The following are true statements involving chapters 2 and 3. Prove these statements by working through the problem and showing all necessary steps.

(1) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = -\frac{1}{2}$

Solution

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{\sqrt{1+t}}{t\sqrt{1+t}}\right)$$

$$= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}$$

$$= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}} \cdot \frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$$

$$= \lim_{t\to 0}\frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{1+t}(1+\sqrt{1+t})}$$

$$= \lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})}$$

$$= \lim_{t\to 0}\frac{\not{1}-\not{1}-t}{t\sqrt{1+t}(1+\sqrt{1+t})}$$

$$= \lim_{t\to 0}\frac{-\not{t}}{\not{t}\sqrt{1+t}(1+\sqrt{1+t})}$$

$$\lim_{t\to 0}\frac{-1}{\sqrt{1+t}(1+\sqrt{1+t})} = \frac{-1}{\sqrt{1+0}(1+\sqrt{1+0})}$$

$$= -\frac{1}{1(1+1)}$$

$$= -\frac{1}{2} \quad \checkmark$$

$\square$

(2) $g(x) = \begin{cases} \frac{4-x^2}{2+x} & x < 1 \\ 1 & x = 1 \\ 2 - x^2 & 1 < x \leq 2 \\ x - 3 & x > 2 \end{cases}$ is continuous at $x = 1$ but not at $x = 2$

Solution

Check $x = 1$

$$\begin{aligned} \lim_{x \to 1^-} g(x) &= \lim_{x \to 1^-} \frac{4 - x^2}{2 + x} \\ &= \frac{4 - (1)^2}{2 + 1} \\ &= \frac{3}{3} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} g(x) &= \lim_{x \to 1^+} 2 - x^2 \\ &= 2 - (1)^2 \\ &= 1 \end{aligned}$$

$$g(1) = 1$$

Since all of these are equal, $g(x)$ is continuous at $x = 1$.

Check $x = 2$

$$\begin{aligned} \lim_{x \to 2^-} g(x) &= \lim_{x \to 2^-} (2 - x^2) \\ &= 2 - (2)^2 \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} g(x) &= \lim_{x \to 2^+} (x - 3) \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

Since the left and right hand limits aren't the same, the limit DNE and $g(x)$ is not continuous at $x = 2$.

□

(3) $\sqrt{x-5} = \dfrac{1}{x+3}$ has at least one solution.

Solution

$$\sqrt{x-5} = \frac{1}{x+3} \Leftrightarrow \sqrt{x-5} - \frac{1}{x+3} = 0$$

Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$. We want to show that $f(x) = 0$ at least once.

$$\begin{aligned} f(6) &= \sqrt{6-5} - \frac{1}{6+3} \\ &= \sqrt{1} - \frac{1}{9} \\ &= \frac{8}{9} > 0 \end{aligned}$$

$$\begin{aligned} f(5) &= \sqrt{5-5} - \frac{1}{5+3} \\ &= \sqrt{0} - \frac{1}{8} \\ &= -\frac{1}{8} < 0 \end{aligned}$$

Since f is continuous on $[5,6]$ with $f(5) < 0$ and $f(6) > 0$, then by the Intermediate Value Theorem, there is at least one solution to $f(x) = 0$ between 5 and 6. ✓

□

(4) $\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x} = -4$

Solution Plugging in we get $\frac{0}{0}$ so we want to check if the absolute values have negative or positive inputs.

Plugging in $x = 0$ to $2x - 1$ gives a negative, therefore as $x \to 0$, $|2x - 1| = -(2x - 1)$
Plugging in $x = 0$ to $2x + 1$ gives a positive, therefore as $x \to 0$, $|2x + 1| = 2x + 1$

$$\begin{aligned}
\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x} &= \lim_{x\to 0} \frac{-(2x-1)-(2x+1)}{x} \\
&= \lim_{x\to 0} \frac{-2x+\cancel{1}-2x-\cancel{1}}{x} \\
&= \lim_{x\to 0} \frac{-2x-2x}{x} \\
&= \lim_{x\to 0} \frac{-4\cancel{x}}{\cancel{x}} \\
&= \lim_{x\to 0} -4 \\
&= -4 \quad \checkmark
\end{aligned}$$

$\square$

(5) The equation of the line tangent to $y = 3 + 4x^2 - 2x^3$ at $x = 1$ is $y = 2x + 3$ and can be found using the limit definition of a derivative.

