

T 67–71. Special curves The following classical curves have been studied by generations of mathematicians. Use analytical methods (including implicit differentiation) and a graphing utility to graph the curves. Include as much detail as possible.

67. $x^{2/3} + y^{2/3} = 1$ (Astroid or hypocycloid with four cusps)

68. $y = \frac{8}{x^2 + 4}$ (Witch of Agnesi)

69. $x^3 + y^3 = 3xy$ (Folium of Descartes)

70. $y^2 = x^3(1 - x)$ (Pear curve)

71. $x^4 - x^2 + y^2 = 0$ (Figure-8 curve)

T 72. Elliptic curves The equation $y^2 = x^3 - ax + 3$, where a is a parameter, defines a well-known family of *elliptic curves*.

- a. Plot a graph of the curve when $a = 3$.

- b. Plot a graph of the curve when $a = 4$.
c. By experimentation, determine the approximate value of a ($3 < a < 4$) at which the graph separates into two curves.

T 73. Lamé curves The equation $|y/a|^n + |x/a|^n = 1$, where n and a are positive real numbers, defines the family of Lamé curves. Make a complete graph of this function with $a = 1$, for $n = \frac{2}{3}, 1, 2$, and 3. Describe the progression that you observe as n increases.

QUICK CHECK ANSWERS

1. Make the window larger in the y -direction.
2. Notice that f and $f + C$ have the same derivatives.

3. $f(-x) = \frac{10(-x)^3}{(-x)^2 - 1} = -\frac{10x^3}{x^2 - 1} = -f(x) \leftarrow$

4.5 Optimization Problems

The theme of this section is *optimization*, a topic arising in many disciplines that rely on mathematics. A structural engineer may seek the dimensions of a beam that maximize strength for a specified cost. A packaging designer may seek the dimensions of a container that maximize the volume of the container for a given surface area. Airline strategists need to find the best allocation of airliners among several hubs to minimize fuel costs and maximize passenger miles. In all these examples, the challenge is to find an *efficient* way to carry out a task, where “efficient” could mean least expensive, most profitable, least time consuming, or, as you will see, many other measures.

To introduce the ideas behind optimization problems, think about pairs of nonnegative real numbers x and y between 0 and 20 with the property that their sum is 20, that is, $x + y = 20$. Of all possible pairs, which has the greatest product?

Table 4.3 displays a few cases showing how the product of two nonnegative numbers varies while their sum remains constant. The condition that $x + y = 20$ is called a **constraint**: It tells us to consider only (nonnegative) values of x and y satisfying this equation.

The quantity that we wish to maximize (or minimize in other cases) is called the **objective function**; in this case, the objective function is the product $P = xy$. From Table 4.3, it appears that the product is greatest if both x and y are near the middle of the interval $[0, 20]$.

This simple problem has all the essential features of optimization problems. At their heart, optimization problems take the following form:

What is the maximum (minimum) value of an objective function subject to the given constraint(s)?

For the problem at hand, this question would be stated as, “What pair of nonnegative numbers maximizes $P = xy$ subject to the constraint $x + y = 20$?” The first step is to use the constraint to express the objective function $P = xy$ in terms of a single variable. In this case, the constraint is

$$x + y = 20, \quad \text{or} \quad y = 20 - x.$$

Substituting for y , the objective function becomes

$$P = xy = x(20 - x) = 20x - x^2,$$

which is a function of the single variable x . Notice that the values of x lie in the interval $0 \leq x \leq 20$ with $P(0) = P(20) = 0$.

Table 4.3

x	y	$x + y$	$P = xy$
1	19	20	19
5.5	14.5	20	79.75
9	11	20	99
13	7	20	91
18	2	20	36

- In this problem, it is just as easy to eliminate x as y . In other problems, eliminating one variable may result in less work than eliminating other variables.

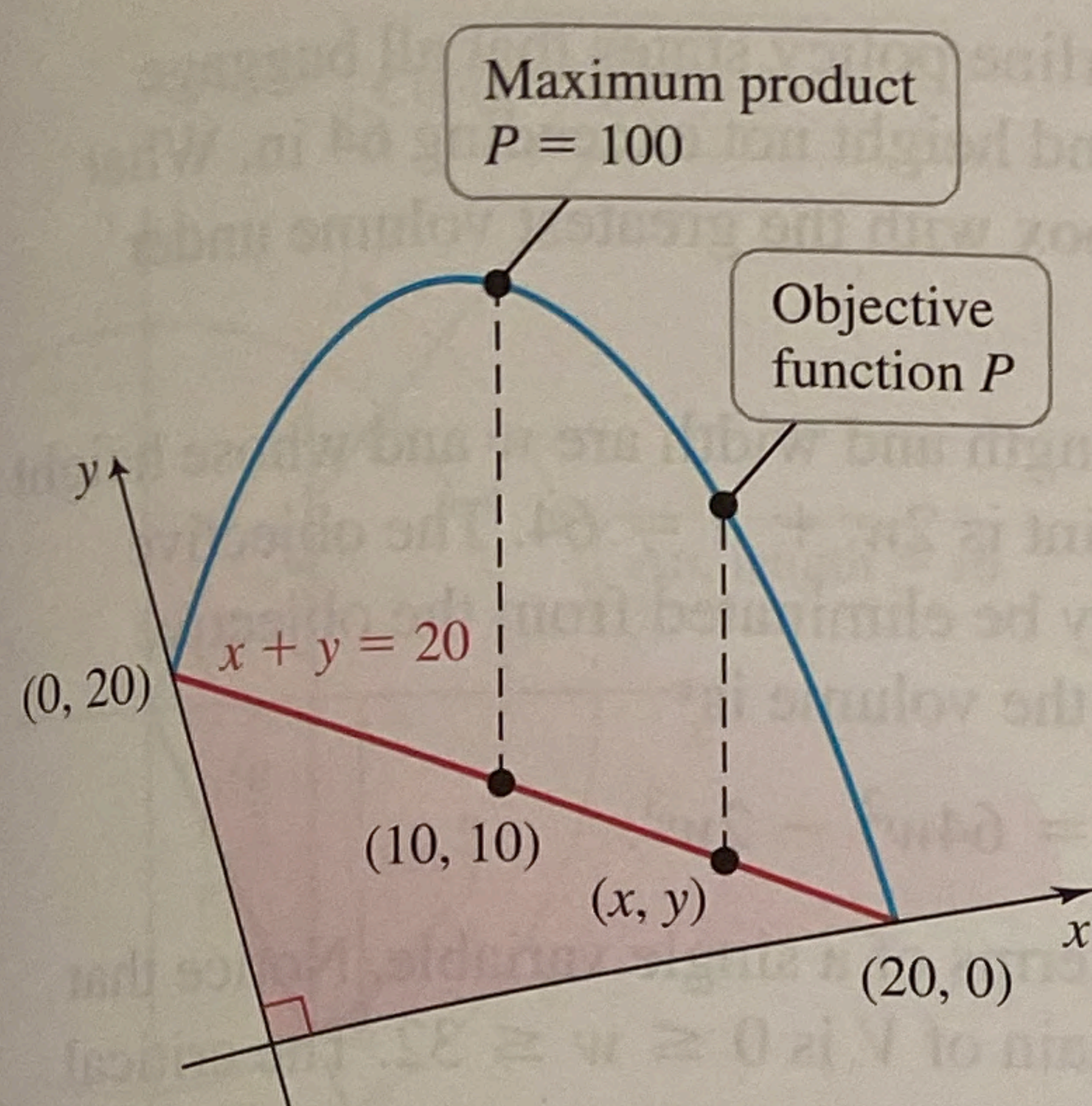


Figure 4.53

QUICK CHECK 1 Verify that in the previous example, the same result is obtained if the constraint $x + y = 20$ is used to eliminate x rather than y . ◀

