Section 1.7: The Precise Definition of a Limit

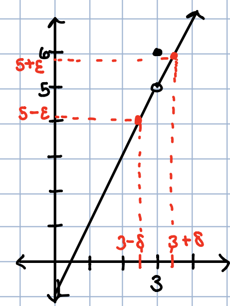
In the previous definitions of limits we used phrases like "f(x) gets close to L as x gets close to a". A question might then be how close?

The precise definition of a limit: Let f be a function defined on some open interval containing the number a. We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ s.t.

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example: $f(x) = \begin{cases} 2x-1, & x \neq 3 \\ 6, & x = 3 \end{cases}$

$\lim_{x\to 3} f(x) = 5$



i.e. $\lim_{x\to a} f(x) = L$ means f(x) can be made as close as we want to L by requiring x to be close enough to a

Example: Prove $\lim_{x\to 3} (4x-5) = 7$

Let $\varepsilon > 0$. We want to find a δ s.t.
$0 < |x-3| < \delta \Rightarrow |4x-5-7| < \varepsilon$
i.e. $0 < |x-3| < \delta \Rightarrow 4|x-3| < \varepsilon$
i.e. $0 < |x-3| < \delta \Rightarrow |x-3| < \frac{\varepsilon}{4}$

Thus $\delta = \varepsilon/4$ works

Left hand and right hand limits

$\lim_{x\to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. $a-\delta < x < a \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x\to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. $a < x < a+\delta$ then $|f(x) - L| < \varepsilon$

Example: Prove $\lim_{x\to 0^+} \sqrt{x} = 0$

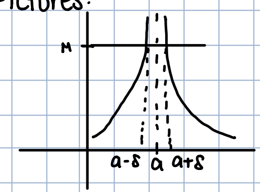
Let $\varepsilon > 0$. We want $0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$
i.e. $0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$
i.e. $0 < x < \delta \Rightarrow x < \varepsilon^2$
so $\delta = \varepsilon^2$ works

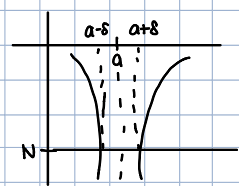
Infinite limits

$\lim_{x \to a} f(x) = \infty$ if for every $M>0$ there is a $\delta>0$ s.t. $0<|x-a|<\delta \Rightarrow f(x)>M$

$\lim_{x \to a} f(x) = -\infty$ if for every $N<0$ there is a $\delta>0$ s.t. $0<|x-a|<\delta \Rightarrow f(x)<N$

Pictures:





Limits at infinity

$\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon>0$ there is an N s.t. if $x>N \Rightarrow |f(x)-L|<\varepsilon$

$\lim_{x \to -\infty} f(x) = L$ if for every $\varepsilon>0$ there is an N s.t. $x<N \Rightarrow |f(x)-L|<\varepsilon$

$\lim_{x \to \infty} f(x) = \infty$ if for every $M>0$ there is an $N>0$ s.t. $x>N \Rightarrow f(x)>M$

Pictures: