogeneous system with n unknowns is a subspace of $\mathbb{R}^n$

Example: Show that $W=\{[x,2x]\mid x\in\mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

Of course W is nonempty of $\mathbb{R}^2$. Let $\vec{u},\vec{v}\in W$ so $\vec{u}=[a,2a]$ and $\vec{v}=[b,2b]$ then $\vec{u}+\vec{v}=[a,2a]+[b,2b]=[a+b,2(a+b)]$ which is in W. Also $c\vec{u}=c[a,2a]=[ca,2(ca)]$ is also in W so W is a subspace of $\mathbb{R}^n$

Theorem: Let $W=sp(\vec{w_1},\vec{w_2},\dots,\vec{w_k})$ be the span of $k>0$ vectors in $\mathbb{R}^n$. Then W is a subspace of $\mathbb{R}^n$

Proof: Let $\vec{u}=r_1\vec{w_1}+\dots+r_k\vec{w_k}$ and $\vec{v}=s_1\vec{w_1}+\dots+s_k\vec{w_k}$ their sum $\vec{u}+\vec{v}=(r_1+s_1)\vec{w_1}+\dots+(r_k+s_k)\vec{w_k}$ which is in W also $c\vec{u}=(cr_1)\vec{w_1}+\dots+(cr_k)\vec{w_k}$ since $k>0$ W is nonempty as needed.

Definition: We say the vectors $\vec{w_1},\vec{w_2},\dots,\vec{w_k}$ span or generate the subspace $sp(\vec{w_1},\vec{w_2},\dots,\vec{w_k})$ of $\mathbb{R}^n$

Example: Express the solution set of

$$\begin{aligned} x_1-2x_2+x_3-x_4&=0\\ 2x_1-3x_2+4x_3-3x_4&=0\\ 3x_1-5x_2+5x_3-4x_4&=0\\ -x_1+x_2-3x_3+2x_4&=0\end{aligned} \sim \left[\begin{array}{cccc|c}1&-2&1&-1&0\\2&-3&4&-3&0\\3&-5&5&-4&0\\-1&1&-3&2&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&-2&1&-1&0\\0&1&2&-1&0\\0&1&2&-1&0\\0&-1&-2&1&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&5&-3&0\\0&1&2&-1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]$$

$$X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-5r+3s\\-2r+s\\r\\s\end{bmatrix}=r\begin{bmatrix}-5\\-2\\1\\0\end{bmatrix}+s\begin{bmatrix}3\\1\\0\\1\end{bmatrix}$$ so the solution set is $$sp\left(\begin{bmatrix}-5\\-2\\1\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\\1\end{bmatrix}\right)$$

Definition: Given an $m\times n$ matrix, there are three natural subspaces of $\mathbb{R}^m$ or $\mathbb{R}^n$

1) The span of the row vectors of A is the row space of A and a subspace of $\mathbb{R}^n$
2) The span of column vectors of A is the column space of A and is a subspace of $\mathbb{R}^m$
3) The solution set of $Ax=0$ is the nullspace of A and is a subspace of $\mathbb{R}^n$ (also called the kernel)

Example: For $A=\begin{bmatrix}1&0&3\\0&1&-1\end{bmatrix}$ the rowspace is $sp([1,0,3],[0,1,-1])$ in $\mathbb{R}^3$

the column space $sp\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}3\\-1\end{bmatrix}\right)$ in $\mathbb{R}^2$. The nullspace is $sp\left(\begin{bmatrix}-3\\1\\1\end{bmatrix}\right)$ in $\mathbb{R}^3$

Theorem: A linear space $Ax=b$ has a solution if and only if b is in the column space of A

Let W be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w_1}, \vec{w_2}, \dots, \vec{w_k}\}$ of W is a basis for W if every vector in W can be expressed uniquely as a linear combination of $\vec{w_1}, \vec{w_2}, \dots, \vec{w_k}$. That is, $x \in W$, $x = \lambda_1 w_1 + \dots + \lambda_k w_k$

↳ If $\{\vec{w_1}, \vec{w_2}, \dots, \vec{w_k}\}$ is a basis for W, then we have $W = sp(\vec{w_1}, \vec{w_2}, \dots, \vec{w_k})$

↳ $\{e_1, e_2, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$

Recall: Every vector in $\mathbb{R}^n$ can be written as a unique combination of $\{e_1, \dots, e_n\}$.