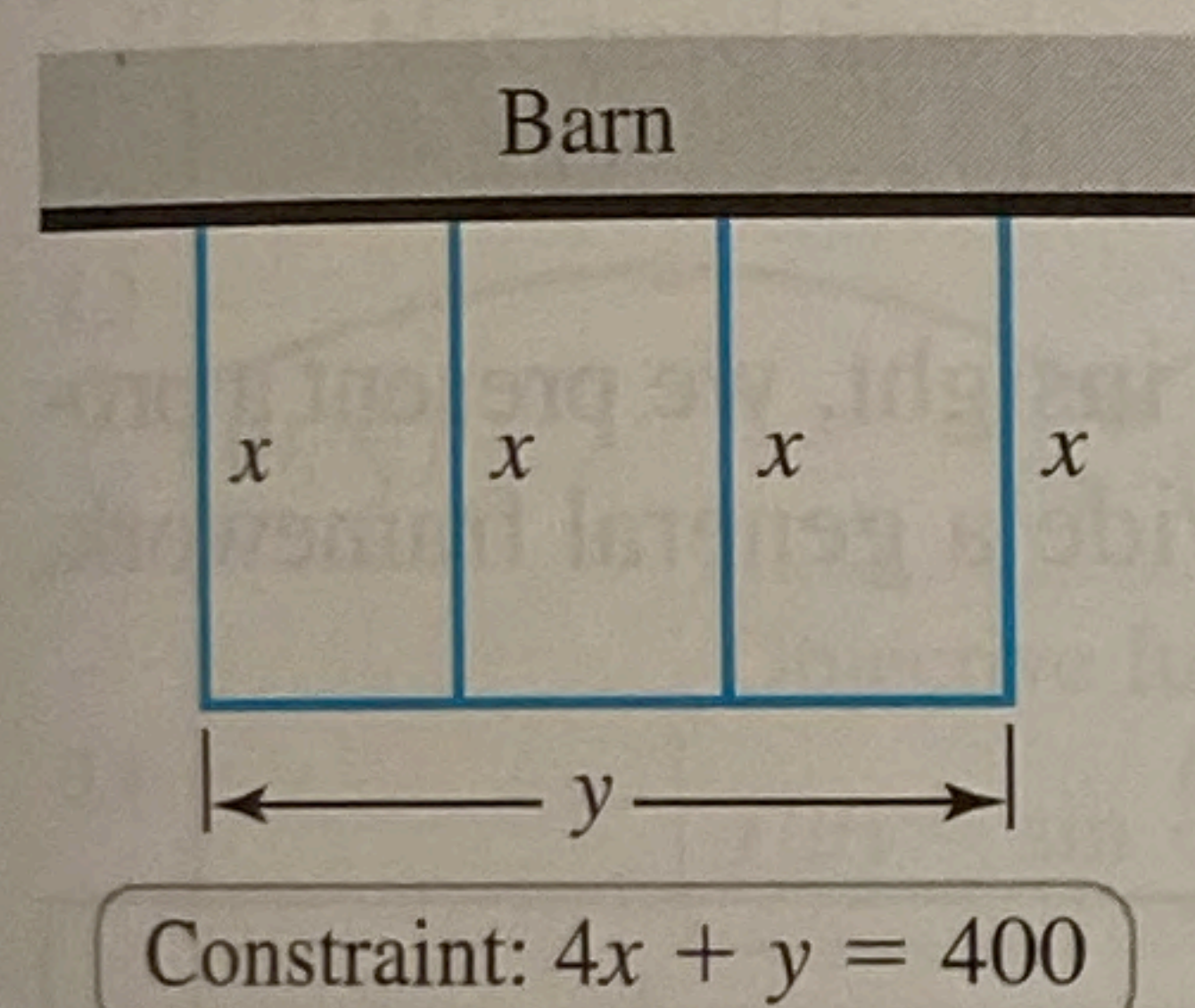


Figure 4.54

► Recall from Section 4.1 that, when they exist, the absolute extreme values occur at critical points or endpoints.

QUICK CHECK 2 Find the objective function in Example 1 (in terms of x)

- if there is no interior fence and
- if there is one interior fence that forms a right angle with the barn, as in Figure 4.54. ◀

To maximize P , we first find the critical points by solving

$$P'(x) = 20 - 2x = 0$$

to obtain the solution $x = 10$. To find the absolute maximum value of P on the interval $[0, 20]$, we check the endpoints and the critical points. Because $P(0) = P(20) = 0$ and $P(10) = 100$, we conclude that P has its absolute maximum value at $x = 10$. By the constraint $x + y = 20$, the numbers with the greatest product are $x = y = 10$, and their product is $P = 100$.

Figure 4.53 summarizes this problem. We see the constraint line $x + y = 20$ in the xy -plane. Above the line is the objective function $P = xy$. As x and y vary along the constraint line, the objective function changes, reaching a maximum value of 100 when $x = y = 10$.

Most optimization problems have the same basic structure as the preceding example: There is an objective function, which may involve several variables, and one or more constraints. The methods of calculus (Sections 4.1 and 4.3) are used to find the minimum or maximum values of the objective function.

EXAMPLE 1 Rancher's dilemma A rancher has 400 ft of fence for constructing a rectangular corral. One side of the corral will be formed by a barn and requires no fence. Three exterior fences and two interior fences partition the corral into three rectangular regions as shown in **Figure 4.54**. What are the dimensions of the corral that maximize the enclosed area? What is the area of that corral?

SOLUTION We first sketch the corral (Figure 4.54), where $x \geq 0$ is the width and $y \geq 0$ is the length of the corral. The amount of fence required is $4x + y$, so the constraint is $4x + y = 400$, or $y = 400 - 4x$.

The objective function to be maximized is the area of the corral, $A = xy$. Using $y = 400 - 4x$, we eliminate y and express A as a function of x :

$$A = xy = x(400 - 4x) = 400x - 4x^2.$$

Notice that the width of the corral must be at least $x = 0$, and it cannot exceed $x = 100$ (because 400 ft of fence are available). Therefore, we maximize $A(x) = 400x - 4x^2$, for $0 \leq x \leq 100$. The critical points of the objective function satisfy

$$A'(x) = 400 - 8x = 0,$$

which has the solution $x = 50$. To find the absolute maximum value of A , we check the endpoints of $[0, 100]$ and the critical point $x = 50$. Because $A(0) = A(100) = 0$ and $A(50) = 10,000$, the absolute maximum value of A occurs when $x = 50$. Using the constraint, the optimal length of the corral is $y = 400 - 4(50) = 200$. Therefore, the maximum area of 10,000 ft² is achieved with dimensions $x = 50$ ft and $y = 200$ ft. The objective function A is shown in **Figure 4.55**.

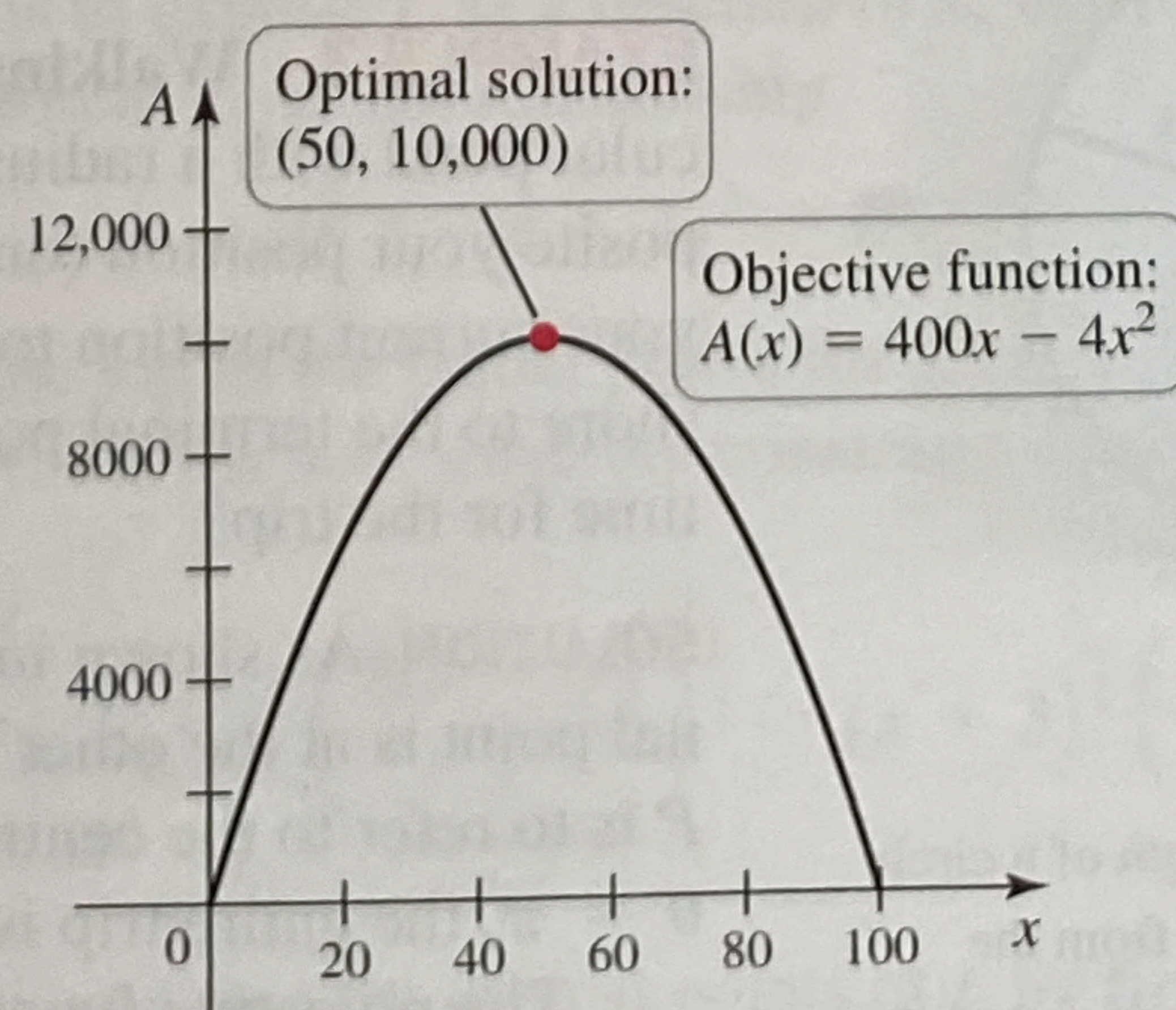
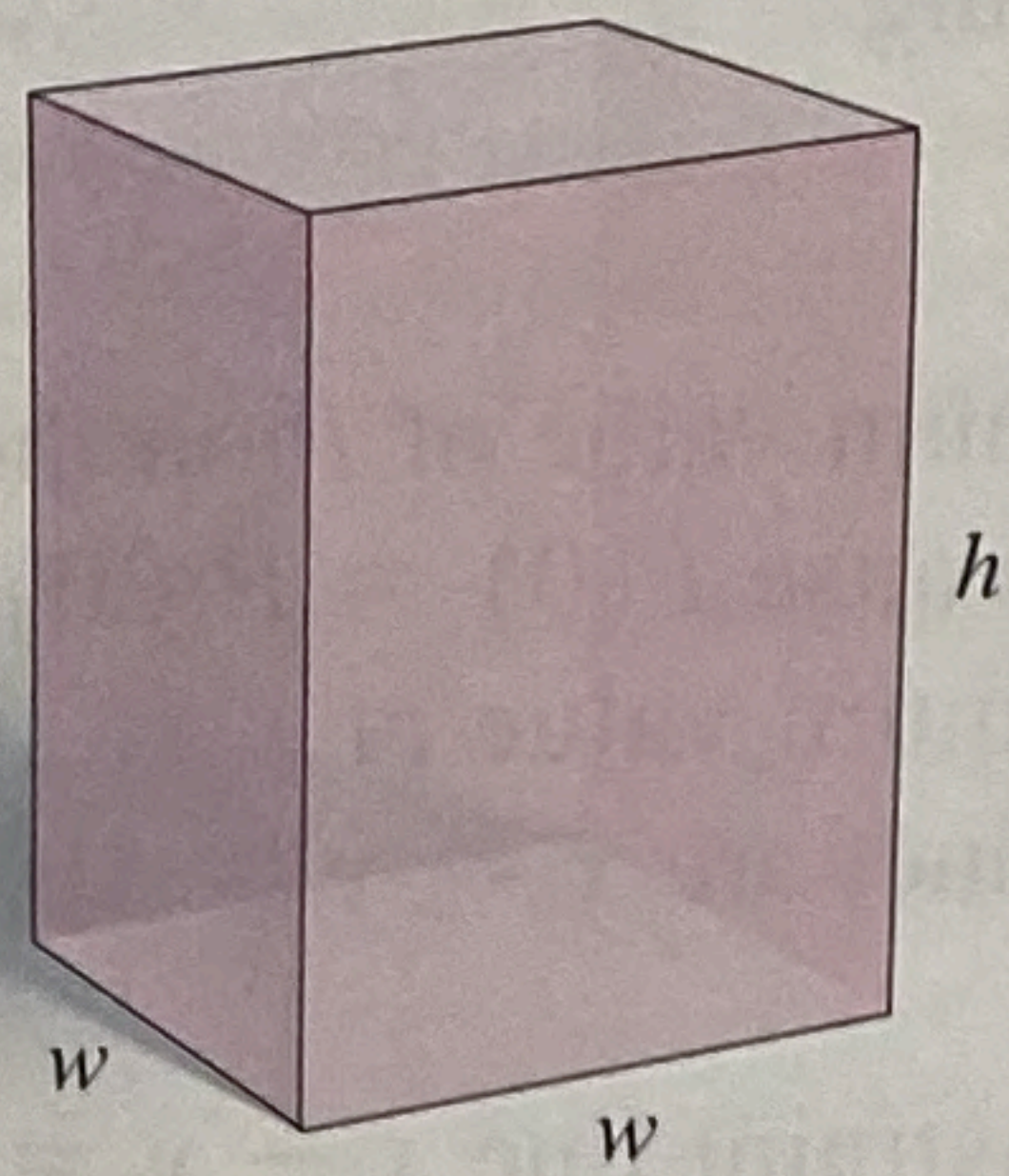