Solution

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{[3 + 4(1+h)^2 - 2(1+h)^3] - [3 + 4(1)^2 - 2(1)^3]}{h} \\
&= \lim_{h\to 0} \frac{[3 + 4(1 + 2h + h^2) - 2(h^3 + 3h^2 + 3h + 1)] - [3 + 4 - 2]}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3} + \cancel{4} + 8h + 4h^2 - 2h^3 - 6h^2 - 6h - \cancel{2} - \cancel{3} - \cancel{4} + \cancel{2}}{h} \\
&= \lim_{h\to 0} \frac{8h + 4h^2 - 2h^3 - 6h^2 - 6h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(8 + 4h - 2h^2 - 6h - 6)}{\cancel{h}} \\
&= \lim_{h\to 0} (8 + 4h - 2h^2 - 6h - 6) \\
&= 8 + 4(0) - 2(0)^2 - 6(0) - 6 \\
&= 8 - 6 \\
&= 2
\end{aligned}
$$

When $x = 1$, $y = 3 + 4 - 2 = 5$. Using point-slope form we have

$$y - 5 = 2(x - 1) \Rightarrow y - 5 = 2x - 2 \Rightarrow y = 2x - 2 + 5 \Rightarrow y = 2x + 3 \checkmark$$

□

(6) $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = -\dfrac{3}{5}$ for $\dfrac{5x}{1+x^2}$

Solution

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0} \frac{\frac{5(2+h)}{1+(2+h)^2} - \frac{5(2)}{1+(2)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{10+5h}{1+(2+h)^2} - \frac{10}{1+4}}{h} \\
&= \lim_{h\to 0} \frac{\frac{10+5h}{1+(2+h)^2} - 2}{h} \cdot \frac{1+(2+h)^2}{1+(2+h)^2} \\
&= \lim_{h\to 0} \frac{10+5h-2(1+(2+h)^2)}{h(1+(2+h)^2)} \\
&= \lim_{h\to 0} \frac{10+5h-2(1+4+4h+h^2)}{h(1+(2+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{10}+5h-\cancel{2}-\cancel{8}-8h-2h^2}{h(1+(2+h)^2)} \\
&= \lim_{h\to 0} \frac{\cancel{h}(5-8-2h)}{\cancel{h}(1+(2+h)^2)} \\
&= \lim_{h\to 0} \frac{-3-2h}{1+(2+h)^2} \\
&= \frac{-3-2(0)}{1+(2+0)^2} \\
&= \frac{-3}{1+4} \\
&= -\frac{3}{5} \quad \checkmark
\end{aligned}$$

□

(7) $y = \dfrac{x^2+4x+3}{\sqrt{x}} \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2+4x-3}{2x^{3/2}}$

Solution

$$\frac{x^2+4x+3}{\sqrt{x}} = x^{-1/2}\left(x^2+4x+3\right) = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2} \\
&= \frac{3x^{1/2}}{2} + 2x^{-1/2} - \frac{3}{2x^{3/2}} \\
&= \frac{3x^{1/2}\cdot x^{3/2}}{2x^{3/2}} + \frac{2x^{-1/2}\cdot 2x^{3/2}}{2x^{3/2}} - \frac{3}{2x^{3/2}} \\
&= \frac{3x^2+4x-3}{2x^{3/2}} \quad \checkmark
\end{aligned}$$

You can also do this using quotient rule.

□

(8) $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3) \Rightarrow F'(y) = \frac{9}{y^4} + \frac{14}{y^2} + 5$

Solution

$$F(y) = (y^{-2} - 3y^{-4})(y + 5y^3) = y^{-1} + 5y - 3y^{-3} - 15y^{-1} = 14y^{-1} - 3y^{-3} + 5y$$

$$F'(y) = 14y^{-2} + 9y^{-4} + 5 = \frac{9}{y^4} + \frac{14}{y^2} + 5\checkmark$$

You can also do this using product rule.

$\square$

(9) $\dfrac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

Solution

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)h(x)] &= \frac{d}{dx}[f(x)g(x)\cdot h(x)]\\ &= \frac{d}{dx}[f(x)g(x)]h(x) + f(x)g(x)h'(x) \leftarrow \text{product rule}\\ &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \leftarrow \text{product rule again}\\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)g'(x) \quad \checkmark\end{aligned}$$

$\square$

(10) $H(\theta) = \theta \sin\theta \Rightarrow H''(\theta) = 2\cos\theta - \theta\sin\theta$

Solution

$$H'(\theta) = \left(\frac{d}{d\theta}\theta\right)\sin\theta + \theta\frac{d}{d\theta}\sin\theta$$
$$= \sin\theta + \theta\cos\theta$$

$$H''(\theta) = \frac{d}{d\theta}(\sin\theta + \theta\cos\theta$$
$$= \cos\theta + \left(\frac{d}{d\theta}(\theta)\right)\cos\theta + \theta\frac{d}{d\theta}\cos\theta$$
$$= \cos\theta + \cos\theta + \theta(-\sin\theta)$$
$$= 2\cos\theta - \theta\sin\theta \quad \checkmark$$