$v = [v_1, \dots, v_n] = v_1 e_1 + \dots + v_n e_n$

Theorem: The set $\{\vec{w_1}, \vec{w_2}, \dots, \vec{w_k}\}$ is a basis for $W \subseteq \mathbb{R}^n$ if and only if

1) $W = sp(w_1, \dots, w_k)$ and
2) If $r_1 \vec{w_1} + \dots + r_k \vec{w_k} = 0$ then $r_1 = r_2 = \dots = r_k = 0$

Theorem: Let A be an $n \times n$ matrix. The following are equivalent

1) The linear system $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$
2) The matrix A is row equivalent to the identity matrix
3) The matrix A is invertible
4) The column vectors of A form a basis for $\mathbb{R}^n$

Example: Determine whether the vectors $v_1 = [1,1,3]$ $v_2 = [3,0,4]$ and $v_3 = [1,4,-1]$ form a basis for $\mathbb{R}^3$.
We must show that the matrix of $[v_1, v_2, v_3]$ is row equivalent to I

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -9 \end{bmatrix} \sim I$$

thus $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$

Definition: A linear system having the same number n of equations as unknowns is called a square system.

Definition: A square matrix with zero entries below the main diagonal is called upper triangular

Example: $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 9 \end{bmatrix}$ is upper triangular

Theorem: Let A be an $m \times n$ matrix. The following are equivalent

i) Each consistent system $Ax = b$ has a unique solution
ii) The reduced row-echelon form of A consists of the $n \times n$ identity matrix followed by $m-n$ row of zeros
   ↳ Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $m = 4$, $n = 3$, $m - n = 1$
iii) The column vectors of A form a basis for the column space of A

Example: Determine whether the vectors $w_1 = [1,2,3,-1]$, $w_2 = [-2,-3,-5,1]$ and $w_3 = [-1,-3,-4,2]$ form a basis for the subspace $sp(w_1, w_2, w_3)$ in $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & -3 \\ 3 & -5 & -4 \\ -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so no

Definition: A linear system having an infinite number of solutions is called underdetermined.

Corollary: If a linear system $Ax=b$ is consistent and has fewer equations than unknowns, then it has an infinite number of solutions

Proof: If $m<n$ then the reduced row echelon form of A cannot contain the $n\times n$ identity matrix so it cannot be the unique case. Since we assume the system is consistent, there must be an infinite number of solutions $\boxtimes$

Corollary:
↳ A homogeneous linear system $Ax=0$ having fewer equations than unknowns has a non trivial solution, that is a solution that is not $\vec{0}$
↳ A square homogeneous system $Ax=0$ has a nontrivial solution if and only if A is not row equivalent to the identity matrix of the same size

Theorem: Let $Ax=b$ be a linear system. If p is any particular solution of $Ax=b$ and h is a solution of the corresponding system $Ax=0$, then $p+h$ is a solution of $Ax=b$. Moreover, every solution of $Ax=b$ has the form $p+h$ so that the general solution is $x=p+h$ where $Ah=0$.