Figure 4.55

Related Exercises 12, 16 ◀



Objective function: $V = w^2h$
Constraint: $2w + h = 64$

Figure 4.56

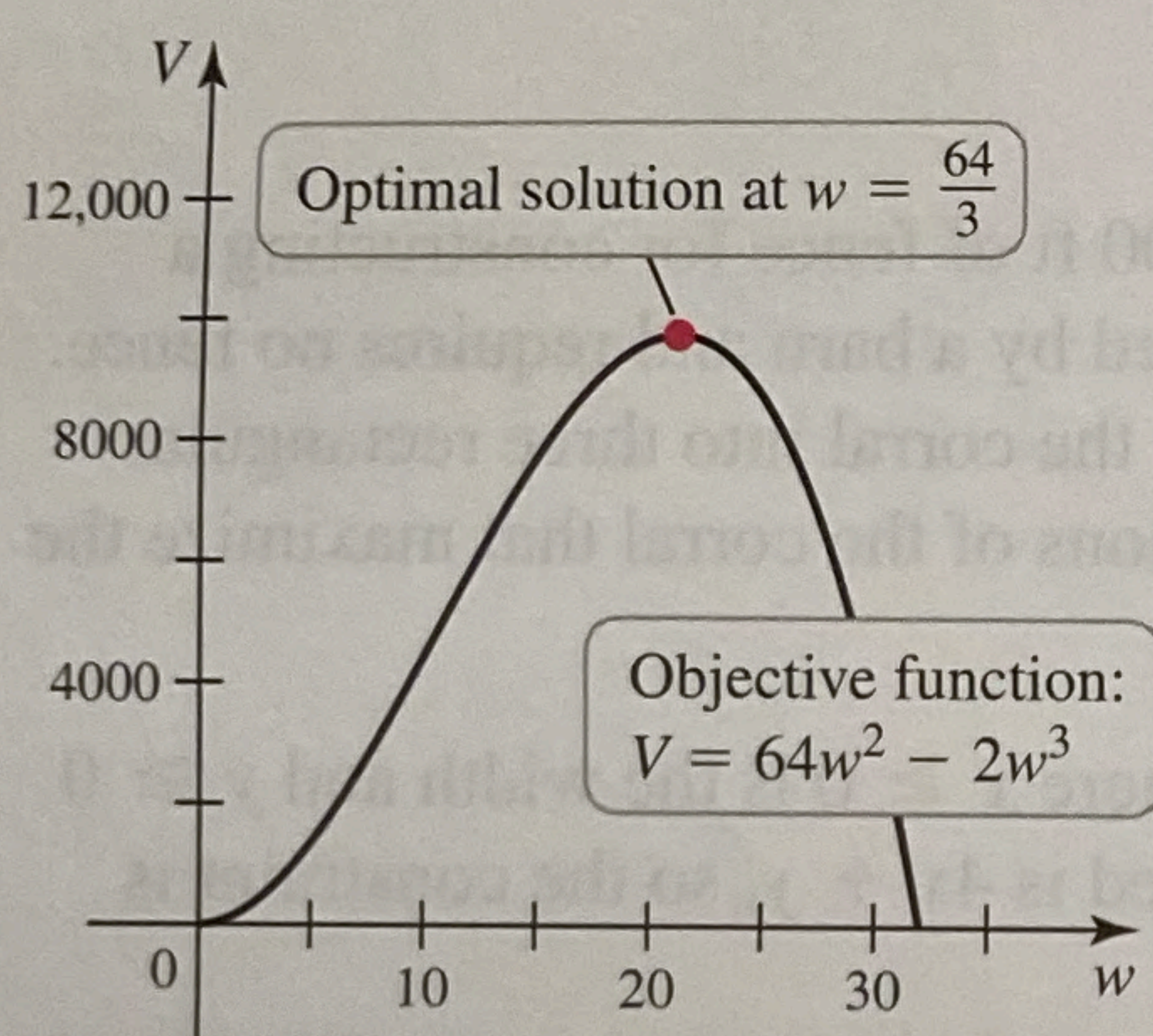


Figure 4.57

QUICK CHECK 3 Find the objective function in Example 2 (in terms of w) if the constraint is that the sum of length and width and height cannot exceed 108 in. ◀

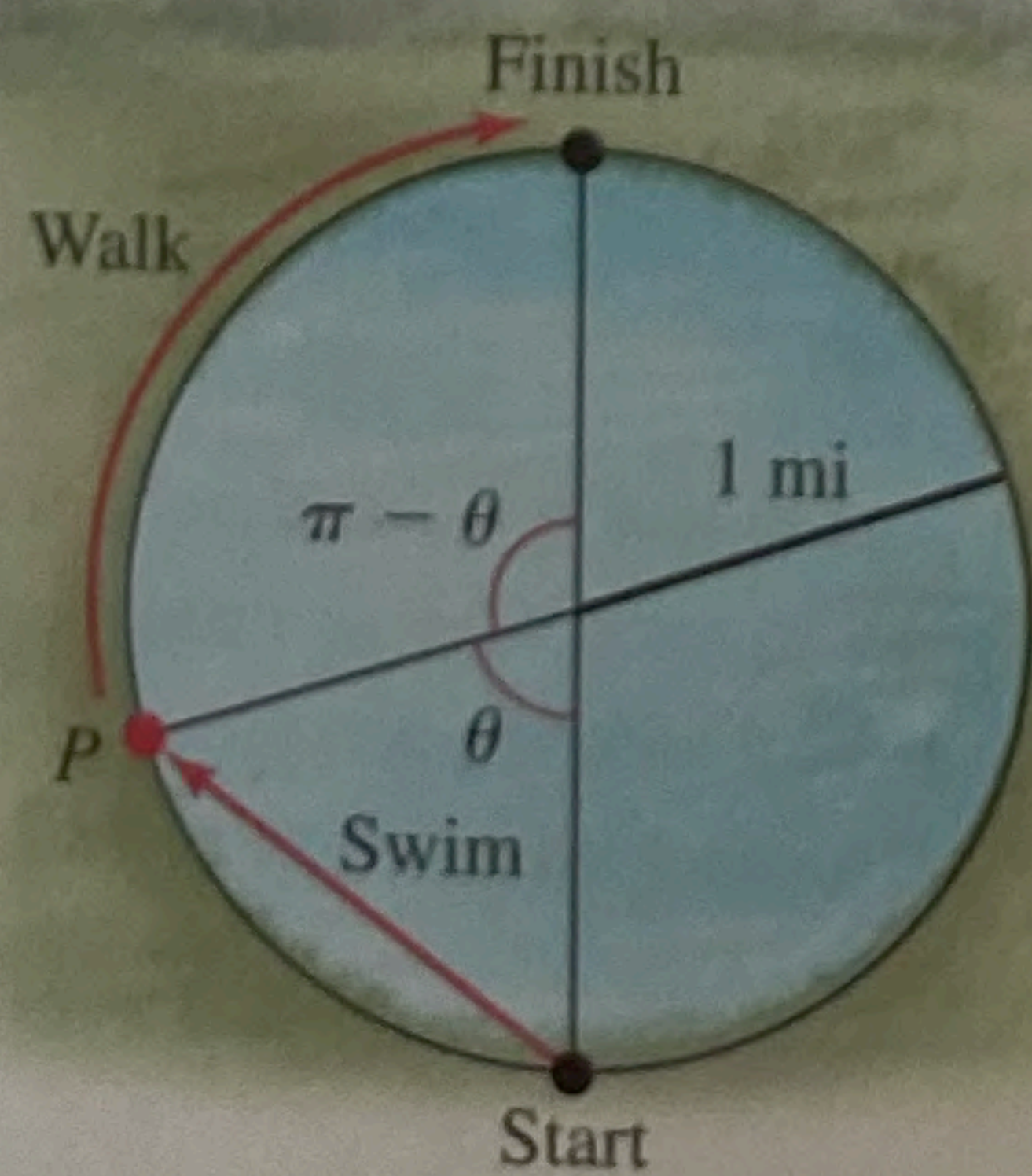


Figure 4.58

► To show that the chord length of a circle is $2r \sin(\theta/2)$, draw a line from the center of the circle to the midpoint of the chord (Figure 4.59). This line bisects the angle θ . Using a right triangle, half the length of the chord is $r \sin(\theta/2)$.

