Math 134 Spring 2018
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Classwork 24 Name: ______________________________

Work on as many problems as you can together with your group members. Towards the end of lecture your group will be asked to present problems correctly to receive classwork points.

1. Use the Rational Root Test to list all possible rational roots for the given function.

   (a) $f(x) = 3x^4 - 19x^3 + 20x^2 + 5x - 6$
   (b) $f(x) = 4x^4 - x^3 + 2x^2 - 5x - 26$
   (c) $f(x) = 9x^4 - x^3 + 2x^2 - 5x - 39$
   (d) $f(x) = 4x^4 - x^3 + 5x^2 - 2x - 10$
   (e) $f(x) = 9x^4 - x^3 + 2x^2 - 2x - 15$

   **Solution**

   (a) Possible numerators: $\pm 1, \pm 2, \pm 3, \pm 6$
   Possible denominators: $\pm 1, \pm 3$
   Possible rational roots: $\boxed{\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{3}, \pm \frac{2}{3}}$

   (b) Possible numerators: $\pm 1, \pm 2, \pm 13, \pm 26$
   Possible denominators: $\pm 1, \pm 2, \pm 4$
   Possible rational roots: $\boxed{\pm 1, \pm 2, \pm 13, \pm 26, \pm \frac{1}{2}, \pm \frac{13}{2}, \pm \frac{1}{4}, \pm \frac{13}{4}}$

   (c) Possible numerators: $\pm 1, \pm 3, \pm 13, \pm 39$
   Possible denominators: $\pm 1, \pm 3, \pm 9$
   Possible rational roots: $\boxed{\pm 1, \pm 3, \pm 13, \pm 39, \pm \frac{1}{3}, \pm \frac{13}{3}, \pm \frac{1}{9}, \pm \frac{13}{9}}$

   (d) Possible numerators: $\pm 1, \pm 2, \pm 5, \pm 10$
   Possible denominators: $\pm 1, \pm 2, \pm 4$
   Possible rational roots: $\boxed{\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{1}{4}, \pm \frac{5}{4}}$

   (e) Possible numerators: $\pm 1, \pm 3, \pm 5 \pm 15$
   Possible denominators: $\pm 1, \pm 3, \pm 9$
   Possible rational roots: $\boxed{\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{3}, \pm \frac{5}{3}, \pm \frac{1}{9}, \pm \frac{5}{9}}$

$\square$

2. The function $f(x)$ is given. List all possible rational roots. Then use synthetic division to test several possible rational roots in order to identify one actual root.

   (a) $f(x) = x^4 - 6x^3 - 3x^2 + 24x - 4$
   (b) $f(x) = x^4 + 2x^3 - 10x^2 - 18x + 9$
   (c) $f(x) = x^3 - 6x - 4$
   (d) $f(x) = x^3 - 37x + 6$
   (e) $f(x) = x^3 - 5x^2 - 9x + 45$

**Solution** Note: There may be more than one answer to each question.

(a) Possible rational roots: $\pm 1, \pm 2, \pm 4$

$$\begin{array}{r|rrrrr} & 1 & -6 & -3 & 24 & -4 \\ 2 & & 2 & -8 & -22 & 4 \\ \hline & 1 & -4 & -11 & 2 & 0 \end{array}$$

Thus $\boxed{x = 2}$ is a root

(b) Possible rational roots: $\pm 1, \pm 3, \pm 9$

$$\begin{array}{r|rrrrr} & 1 & 2 & -10 & -18 & 9 \\ 3 & & 3 & 15 & 15 & -9 \\ \hline & 1 & 5 & 5 & -3 & 0 \end{array}$$

Thus $\boxed{x = 3}$ is a root

(c) Possible rational roots: $\pm 1, \pm 2, \pm 4$

$$\begin{array}{r|rrrr} & 1 & 0 & -6 & -4 \\ -2 & & -2 & 4 & 4 \\ \hline & 1 & -2 & -2 & 0 \end{array}$$

Thus $\boxed{x = -2}$ is a root

(d) Possible rational roots: $\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} & 1 & 0 & -37 & 6 \\ 6 & & 6 & 36 & -6 \\ \hline & 1 & 6 & -1 & 0 \end{array}$$

Thus $\boxed{x = 6}$ is a root

(e) Possible rational roots: $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

$$\begin{array}{r|rrrr} & 1 & -5 & -9 & 45 \\ 3 & & 3 & -6 & -45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Thus $\boxed{x = 3}$ is a root

$\square$

3. Find an $n$th-degree polynomial function $f(x)$ with real coefficients satisfying the given conditions.

   (a) $n = 3$; 4 and $4i$ are roots of $f$; $f(2) = -40$
   (b) $n = 3$; 2 and $2i$ are roots of $f$; $f(1) = 10$
   (c) $n = 4$; $i$ and $2i$ are roots of $f$; $f(-1) = 130$
   (d) $n = 4$; $i$ and $2i$ are roots of $f$; $f(-1) = 10$
   (e) $n = 4$; $2i$ and $3i$ are roots of $f$; $f(-2) = 104$