Examples of subspaces:
1) In $\mathbb{R}^2$: $\{0\}$, straight lines through origin, $\mathbb{R}^2$
2) In $\mathbb{R}^3$: $\{0\}$, lines through the origin, planes through the origin, $\mathbb{R}^3$

Class

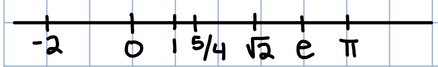
Notes

covering chapters 1-5,7
↳ maybe 6,8,9 if time permits

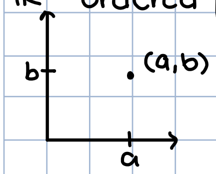
Euclidean Spaces

$\mathbb{R}$ = set of all real numbers
↳ Example: $1, 0, -2, 5/4, \sqrt{2}, \pi, e$
↳ Natural: 1, 0
Integers: 1, 0, -2
Rational: 1, 0, -2, 5/4
Transcendental: π, e
Algebraic: $1, 0, -2, 5/4, \sqrt{2}$
↳ $\sqrt{2}$ is algebraic because the root of $x^2-2=0$ is $\sqrt{2}$
↳ NOT IMAGINARY NUMBERS
↳ Example: $\sqrt{-9}$

Euclidean 1-space (line)



Euclidean 2-space (plane)
$\mathbb{R}^2$ = ordered pairs (a, b) of real numbers

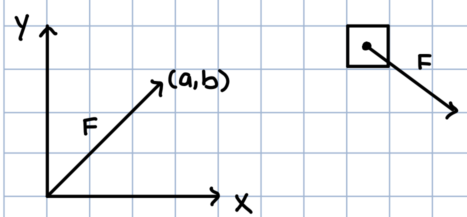


Euclidean 3-space
$\mathbb{R}^3$ = set of (x, y, z)

Euclidean N-space
$\mathbb{R}^N$ = set of N-tuples of real numbers
$(x_1, x_2, \dots, x_N)$

Vectors

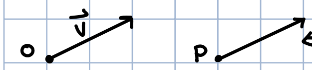
From physics (forces)
Have magnitude and direction



vectors and points are both elements of $\mathbb{R}^N$ but viewed differently
↳ $(1, \sqrt{2}, -1)$ = point $[1, \sqrt{2}, -1]$ = vector
$\vec{v} = [v_1, v_2, \dots, v_N]$, v_i = ith component

If $\vec{v} = [v_1, \ldots, v_N]$ and $\vec{w} = [w_1, \ldots, w_N]$, then $\vec{v} = \vec{w}$ if $N = M$ and $v_i = w_i$ for all i

$\vec{0} = [0, 0, \ldots, 0]$

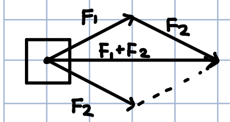


← translation of $\vec{v}$

vector Algebra

From physics (resultant force)

Addition

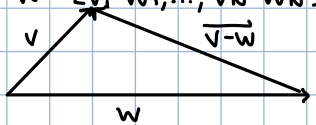


parallelogram rule

In terms of N-tuples: $\vec{v} = [v_1, \ldots, v_N]$, $W = [w_1, \ldots, w_N] \Rightarrow \overrightarrow{v+w} = [v_1 + w_1, v_2 + w_2, \ldots, v_N + w_N]$

Subtraction

$\vec{v} - \vec{w} = [v_1 - w_1, \ldots, v_N - w_N]$

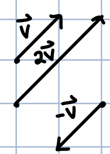


Scalar multiplication

$c \in \mathbb{R}, \vec{v} \in \mathbb{R}^N \Rightarrow c \cdot \vec{v} \in \mathbb{R}^N$

$c \cdot [v_1, v_2, \ldots, v_N] = [cv_1, \ldots, cv_N]$

$\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = c\vec{w}$ or $\vec{w} = c\vec{v}$



Properties

$\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^N$; $r, s \in \mathbb{R}$

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ Associative

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutative

$\vec{0} + \vec{v} = \vec{v}$ additive identity

$\vec{v} + (-\vec{v}) = \vec{0}$ additive inverse

$\left.\begin{array}{l} r \cdot (\vec{v} + \vec{w}) = r\vec{v} + r\vec{w} \\ (r+s)\vec{v} = r\vec{v} + s\vec{v} \end{array}\right\}$ distributive

$r \cdot (s \cdot \vec{v}) = (r \cdot s)\vec{v}$

$1 \cdot \vec{v} = \vec{v}$

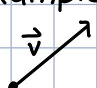
Linear Combinations

$\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^N \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R}$