EXAMPLE 2 Airline regulations Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?

SOLUTION We sketch a square-based box whose length and width are w and whose height is h (Figure 4.56). By the airline policy, the constraint is $2w + h = 64$. The objective function is the volume, $V = w^2h$. Either w or h may be eliminated from the objective function; the constraint $h = 64 - 2w$ implies that the volume is

$$V = w^2h = w^2(64 - 2w) = 64w^2 - 2w^3.$$

The objective function has now been expressed in terms of a single variable. Notice that w is nonnegative and cannot exceed 32, so the domain of V is $0 \leq w \leq 32$. The critical points satisfy

$$V'(w) = 128w - 6w^2 = 2w(64 - 3w) = 0,$$

which has roots $w = 0$ and $w = \frac{64}{3}$. By the First (or Second) Derivative Test, $w = \frac{64}{3}$ corresponds to a local maximum. At the endpoints, $V(0) = V(32) = 0$. Therefore, the volume function has an absolute maximum of $V(64/3) \approx 9709 \text{ in}^3$. The dimensions of the optimal box are $w = 64/3$ in and $h = 64 - 2w = 64/3$ in, so the optimal box is a cube. A graph of the volume function is shown in Figure 4.57.

Related Exercises 19–20 ◀

Optimization Guidelines With two examples providing some insight, we present a procedure for solving optimization problems. These guidelines provide a general framework, but the details may vary depending on the problem.

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

EXAMPLE 3 Walking and swimming Suppose you are standing on the shore of a circular pond with a radius of 1 mile and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 mi/hr from your current position to another point P on the shore and then walk at 3 mi/hr along the shore to the terminal point (Figure 4.58). How should you choose P to minimize the total time for the trip?

SOLUTION As shown in Figure 4.58, the initial point is chosen arbitrarily, and the terminal point is at the other end of a diameter. The easiest way to describe the transition point P is to refer to the central angle θ . If $\theta = 0$, then the entire trip is done by walking; if $\theta = \pi$, the entire trip is done by swimming. So the interval of interest is $0 \leq \theta \leq \pi$. The objective function is the total travel time as it varies with θ . For each leg of the trip (swim and walk), the travel time is the distance traveled divided by the speed. We need a few facts from circular geometry. The length of the swimming leg is the length of the chord of the circle corresponding to the angle θ . For a circle of radius r , this chord

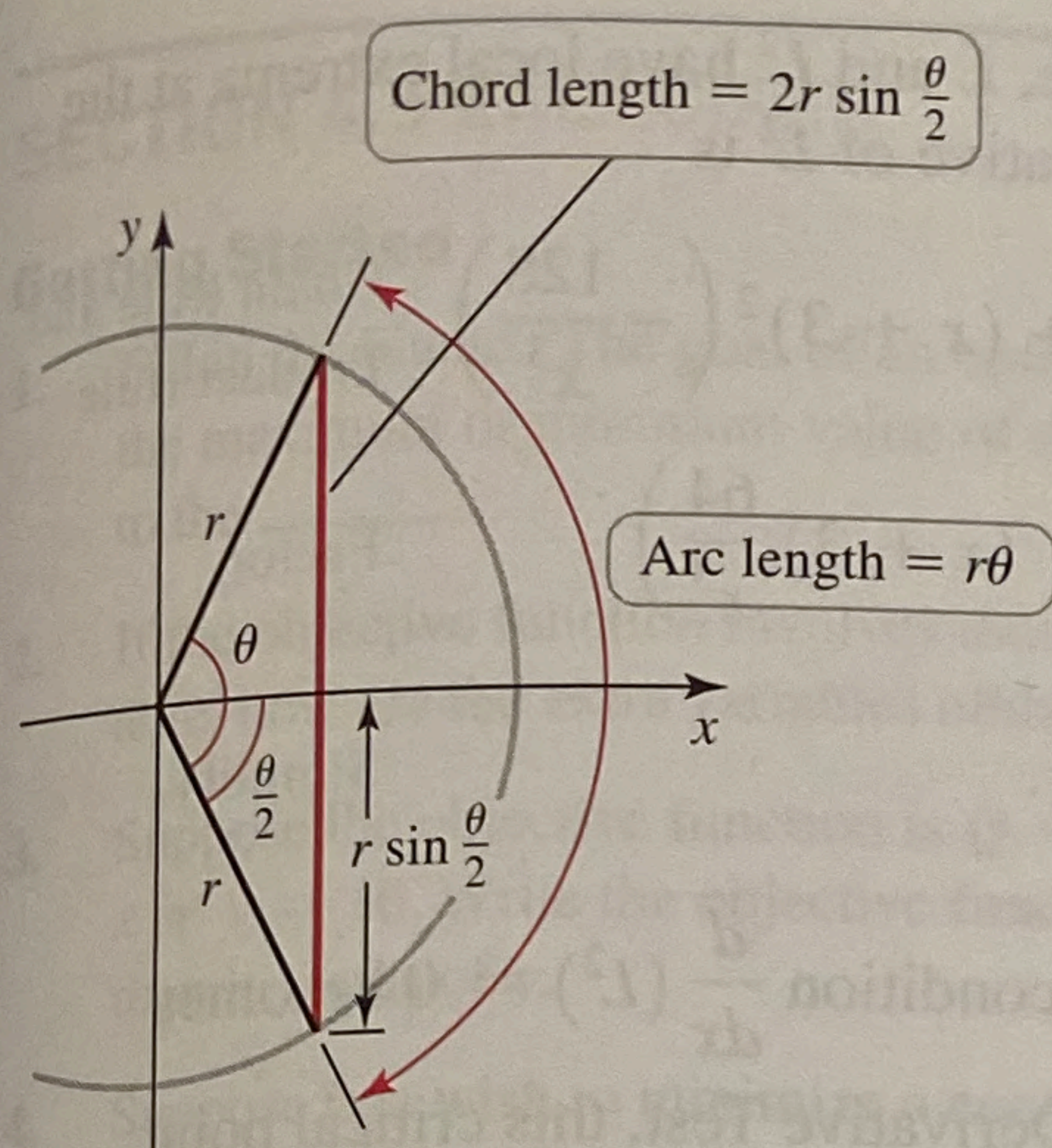


Figure 4.59

- You can check two special cases: If the entire trip is done walking, the travel time is $(\pi \text{ mi}) / (3 \text{ mi/hr}) \approx 1.05 \text{ hr}$. If the entire trip is done swimming, the travel time is $(2 \text{ mi}) / (2 \text{ mi/hr}) = 1 \text{ hr}$.

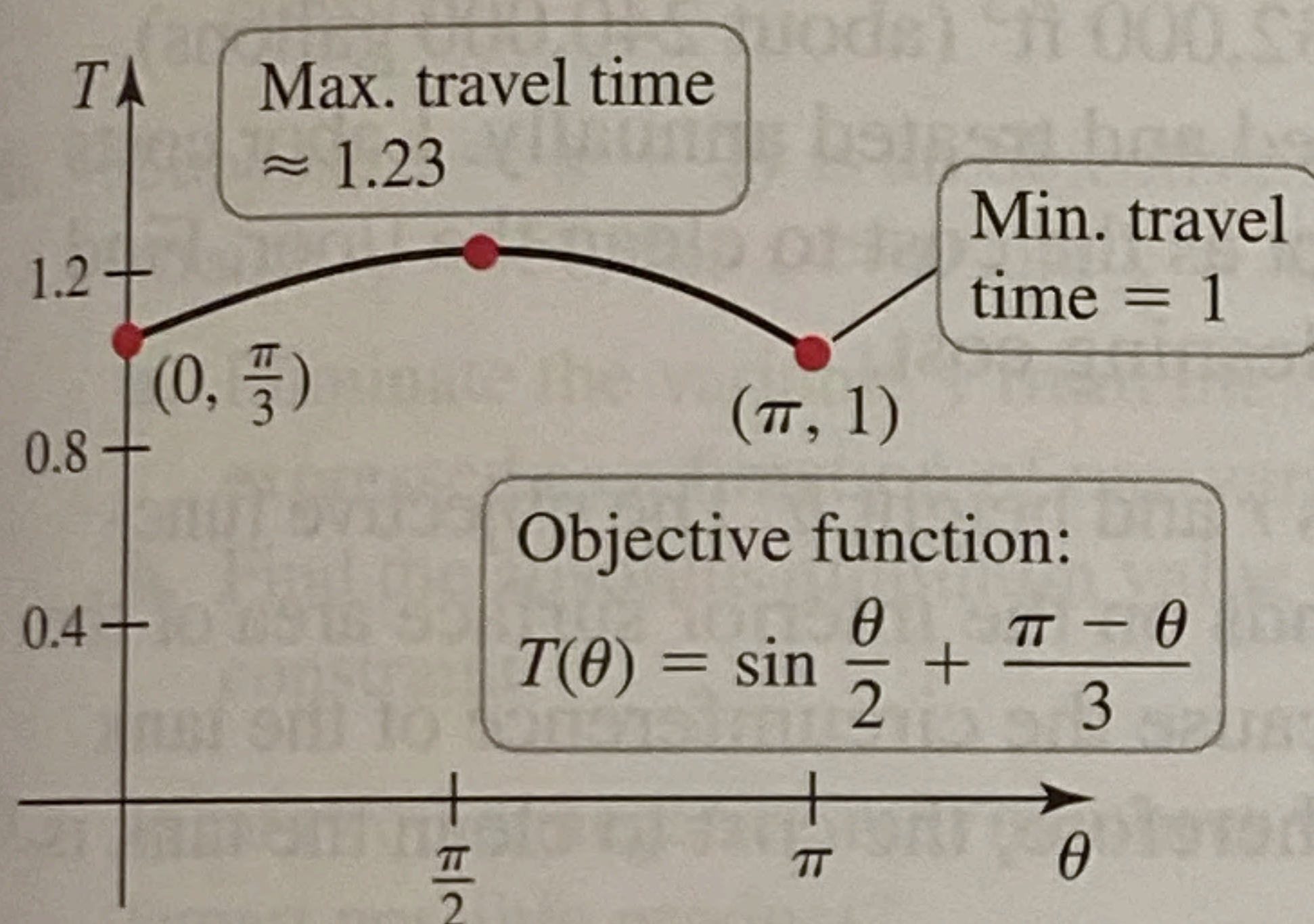


Figure 4.60

length is given by $2r \sin(\theta/2)$ (Figure 4.59). So the time for the swimming leg (with $r = 1$ and a speed of 2 mi/hr) is