**Solution**

(a) We know the roots are $x = 4, 4i, -4i$ so the polynomial is:

$$f(x) = a_n(x-4)(x-4i)(x+4i) = a_n(x-4)(x^2+16)$$

$$\begin{aligned} f(2) = -40 &\Leftrightarrow -40 = a_n(-2)(4+16) \\ &\Leftrightarrow -40 = -40a_n \\ &\Leftrightarrow a_n = 1 \end{aligned}$$

Thus $f(x) = (x-4)(x^2+16) = \boxed{x^3 - 4x^2 + 16x - 64}$

(b) We know the roots are $x = 2, 2i, -2i$ so the polynomial is:

$$f(x) = a_n(x-2)(x-2i)(x+2i) = a_n(x-2)(x+4)$$

$$\begin{aligned} f(1) = 10 &\Leftrightarrow 10 = a_n(1-2)(1+4) \\ &\Leftrightarrow 10 = -5a_n \\ &\Leftrightarrow a_n = -2 \end{aligned}$$

Thus $f(x) = -2(x-2)(x+4) = \boxed{-2x^2 - 4x + 16}$

(c) We know the roots are $x = i, -i, 2i, -2i$ so the polynomial is:

$$f(x) = a_n(x-i)(x+i)(x-2i)(x+2i) = a_n(x^2+1)(x^2+4)$$

$$\begin{aligned} f(-1) = 130 &\Leftrightarrow 130 = a_n(1+1)(1+4) \\ &\Leftrightarrow 130 = 10a_n \\ &\Leftrightarrow a_n = 13 \end{aligned}$$

Thus $f(x) = 13(x^2+1)(x^2+4) = \boxed{13x^4 + 65x^2 + 52}$

(d) We know the roots are $x = i, -i, 2i, -2i$ so the polynomial is:

$$f(x) = a_n(x-i)(x+i)(x-2i)(x+2i) = a_n(x^2+1)(x^2+4)$$

$$\begin{aligned} f(-1) = 10 &\Leftrightarrow 10 = a_n(1+1)(1+4) \\ &\Leftrightarrow 10 = 10a_n \\ &\Leftrightarrow a_n = 1 \end{aligned}$$

Thus $f(x) = (x^2+1)(x^2+4) = \boxed{x^4 + 5x^2 + 4}$

(e) We know the roots are $x = 2i, -2i, 3i, -3i$ so the polynomial is:

$$f(x) = a_n(x - 2i)(x + 2i)(x - 3i)(x + 3i) = a_n(x^2 + 4)(x^2 + 9)$$

$$\begin{aligned} f(-2) = 104 &\Leftrightarrow 104 = a_n(4 + 4)(4 + 9) \\ &\Leftrightarrow 104 = 104a_n \\ &\Leftrightarrow a_n = 1 \end{aligned}$$

Thus $f(x) = (x^2 + 4)(x^2 + 9) = \boxed{x^4 + 13x^2 + 36}$

□

4. Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real roots of $f(x)$

   (a) $f(x) = x^3 + 2x^2 + 3x + 7$
   (b) $f(x) = -6x^3 + 2x^2 - 3x + 5$
   (c) $f(x) = x^3 + 2x^2 + 5x + 8$
   (d) $f(x) = 5x^4 - 9x^3 - 6x^2 - 7x + 8$
   (e) $f(x) = x^3 - 3x^2 - 33x + 35$

   **Solution**

   (a) $f(x)$ has no sign changes so there are no positive real roots.
   $f(-x) = -x^2 + 2x^2 - 3x + 7$ has 3 sign changes so there are either 3 or 1 negative real root(s).

   (b) $f(x)$ has 3 sign changes so there are either 3 or 1 positive real root(s).
   $f(-x) = 6x^3 + 2x^2 + 3x + 5$ has no sign changes so there are no negative real roots.

   (c) $f(x)$ has no sign changes so there are no positive real roots.
   $f(-x) = -x^3 + 2x^2 - 5x + 8$ has 3 sign changes so there are either 3 or 1 negative real root(s).

   (d) $f(x)$ has 2 sign changes so there are either 2 or 0 positive real root(s).
   $f(-x) = 5x^5 + 9x^3 - 6x^2 + 7x + 8$ has 2 sign changes so there are either 2 or 0 negative real root(s).

   (e) $f(x)$ has 2 sign changes so there are either 2 or 0 positive real root(s).
   $f(-x) = -x^3 - 3x^2 + 33x + 35$ has 1 sign change so there is 1 negative real root.

□

5. Find all the roots of the polynomial function. Use the Rational Root Test and Descartes's Rule of Signs to obtain the first root.