$\lambda_1 \vec{v}_1 + \ldots + \lambda_k \vec{v}_k \in \mathbb{R}^N$ is called the linear combination of $\vec{v}_1, \ldots, \vec{v}_k$ with coefficients $\lambda_1, \ldots, \lambda_k$

The span of $\vec{v}_1, \ldots, \vec{v}_k = SP(\vec{v}_1, \ldots, \vec{v}_k) =$ set of all linear combinations $= \{\lambda_1 \vec{v}_1 + \ldots + \lambda_k \vec{v}_k : \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$

↳ Example: The span of one vector



$sp(\vec{v}) =$ all scalar multiples of $\vec{v} = \{\lambda \vec{v} : \lambda \in \mathbb{R}\}$

If $\vec{v} \neq \vec{0}$, $sp(\vec{v})$ is a line. If $\vec{v} = \vec{0}$, $sp(\vec{v})$ is $\vec{0}$

↳ Example: The span of two vectors
↳ If $\vec{v} \parallel \vec{w}$ then $sp(\vec{v}, \vec{w}) =$ line
↳ If $\vec{v}, \vec{w}$ are not parallel, then $sp(\vec{v}, \vec{w}) =$ plane

System of Linear Equations

Example: Is $[1,2,3]$ a linear combination of $[0,1,2]$ and $[2,2,2]$?

$[1,2,3] = \lambda_1 [0,1,2] + \lambda_2 [2,2,2] \iff [2\lambda_2, \lambda_1 + 2\lambda_2, 2\lambda_1 + 2\lambda_2] = [1,2,3]$

$2\lambda_2 = 1 \Rightarrow \lambda_2 = 1/2$

$\lambda_1 + 2\lambda_2 = 2 \Rightarrow \lambda_2 = 1$

$\Rightarrow 2\lambda_1 + 2\lambda_2 = 3$ ✓

Thus $[1,2,3] = [0,1,2] + \frac{1}{2}[2,2,2]$

Definition: A row vector is of the form $[x_1, x_2, \dots, x_n]$. A column vector is of the form $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

The standard basis of $\mathbb{R}^N$ is $\vec{e}_1, \dots, \vec{e}_N$ where $\vec{e}_i = [0,0,\dots,0,\underset{\text{ith component}}{1},0,\dots,0]$

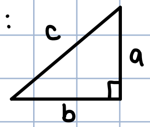
Example: $\mathbb{R}^3$ has a basis of $\vec{e}_1 = [1,0,0]$, $\vec{e}_2 = [0,1,0]$, $\vec{e}_3 = [0,0,1]$

Every vector can be written uniquely as a linear combination of $e_1, \dots, e_N$

$\vec{v} = [v_1, \dots, v_N] = v_1 \vec{e}_1 + \dots + v_N \vec{e}_N$

Norm and Dot Product

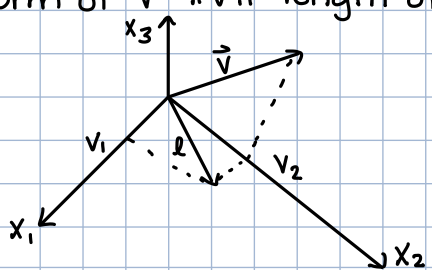
Pythagoras:



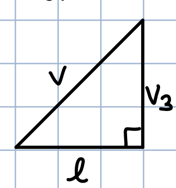
$c = \sqrt{a^2 + b^2}$

$\vec{v} \in \mathbb{R}^2$, $\vec{v} = [v_1, v_2]$

Norm of $\vec{v} = \|\vec{v}\| =$ length of $\vec{v} = \sqrt{v_1^2 + v_2^2} = \sqrt{\sum_{i=1}^{N} (v_i)^2}$ in $\mathbb{R}^3$