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{2 \sin(\theta/2)}{2} = \sin \frac{\theta}{2}.$$

The length of the walking leg is the length of the arc of the circle corresponding to the angle $\pi - \theta$. For a circle of radius r , the arc length corresponding to an angle θ is $r\theta$ (Figure 4.59). Therefore, the time for the walking leg (with an angle $\pi - \theta$, $r = 1$, and a speed of 3 mi/hr) is

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{\pi - \theta}{3}.$$

The total travel time for the trip (in hours) is the objective function

$$T(\theta) = \sin \frac{\theta}{2} + \frac{\pi - \theta}{3}, \quad \text{for } 0 \leq \theta \leq \pi.$$

We now analyze the objective function. The critical points of T satisfy

$$\frac{dT}{d\theta} = \frac{1}{2} \cos \frac{\theta}{2} - \frac{1}{3} = 0 \quad \text{or} \quad \cos \frac{\theta}{2} = \frac{2}{3}.$$

Using a calculator, the only solution in the interval $[0, \pi]$ is $\theta \approx 1.68 \text{ rad} \approx 96^\circ$, which is the critical point.

Evaluating the objective function at the critical point and at the endpoints, we find that $T(1.68) \approx 1.23 \text{ hr}$, $T(0) = \pi/3 \approx 1.05 \text{ hr}$, and $T(\pi) = 1 \text{ hr}$. We conclude that the minimum travel time is $T(\pi) = 1 \text{ hr}$ when the entire trip is done swimming. The maximum travel time, corresponding to $\theta \approx 96^\circ$, is $T \approx 1.23 \text{ hr}$.

The objective function is shown in Figure 4.60. In general, the maximum and minimum travel times depend on the walking and swimming speeds (Exercise 26).

Related Exercises 26–27 ◀

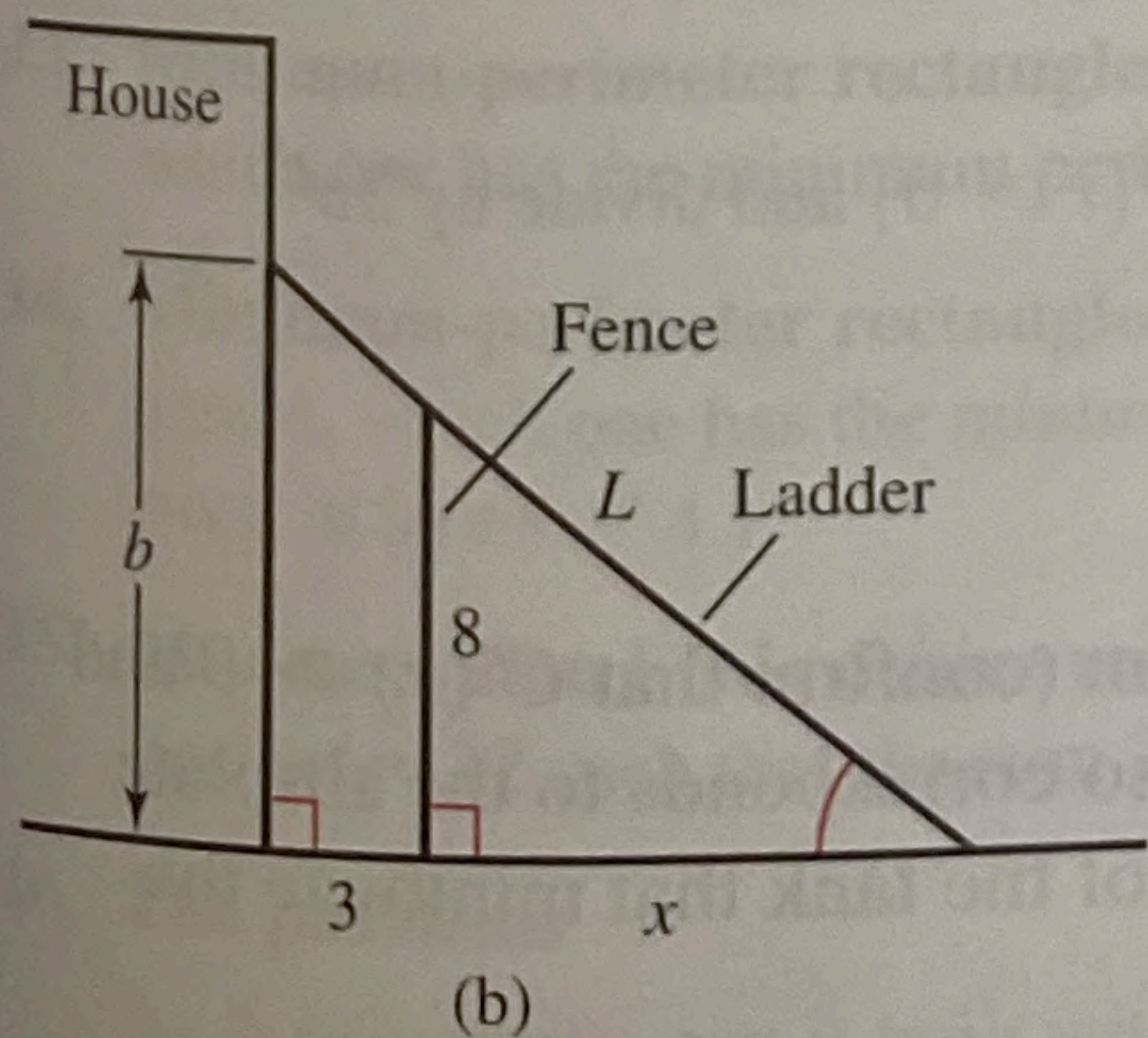
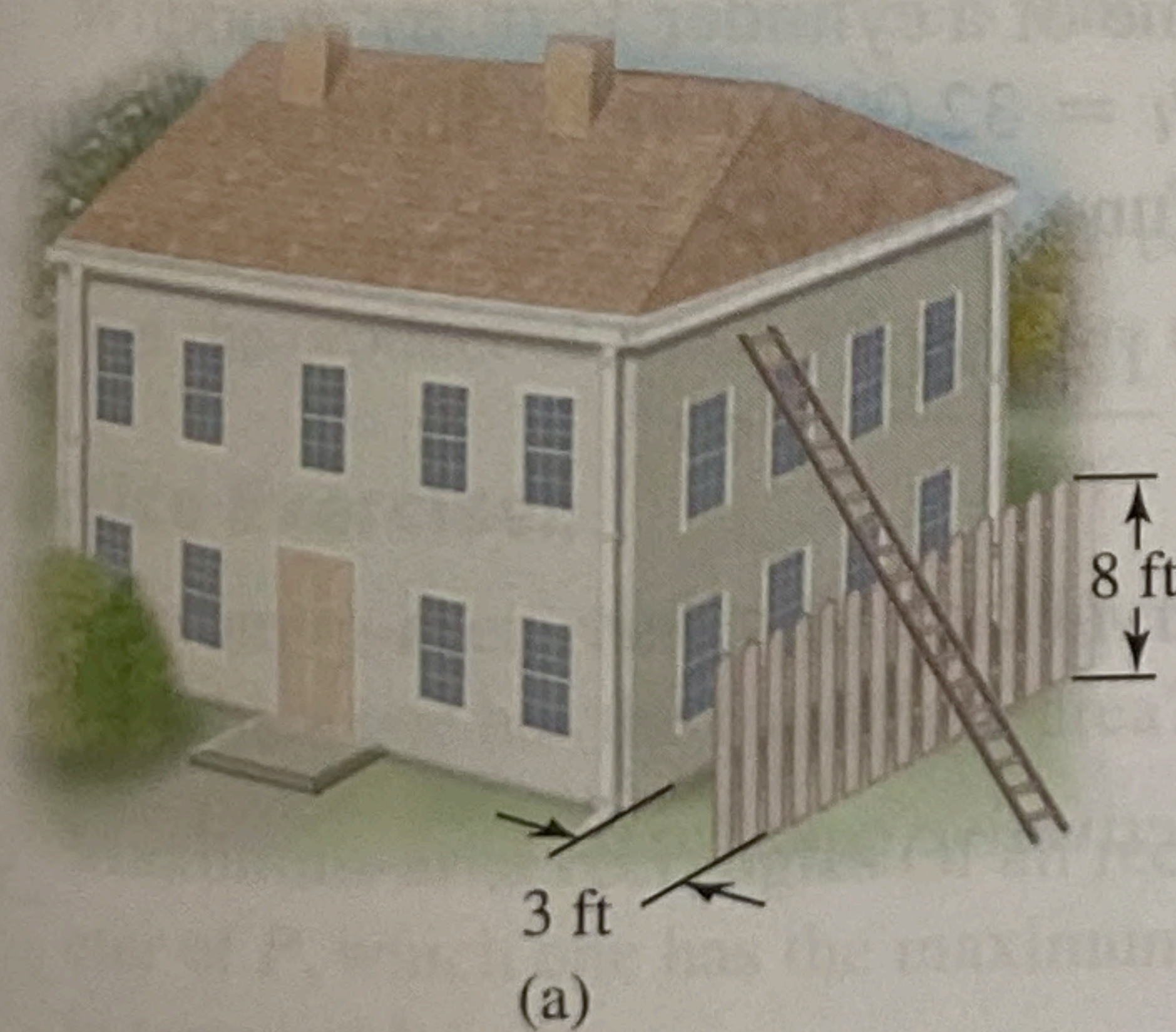


Figure 4.61

EXAMPLE 4 Ladder over the fence An 8-foot-tall fence runs parallel to the side of a house 3 feet away (Figure 4.61a). What is the length of the shortest ladder that clears the fence and reaches the house? Assume the vertical wall of the house and the horizontal ground have infinite extent (see Exercise 31 for more realistic assumptions).