□

(11) $\lim_{t\to 0} \frac{\tan(6t)}{\sin(2t)} = 3$

Solution

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan(6t)}{\sin(2t)} &= \lim_{t\to 0} \tan(6t) \cdot \frac{1}{\sin(2t)} \\
&= \lim_{t\to 0} \frac{\sin(6t)}{\cos(6t)} \cdot \frac{1}{\sin(2t)} \\
&= \lim_{t\to 0} \sin(6t) \cdot \frac{1}{\sin(2t)} \cdot \frac{1}{\cos(6t)} \\
&= \lim_{t\to 0} \frac{6t\sin(6t)}{6t} \cdot \frac{2t}{2t\sin 2t} \cdot \frac{1}{\cos(6t)} \\
&= \lim_{t\to 0} \frac{\sin(6t)}{6t} \cdot \frac{2t}{\sin(2t)} \cdot \frac{1}{\cos(6t)} \cdot \frac{6\not{t}}{2\not{t}} \\
&= \lim_{t\to 0} \frac{\sin(6t)}{6t} \cdot \lim_{t\to 0} \frac{2t}{\sin(2t)} \cdot \lim_{t\to 0} \frac{1}{\cos(6t)} \cdot \lim_{t\to 0} 3 \\
&= 1 \cdot 1 \cdot \frac{1}{\cos 0} \cdot 3 \\
&= \boxed{3} \quad \checkmark
\end{aligned}$$

□

(12) $\tan\left(\frac{x}{y}\right) = x + y \Rightarrow \frac{dy}{dx} = \frac{y\sec^2\left(\frac{x}{y}\right) - y^2}{y^2 + x\sec^2\left(\frac{x}{y}\right)}$

Solution Taking the derivative of both sides with respect to x, we have:

$$\sec^2\left(\frac{x}{y}\right)\cdot\frac{d}{dx}\left(\frac{x}{y}\right) = 1 + \frac{dy}{dx} \Rightarrow \sec^2\left(\frac{x}{y}\right)\cdot\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = 1 + \frac{dy}{dx}$$

$$\Rightarrow \sec^2\left(\frac{x}{y}\right)(y - x\frac{dy}{dx}) = y^2\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow y\sec^2\left(\frac{x}{y}\right) - x\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx} = y^2 + y^2\frac{dy}{dx}$$

$$\Longrightarrow y\sec^2\left(\frac{x}{y}\right) - y^2 = y^2\frac{dy}{dx} + x\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow y\sec^2\left(\frac{x}{y}\right) - y^2 = \frac{dy}{dx}\left(y^2 + x\sec^2\left(\frac{x}{y}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sec^2\left(\frac{x}{y}\right) - y^2}{y^2 + x\sec^2\left(\frac{x}{y}\right)} \quad \checkmark$$

$\square$

(13) If a particle has position function $s(t) = t^3 - 12t^2 + 36t$, then it is speeding up on the intervals $(2,4)$, $(6,\infty)$. and slowing down on the intervals $[0,2)$, $(4,6)$.

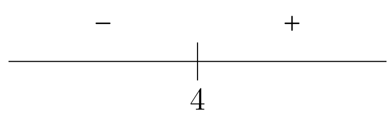
Solution

$$v(t) = s'(t) = 3t^2 - 24t + 36 = 3(t^2 - 8t + 12) = 3(t-6)(t-2)$$

$$a(t) = v'(t) = 6t - 24 = 6(t-4)$$

Velocity is 0 when $t = 2, 6$. Testing the intervals we have

Acceleration is 0 when $t = 4$. Testing the intervals we have



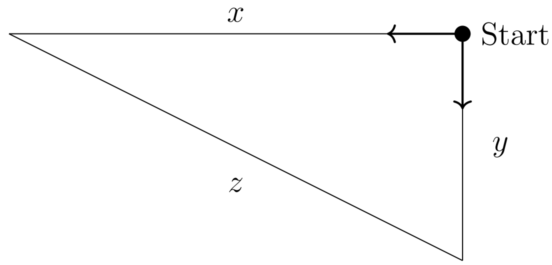
(Note that these lines should start at $t = 0$ since t is time)

The particle is speeding up when the two number lines have the same sign. i.e. on $(2,4)$, $(6,\infty)$ and slowing down when the number lines have opposite signs. i.e. on $[0,2)$, $(4,6)$ ✓

□

(14) If two cars start moving from the same point with Car A traveling south at 60mph and Car B traveling west at 25mph, then their distance is increasing at a rate of 65mph 2 hours later.

Solution



We are given that $\frac{dx}{dt} = 25$ and $\frac{dy}{dt} = 60$. We want to show that $\frac{dz}{dt} = 65$.

Using the pythagorean theorem we have

$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

After two hours, Car A has travelled 120 miles, so $y = 120$ and Car B has traveled 50 miles so $x = 50$. Again, using the pythagorean theorem, we have

$$50^2 + 120^2 = z^2 \Rightarrow z = 130$$

Plugging all of this into the equation, we have

$$\begin{aligned} 2(50)(25) + 2(120)(6) = 2(130)\frac{dz}{dt} &\Leftrightarrow 2500 + 14400 = 260\frac{dz}{dt} \\ &\Leftrightarrow 16900 = 260\frac{dz}{dt} \\ &\Leftrightarrow \frac{dz}{dt} = 65 \text{ mph} \quad \checkmark \end{aligned}$$

$\square$