   (a) $f(x) = x^3 - 6x^2 - 9x + 14$
   (b) $f(x) = x^3 + 2x^2 - 5x - 6$
   (c) $f(x) = x^3 - 13x^2 + 47x - 35$
   (d) $f(x) = x^4 - 2x^3 + x^2 + 12x + 8$
   (e) $f(x) = 2x^3 - 17x^2 - 11x - 1$

**Solution**

(a) Possible rational roots: $\pm 1, \pm 2, \pm 7, \pm 14$
$f(x)$ has 2 sign changes so there are either 2 or 0 positive real root(s).
$f(-x) = -x^3 - 6x^2 + 9x + 14$ has 1 sign change so there is 1 negative real root.

$$\begin{array}{r|rrrr} & 1 & -6 & -9 & 14 \\ -2 & & -2 & 16 & -14 \\ \hline & 1 & -8 & 7 & 0 \end{array}$$

$$x^3 - 6x^2 - 9x + 14 = (x+2)(x^2 - 8x + 7) = (x+2)(x-7)(x-1)$$

Thus the solutions are $\boxed{x = -2, 7, 1}$

(b) Possible rational roots: $\pm 1, \pm 2, \pm 3, \pm 6$
$f(x)$ has 1 sign change so there is 1 possible real root.
$f(-x) = -x^3 + 2x^2 + 5x - 6$ has 2 sign changes so there are either 2 or 0 negative real root(s).

$$\begin{array}{r|rrrr} & 1 & 2 & -5 & -6 \\ 2 & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$$x^3 + 2x^2 - 5x - 6 = (x-2)(x^2 + 4x + 3) = (x-2)(x+3)(x+1)$$

Thus the solutions are $\boxed{x = 2, -3, -1}$

(c) Possible rational roots: $\pm 1, \pm 5, \pm 7, \pm 35$
$f(x)$ has 3 sign changes so there are either 3 or 1 positive real root(s).
$f(-x) = -x^3 - 13x^2 - 47x - 35$ so there are no negative real root(s).

$$\begin{array}{r|rrrr} & 1 & -13 & 47 & -35 \\ 1 & & 1 & -12 & 35 \\ \hline & 1 & -12 & 35 & 0 \end{array}$$

$$x^3 - 13x^2 + 47x - 35 = (x-1)(x^2 - 12x + 35) = (x-1)(x-7)(x-5)$$

Thus the solutions are $\boxed{x = 1, 7, 5}$

(d) Possible rational roots: $\pm 1, \pm 2, \pm 4, \pm 8$

$f(x)$ has 2 sign changes so there are either 2 or 0 positive real root(s).

$f(-x) = x^4 + 2x^3 + x^2 - 12x + 8$ has 2 sign changes so there are either 2 or 0 negative real root(s).

$$\begin{array}{r|rrrrr} & 1 & -2 & 1 & 12 & 8 \\ -1 & & -1 & 3 & -4 & -8 \\ \hline & 1 & -3 & 4 & 8 & 0 \end{array}$$

$$x^4 - 2x^3 + x^2 + 12x + 8 = (x+1)(x^3 - 3x^2 + 4x + 8)$$

Using synthetic division on $x^3 - 3x^2 + 4x + 8$ gives

$$\begin{array}{r|rrrr} & 1 & -3 & 4 & 8 \\ -1 & & -1 & 4 & -8 \\ \hline & 1 & -4 & 8 & 0 \end{array}$$

$$x^4 - 2x^3 + x^2 + 12x + 8 = (x+1)^2(x^2 - 4x + 8)$$

Using the quadratic formula we have

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 4(1)(8)}}{2} \\ &= \frac{4 \pm \sqrt{-16}}{2} \\ &= \frac{4 \pm 4i}{2} \\ &= 2 \pm 2i \end{aligned}$$

So the solutions are $x = -1, 2 \pm 2i$

(e) Possible rational roots: $\pm 1, \pm \frac{1}{2}$

$f(x)$ has 1 sign change so there is 1 positive real root. $f(-x) = -2x^3 - 17x^2 + 11x - 1$ has 2 sign changes so there is either 2 or 0 negative real roots.

$$\begin{array}{r|rrrr} & 2 & -17 & -11 & -1 \\ -\frac{1}{2} & & -1 & 9 & 1 \\ \hline & 2 & -18 & -2 & 0 \end{array}$$

$$2x^3 - 17x^2 - 11x - 1 = \left(x + \frac{1}{2}\right)(2x^2 - 18x - 2) = 2\left(x + \frac{1}{2}\right)(x^2 - 9x - 1)$$

Using the quadratic formula we have

$$\begin{aligned} x &= \frac{9 \pm \sqrt{81 - 4(1)(-1)}}{2} \\ &= \frac{9 \pm \sqrt{85}}{2} \end{aligned}$$

So the solutions are $\boxed{x = -\frac{1}{2}, \frac{9 \pm \sqrt{85}}{2}}$

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