SOLUTION Let's first ask why we expect a minimum ladder length. You could put the foot of the ladder far from the fence so that it clears the fence at a shallow angle, but the ladder would be long. Or you could put the foot of the ladder close to the fence so that it clears the fence at a steep angle, but again, the ladder would be long. Somewhere between these extremes is a ladder position that minimizes the ladder length.

The objective function in this problem is the ladder length L . The position of the ladder is specified by x , the distance between the foot of the ladder and the fence (Figure 4.61b). The goal is to express L as a function of x , where $x > 0$.

The Pythagorean theorem gives the relationship

$$L^2 = (x + 3)^2 + b^2,$$

where b is the height of the top of the ladder above the ground. Similar triangles give the constraint $8/x = b/(3 + x)$. We now solve the constraint equation for b and substitute to express L^2 in terms of x :

$$L^2 = (x + 3)^2 + \left(\frac{8(x + 3)}{x} \right)^2 = (x + 3)^2 \left(1 + \frac{64}{x^2} \right).$$

At this juncture, we could find the critical points of L by first solving the preceding equation for L and then solving $L' = 0$. However, the solution is simplified considerably

if we note that L is a nonnegative function. Therefore, L and L^2 have local extrema at the same points, so we choose to minimize L^2 . The derivative of L^2 is

$$\begin{aligned} \frac{d}{dx} \left((x+3)^2 \left(1 + \frac{64}{x^2} \right) \right) &= 2(x+3) \left(1 + \frac{64}{x^2} \right) + (x+3)^2 \left(-\frac{128}{x^3} \right) && \text{Chain Rule and Product Rule} \\ &= 2(x+3) \left(1 + \frac{64}{x^2} - (x+3) \frac{64}{x^3} \right) && \text{Factor.} \\ &= \frac{2(x+3)(x^3 - 192)}{x^3}. && \text{Simplify.} \end{aligned}$$

Because $x > 0$, we have $x + 3 \neq 0$; therefore, the condition $\frac{d}{dx}(L^2) = 0$ becomes $x^3 - 192 = 0$, or $x = 4\sqrt[3]{3} \approx 5.77$. By the First Derivative Test, this critical point corresponds to a local minimum. By Theorem 4.11, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore, the minimum ladder length occurs when the foot of the ladder is approximately 5.77 ft from the fence. We find that $L^2(5.77) \approx 224.77$ and the minimum ladder length is $\sqrt{224.77} \approx 15$ ft.

Related Exercises 30–31 ◀

EXAMPLE 5 Water tower A water storage tank in a small community is built in the shape of a right circular cylinder with a capacity of 32,000 ft³ (about 240,000 gallons). The interior wall and floor of the tank must be cleaned and treated annually. Labor costs for cleaning the wall are twice as high per square foot as the cost to clean the floor. Find the radius and height of the tank that minimize the cleaning cost.

SOLUTION Figure 4.62 shows a water tank with radius r and height h . The objective function describes the cost to clean the tank, which depends on the interior surface area of the tank. The area of the tank's circular floor is πr^2 . Because the circumference of the tank is $2\pi r$, the lateral surface area of the wall is $2\pi rh$. Therefore, the cost to clean the tank is proportional to

$$C = 2 \cdot \underbrace{2\pi rh}_{\text{area of wall}} + \underbrace{\pi r^2}_{\text{area of floor}}. \quad \text{It costs twice as much per square foot to clean the wall, so the area of the wall is multiplied by 2.}$$

To find a constraint equation, we start with the volume of a cylinder $V = \pi r^2 h$ and replace V with 32,000 to obtain $\pi r^2 h = 32,000$, or $h = 32,000/(\pi r^2)$. Substituting this expression for h into the cost equation, we obtain a function of the radius:

$$C(r) = 4\pi r \frac{32,000}{\pi r^2} + \pi r^2 = \frac{128,000}{r} + \pi r^2.$$

To minimize C , we find its critical points on the interval $0 < r < \infty$:

$$C'(r) = -\frac{128,000}{r^2} + 2\pi r = 0 \quad \text{Set the derivative equal to 0.}$$

$$2\pi r = \frac{128,000}{r^2} \quad \text{Rearrange equation.}$$

$$r^3 = \frac{64,000}{\pi} \quad \text{Multiply by } r^2 (r > 0) \text{ and divide by } 2\pi.$$

$$r = \frac{40}{\sqrt[3]{\pi}} \approx 27.3 \text{ ft.} \quad \text{Solve for } r.$$

This solitary critical point corresponds to a local minimum (confirm that $C''(r) > 0$ and use the Second Derivative Test), so by Theorem 4.9, it also corresponds to the absolute minimum. Because $h = 32,000/(\pi r^2)$, the dimensions of the tank that minimize the cleaning cost are

$$r \approx 27.3 \text{ ft} \quad \text{and} \quad h = 32,000/(\pi r^2) \approx 13.7 \text{ ft.}$$

A bit of algebra shows that r is exactly twice as large as h , so the optimal tank is 4 times as wide as it is high.

Related Exercises 21, 35, 42 ◀



Figure 4.62

- We don't need to know the actual cost to clean the tank. Whatever it costs (per square foot) to clean the floor, it costs twice as much to clean the wall, which implies that the objective function in Example 5 is a constant multiple of the actual cleaning cost. Because $\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$, both $f(x)$ and $kf(x)$ share the same critical points, so either function yields the solution.

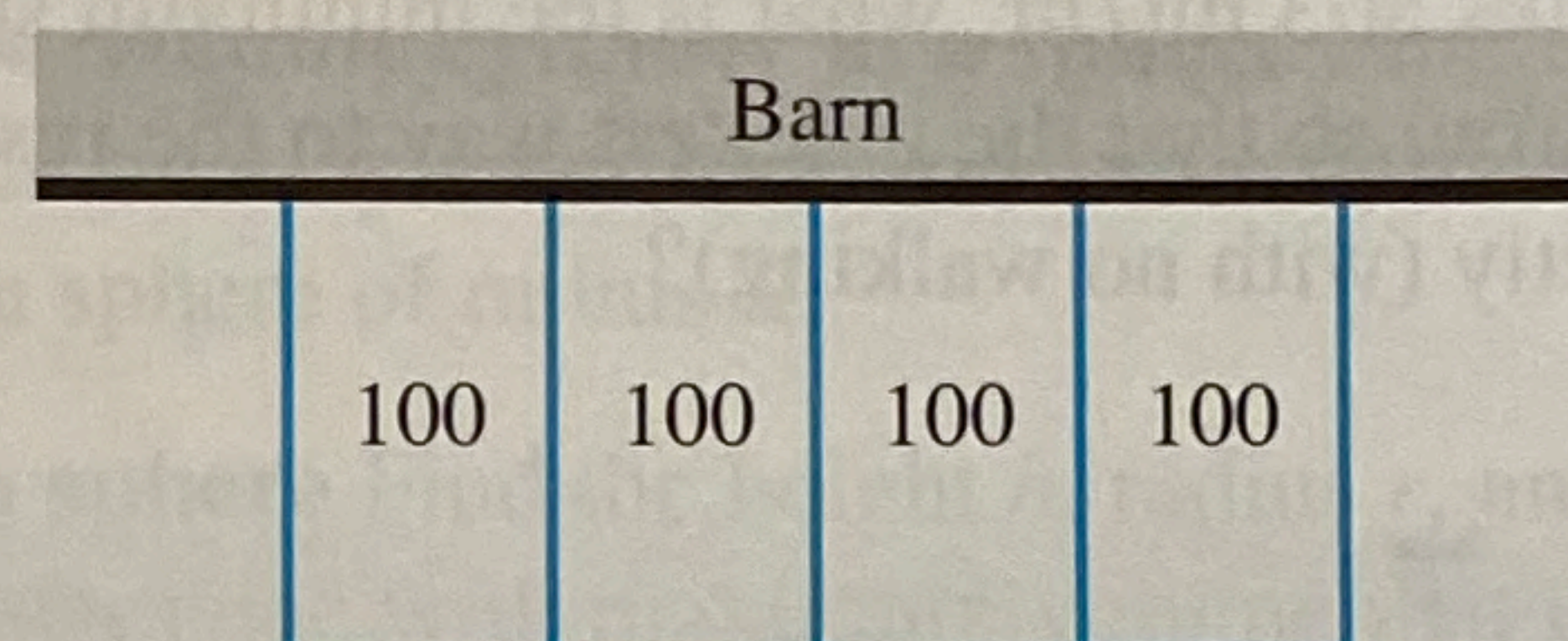
SECTION 4.5 EXERCISES

Getting Started

- Fill in the blanks: The goal of an optimization problem is to find the maximum or minimum value of the _____ function subject to the _____.
- If the objective function involves more than one independent variable, how are the extra variables eliminated?
- Suppose the objective function is $Q = x^2y$ and you know that $x + y = 10$. Write the objective function first in terms of x and then in terms of y .
- Suppose you wish to minimize a continuous objective function on a closed interval, but you find that it has only a single local maximum. Where should you look for the solution to the problem?
- Suppose the objective function $P = xy$ is subject to the constraint $10x + y = 100$, where x and y are real numbers.
 - Eliminate the variable y from the objective function so that P is expressed as a function of one variable x .
 - Find the absolute maximum value of P subject to the given constraint.
- Suppose $S = x + 2y$ is an objective function subject to the constraint $xy = 50$, for $x > 0$ and $y > 0$.
 - Eliminate the variable y from the objective function so that S is expressed as a function of one variable x .
 - Find the absolute minimum value of S subject to the given constraint.
- What two nonnegative real numbers with a sum of 23 have the largest possible product?
- What two nonnegative real numbers a and b whose sum is 23 maximize $a^2 + b^2$? Minimize $a^2 + b^2$?
- What two positive real numbers whose product is 50 have the smallest possible sum?
- Find numbers x and y satisfying the equation $3x + y = 12$ such that the product of x and y is as large as possible.