$l = \sqrt{v_1^2 + v_2^2}$



$\|v\| = \sqrt{V_3^2 + l^2} = \sqrt{V_1^2 + V_2^2 + V_3^2}$

In $\mathbb{R}^N$, $\|v\| = \sqrt{V_1^2 + V_2^2 + \dots + V_N^2}$

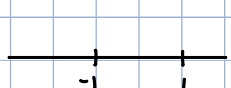
Properties

↳ $\|\vec{v}\| \geq 0$, $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$ (positivity)
↳ $\|r \cdot \vec{v}\| = |r| \cdot \|\vec{v}\|$
↳ $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (triangle inequality)

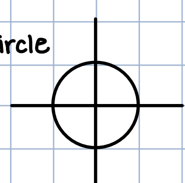
Definition: A unit vector is a vector $\vec{v}$ such that $\|\vec{v}\| = 1$

Definition: A sphere is a set of all unit vectors

In $\mathbb{R}^1$:



In $\mathbb{R}^2$: circle



In $\mathbb{R}^3$: sphere

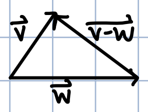
Dot Product (inner/scalar product)

let $\vec{v},\vec{w}\in\mathbb{R}^N$ $\quad \vec{v}\cdot\vec{w}=v_1w_1+v_2w_2+\dots+v_Nw_N=\sum_{i=1}^{N}v_iw_i$

Example: $[1,2,3]\cdot[-1,0,4]=-1+0+12=11\in\mathbb{R}$

Note: $\vec{v}\cdot\vec{v}=v_1^2+\dots+v_N^2=\|\vec{v}\|^2$

Geometric Interpretation



$$\|\vec{v}\|^2+\|\vec{w}\|^2=\|\vec{v}-\vec{w}\|^2+2\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$$
$$v_1^2+\dots+v_N^2+w_1^2+\dots+w_N^2=(v_1-w_1)^2+\dots+(v_N-w_N)^2+2\|\vec{v}\|\|\vec{w}\|\cos\theta$$
$$v_1^2+\dots+v_N^2+w_1^2+\dots+w_N^2-2(v_1w_1+v_2w_2+\dots+v_Nw_N)+2\cos\theta\|\vec{v}\|\|\vec{w}\|$$
$$0=-2(\vec{v}\cdot\vec{w})+2\cos\theta\|\vec{v}\|\|\vec{w}\|$$
$$\cos\theta=\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|\|\vec{v}\|}$$ (if $\vec{v},\vec{w}$ are unit vectors, $\cos\theta=\vec{v}\cdot\vec{w}$)

Example: Find the angle between $[\sqrt{3},1,0]$ and $[0,\sqrt{2},0]$

$\sqrt{3+1}=2 \qquad \cos\theta=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2} \qquad \theta=\frac{\pi}{3}$

$\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v}\cdot\vec{w}=0$

Properties of dot product

↳ $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$
↳ $\vec{v}\cdot(\vec{u}+\vec{w})=\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{w}$
↳ $r\cdot(\vec{v}\cdot\vec{w})=(r\vec{v})\cdot\vec{w}=(r\vec{w})\cdot\vec{v}$
↳ $\vec{v}\cdot\vec{v}\geq 0$, $\vec{v}\cdot\vec{v}=0 \iff \vec{v}=\vec{0}$

Schwartz Inequality: $|\vec{v}\cdot\vec{w}|\leq\|\vec{v}\|\cdot\|\vec{w}\|$

Proof: If $\vec{v}=\vec{0}$ or $\vec{w}=\vec{0}$ then both sides equal 0

If $\vec{v}\neq 0$, $\vec{w}\neq\vec{0}$ then let $z=\vec{v}-\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\cdot\vec{w}$ (note: $\vec{z}\cdot\vec{w}=0$)

$$\left(\vec{v}-\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\cdot\vec{w}\right)\cdot\vec{w}=\vec{v}\cdot\vec{w}-\frac{\vec{v}\cdot\vec{w}}{\cancel{\vec{w}\cdot\vec{w}}}\cancel{\vec{w}\cdot\vec{w}}=0$$

$$\|\vec{v}\|^2=\|z\|^2+\left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)^2\|\vec{w}\|^2=\|z\|^2+\frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{w}\|^2}\geq\frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{w}\|^2}$$