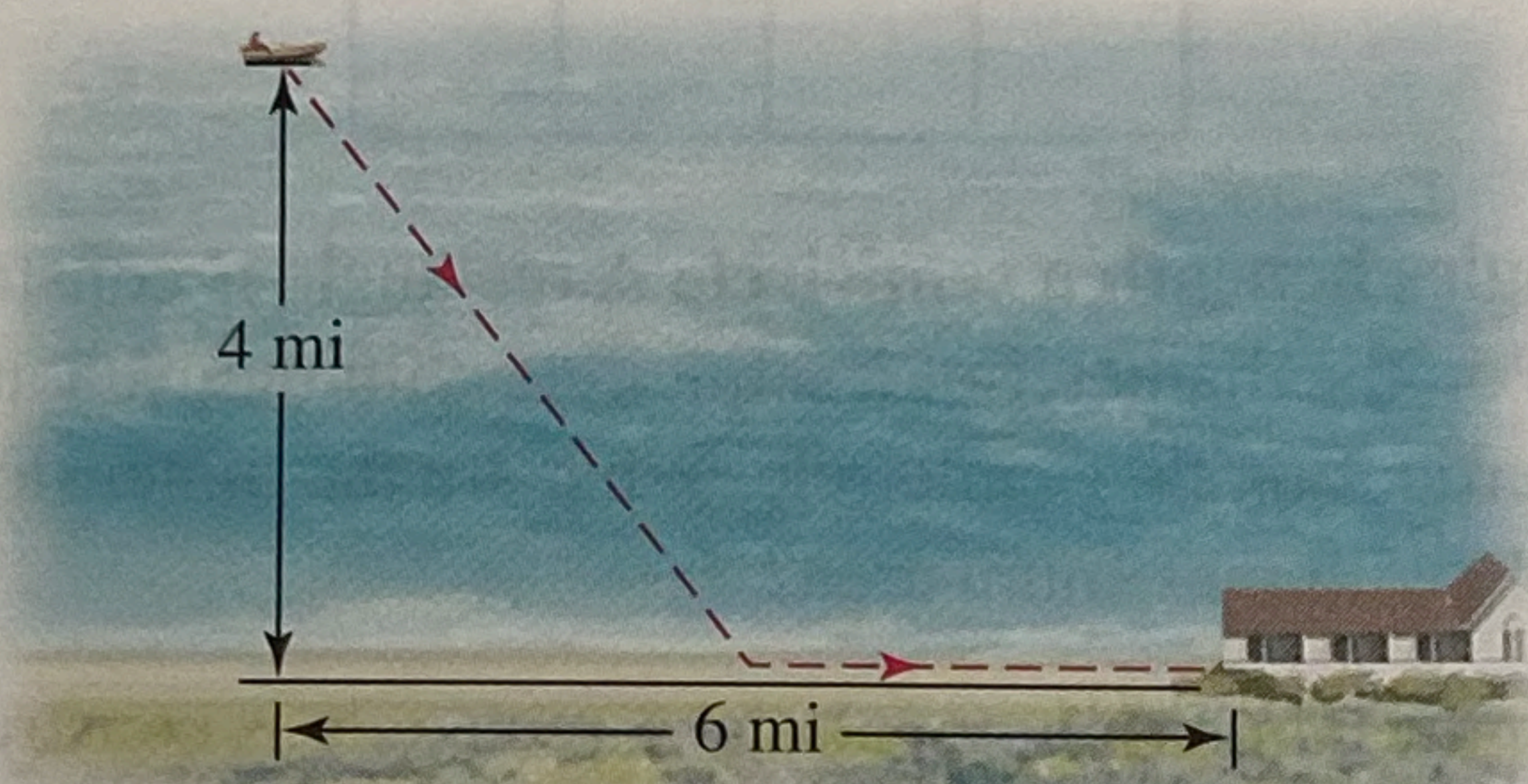
Practice Exercises

- Maximum-area rectangles** Of all rectangles with a perimeter of 10, which one has the maximum area? (Give the dimensions.)
- Maximum-area rectangles** Of all rectangles with a fixed perimeter of P , which one has the maximum area? (Give the dimensions in terms of P .)
- Minimum-perimeter rectangles** Of all rectangles of area 100, which one has the minimum perimeter?
- Minimum-perimeter rectangles** Of all rectangles with a fixed area A , which one has the minimum perimeter? (Give the dimensions in terms of A .)
- Minimum sum** Find positive numbers x and y satisfying the equation $xy = 12$ such that the sum $2x + y$ is as small as possible.
- Pen problems**
 - A rectangular pen is built with one side against a barn. Two hundred meters of fencing are used for the other three sides of the pen. What dimensions maximize the area of the pen?
 - A rancher plans to make four identical and adjacent rectangular pens against a barn, each with an area of 100 m^2 (see figure). What are the dimensions of each pen that minimize the amount of fence that must be used?