$\|\vec{v}\|^2\|\vec{w}\|^2\geq(\vec{v}\cdot\vec{w})^2 \qquad |\vec{v}\cdot\vec{w}|\leq\|\vec{v}\|\|\vec{w}\|$

Proving the Triangle Inequality: $\|\vec{v}+\vec{w}\|\leq\|\vec{v}\|+\|\vec{w}\|$

Proof: $\|\vec{v}+\vec{w}\|^2=(\vec{v}+\vec{w})(\vec{v}+\vec{w})=\vec{v}\cdot\vec{v}+2\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{w}\leq\vec{v}\cdot\vec{v}+2\|\vec{v}\|\|\vec{w}\|+\vec{w}\cdot\vec{w}\leq(\|\vec{v}\|+\|\vec{w}\|)^2\Rightarrow\|\vec{v}+\vec{w}\|\leq\|\vec{v}\|+\|\vec{w}\|$

Matrix Algebra

Definition: An $N\times M$ matrix is an array of real numbers with N rows and M columns

Example: $A=\begin{bmatrix}1&0&\sqrt{2}\\2&-1&3\end{bmatrix}$ 2×3 matrix $\qquad a_{ij}$ = entry in row i, column j

$a_{11}=1$, $a_{12}=0$, $a_{23}=3$

Multiplication

Let A be an $N\times M$ matrix and $x\in\mathbb{R}^N$ $\quad x=\begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$ then $Ax\in\mathbb{R}^N$

$$Ax=\begin{bmatrix} A_{11}x_1+A_{12}x_2+\dots+A_{1M}x_M \\ \vdots \\ A_{N1}x_1+\dots\dots+A_{NM}x_M \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}=\begin{bmatrix} 1\cdot 0+0\cdot 1+\sqrt{2}\cdot 4 \\ 2\cdot 0+-1\cdot 1+3\cdot 4 \end{bmatrix}=\begin{bmatrix} 4\sqrt{2} \\ 11 \end{bmatrix}$

Note:

↳ components of Ax are dot products of rows of A with x

$$Ax=\begin{bmatrix} (\text{1st row})\cdot x \\ (N^{th}\text{ row})\cdot x \end{bmatrix}$$

↳ Ax = linear combination of columns of A with coefficients x_i

$$Ax=x_1(\text{1st column})+\dots+x_M(M\text{th column})=0\begin{bmatrix} 1 \\ 2 \end{bmatrix}+1\begin{bmatrix} 0 \\ -1 \end{bmatrix}+4\begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

↳ Systems of linear equations can be written as $Ax=B$

$$\begin{cases} x_1+\sqrt{2}x_3=0 \\ 2x_1-x_2+3x_3=1 \end{cases} \qquad \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let A be an $N\times M$ matrix and B be an $M\times L$ matrix, $C=AB$ is an $N\times L$ matrix

The jth column is $A\cdot(j\text{th column of } B)$

$$c_{ij}=(i\text{th row of } A)\cdot(j\text{th column of } B)=\sum_{k=1}^{m} a_{ik}b_{kj}$$

Example: $\begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} \sqrt{2} & 3+2\sqrt{2} \\ 4 & 12 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}$ is not defined

Warning: NOT commutative

Example: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It is associative: $(AB)C=A(BC)$

Definition: An $N\times N$ matrix is a square matrix

A square matrix has main diagonal $a_{11},a_{22},\dots,a_{NN}$

Definition: A diagonal matrix is a square matrix where all off-diagonal entries are zero

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad$ Identity Matrix $\underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{I_N}$ $\qquad I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I \cdot A = A,\ B \cdot I = B$

Addition and Multiplication by Scalar

$A + B = C,\ C_{ij} = a_{ij} + b_{ij}$

$R \cdot A = C,\ C_{ij} = Ra_{ij}$

Transpose

Let A be an $N \times M$ matrix $\Rightarrow A^T$ is an $M \times N$ matrix (switch rows and columns)

If $C = A^T$, $C_{ij} = a_{ji}$

Example: $\begin{bmatrix} 1 & 0 & \sqrt{2} \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ \sqrt{2} & 3 \end{bmatrix}$

Definition: A symmetric matrix A is an $N \times N$ matrix where $A^T = A$ i.e. $a_{ij} = a_{ji}$

Example: $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

Properties

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

System of Linear Equations

$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$\vdots$

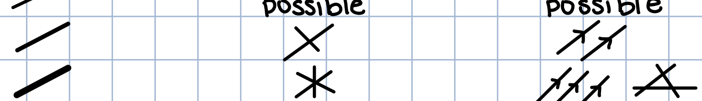
$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{Nn}x_n = b_N$

If $a_{i1} \neq 0$ or $a_{i2} \neq 0$, $a_{i1}x_1 + a_{i2}x_2 = b_i$ determines a line

The solution for the system is the intersection of all these N lines

Example:

N	line solution	single solution	no solution
1		not possible	not possible
2			
3			



Algorithm

$Ax = b$

Augmented matrix $(A|b)$

Example: $2x_1 + x_3 = 1$

$x_1 + 2x_2 - x_3 = 0$

$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right)$

Elementary row operations

R1: interchange two rows

R2: multiply one row by $c \neq 0$

R3: add a multiple of a row to a different row

Example: $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{pmatrix}$

R1 $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$ R2 $\begin{pmatrix} 6 & 0 & 3 & 3 \\ 1 & 2 & -1 & 0 \end{pmatrix}$ R3 $(-2)\times$(2nd row)+(first row) $\begin{pmatrix} 0 & -4 & 3 & 1 \\ 1 & 2 & -1 & 0 \end{pmatrix}$

Note:

↳ each of these are invertible

↳ these do not change the solution set

Theorem: If $(A|b) \sim (H|c)$ (equivalent or can get from one to the other using elementary row operations) then $Ax=b$ and $Hx=c$ have the same solution set

Example: $\begin{pmatrix} 0 & 2 & -4 \\ 3 & 6 & 12 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 0 & 2 & -4 & 0 \\ 3 & 6 & 12 & -9 \\ 2 & 0 & 4 & 0 \end{array}\right)$ $R_1 \leftrightarrow R_2$ $\left(\begin{array}{ccc|c} 3 & 6 & 12 & -9 \\ 0 & 2 & -4 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right)$ $\frac{1}{3}R_1$ $\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 2 & -4 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right)$ $\frac{1}{2}R_2$ $\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right)$

$-2R_1+R_3$ $\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & -4 & -4 & 6 \end{array}\right)$ $4R_2+R_3$ $\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -12 & 6 \end{array}\right)$

equivalent to: $x_1+2x_2+4x_3=-3$ $\quad x_1 = 1$

$x_2 - 2x_3 = 0$ $\quad x_2 = -1$

$-12x_3 = 6$ $\quad x_3 = -1/2$

Row Echelon Form

Definition: Matrix is in row echelon form if in each row, the first non-zero entry appears to the right of the previous row's first non-zero entry. The non-zero entry is called the pivot.

Example: $\begin{bmatrix} 0 & 1 & 5 & 10 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

note: If $[A|b]$ is in row echelon form, then $Ax=b$ is easily solved by back substitution

Example: $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right] \rightarrow \begin{matrix} x_2+x_3=3 \\ 2x_3=1 \end{matrix} \rightarrow x_3 = \frac{1}{2} \Rightarrow x_2 = \frac{5}{2}$, x_1 is free

Theorem: You can always reduce a matrix to row echelon form using elementary row operation.

Gaussian Elimination