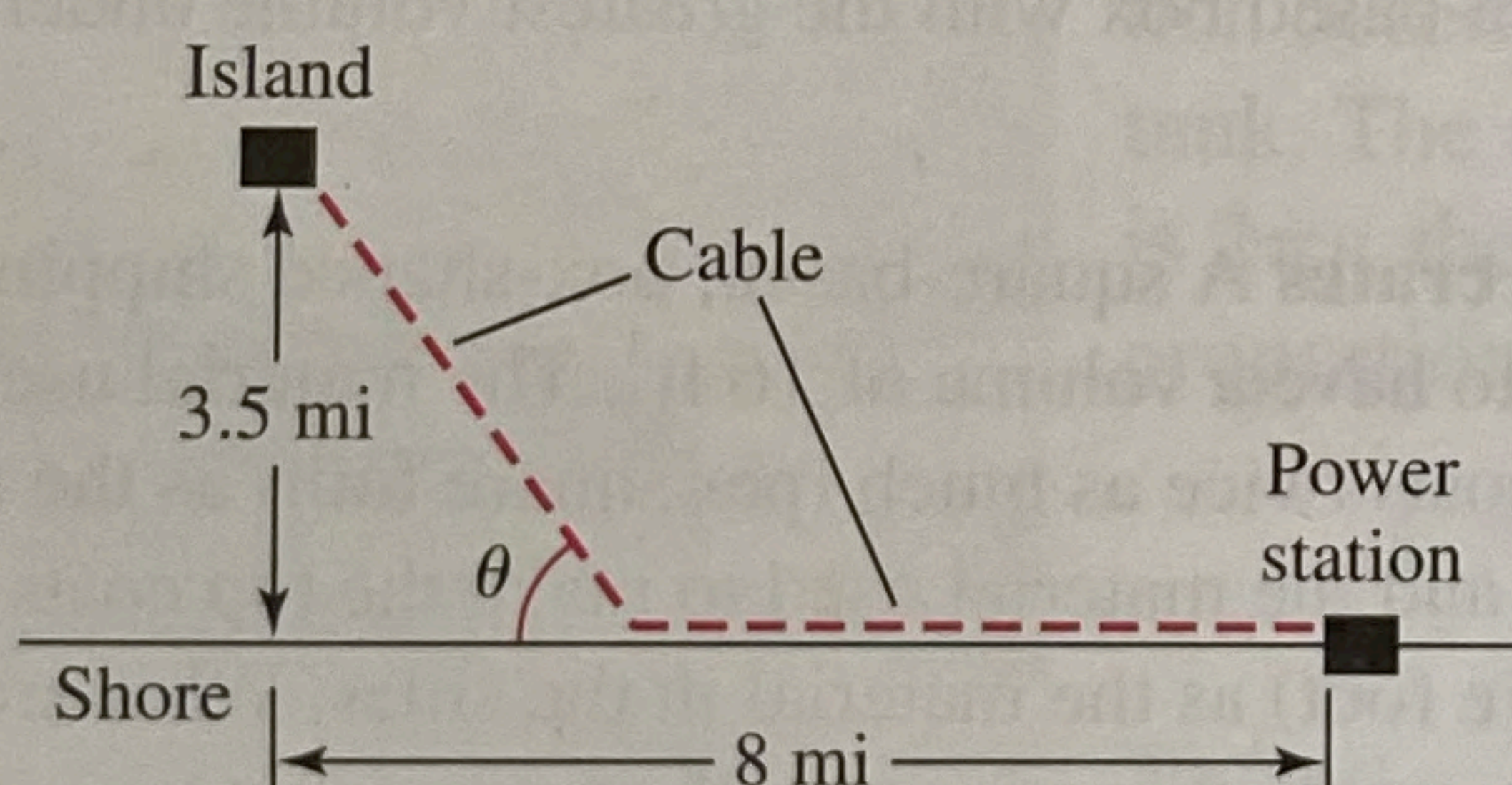


- Rectangles beneath a semicircle** A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?
- Rectangles beneath a parabola** A rectangle is constructed with its base on the x -axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?
- Minimum-surface-area box** Of all boxes with a square base and a volume of 8 m^3 , which one has the minimum surface area? (Give its dimensions.)
- Maximum-volume box** Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 108 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?
- Shipping crates** A square-based, box-shaped shipping crate is designed to have a volume of 16 ft^3 . The material used to make the base costs twice as much (per square foot) as the material in the sides, and the material used to make the top costs half as much (per square foot) as the material in the sides. What are the dimensions of the crate that minimize the cost of materials?
- Closest point on a line** What point on the line $y = 3x + 4$ is closest to the origin?
- Closest point on a curve** What point on the parabola $y = 1 - x^2$ is closest to the point $(1, 1)$?
- Minimum distance** Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$? (*Hint:* Use synthetic division.)
- Minimum distance** Find the point P on the line $y = 3x$ that is closest to the point $(50, 0)$. What is the least distance between P and $(50, 0)$?
- Walking and swimming** A man wishes to get from an initial point on the shore of a circular pond with radius 1 mi to a point on the shore directly opposite (on the other end of the diameter). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.
 - If he swims at 2 mi/hr and walks at 4 mi/hr, what are the maximum and minimum times for the trip?
 - If he swims at 2 mi/hr and walks at 1.5 mi/hr, what are the maximum and minimum times for the trip?
 - If he swims at 2 mi/hr, what is the minimum walking speed for which it is quickest to walk the entire distance?

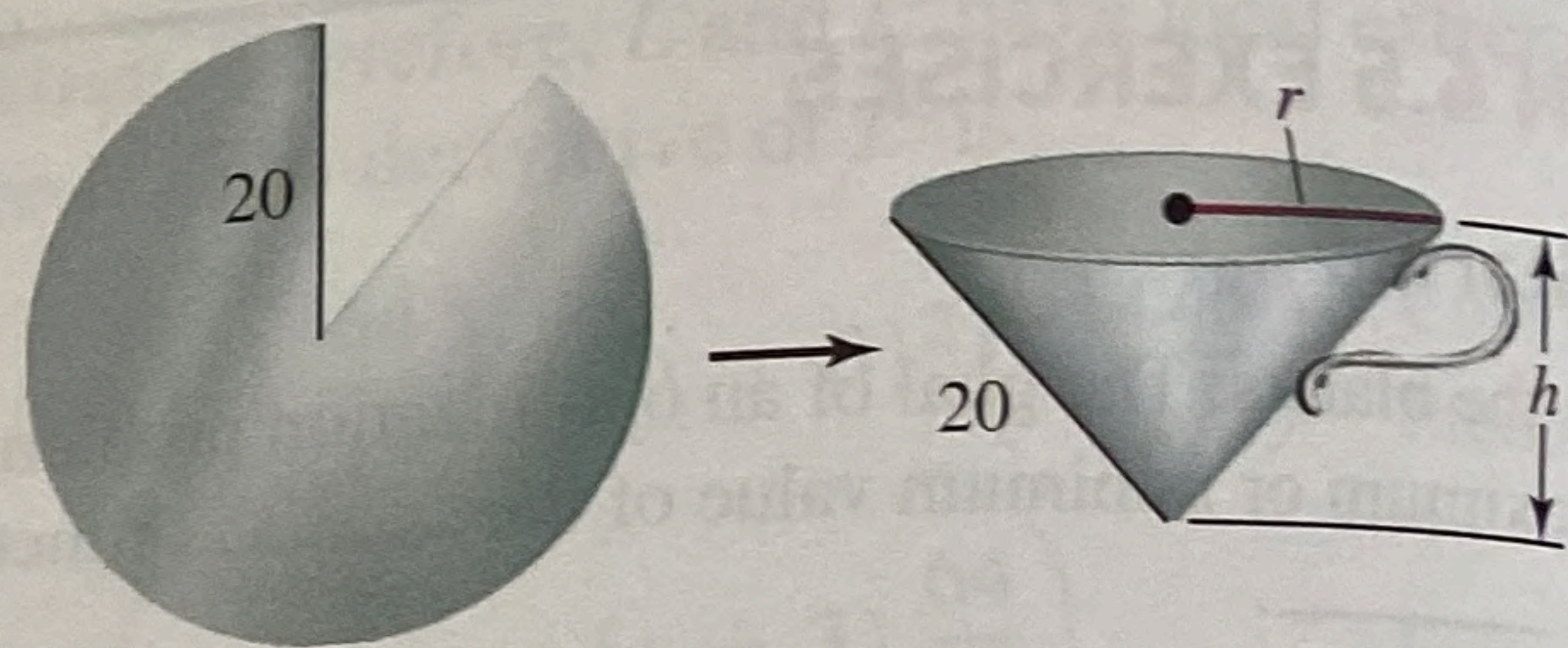
27. **Walking and rowing** A boat on the ocean is 4 mi from the nearest point on a straight shoreline; that point is 6 mi from a restaurant on the shore (see figure). A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.
- If she walks at 3 mi/hr and rows at 2 mi/hr, at which point on the shore should she land to minimize the total travel time?
 - If she walks at 3 mi/hr, what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?



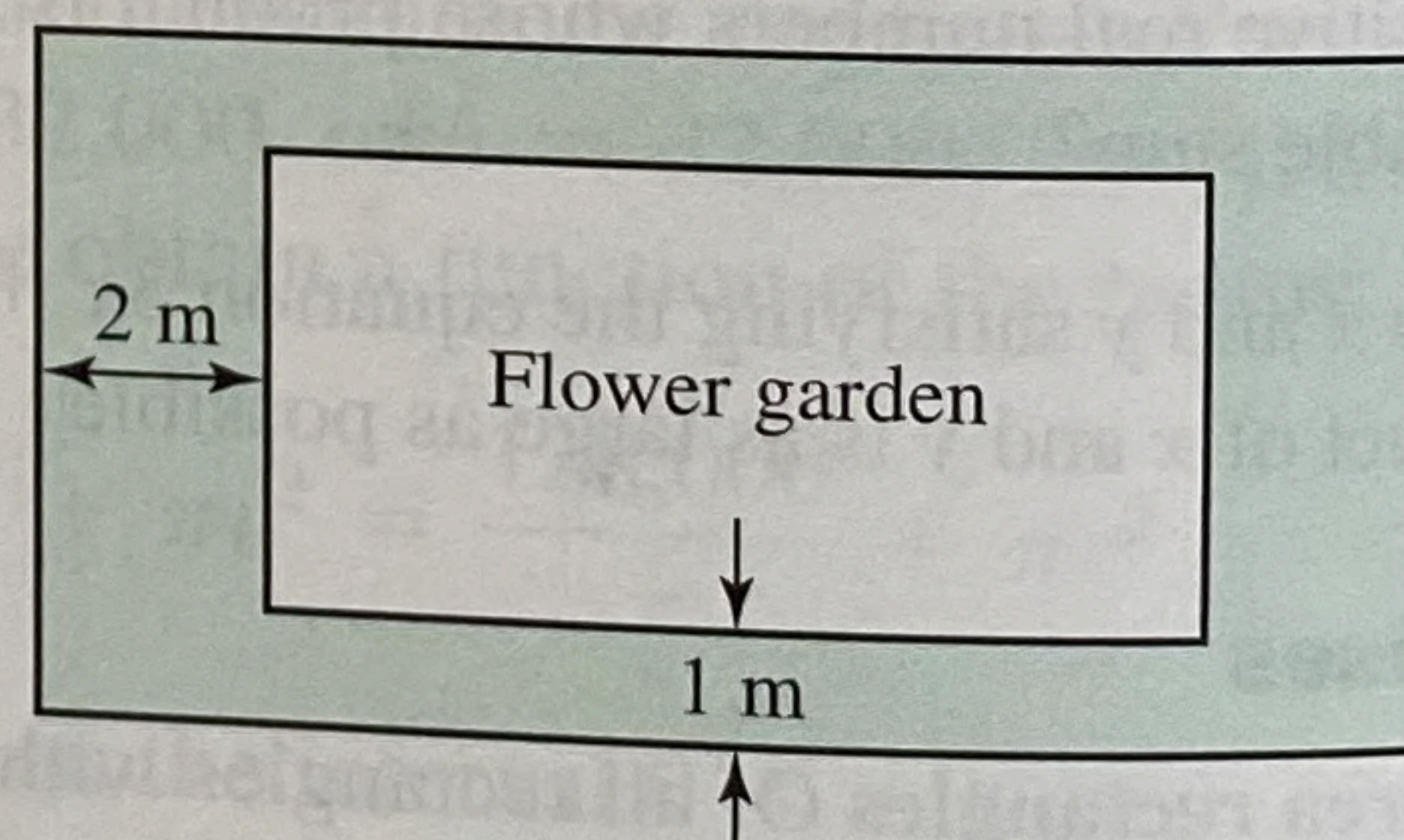
28. **Laying cable** An island is 3.5 mi from the nearest point on a straight shoreline; that point is 8 mi from a power station (see figure). A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume it costs \$2400/mi to lay underwater cable and \$1200/mi to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the project?



29. **Laying cable again** Solve the problem in Exercise 28, but this time minimize the cost with respect to the smaller angle θ between the underwater cable and the shore. (You should get the same answer.)
30. **Shortest ladder** A 10-ft-tall fence runs parallel to the wall of a house at a distance of 4 ft. Find the length of the shortest ladder that extends from the ground to the house without touching the fence. Assume the vertical wall of the house and the horizontal ground have infinite extent.
31. **Shortest ladder—more realistic** An 8-ft-tall fence runs parallel to the wall of a house at a distance of 5 ft. Find the length of the shortest ladder that extends from the ground to the house without touching the fence. Assume the vertical wall of the house is 20 ft high and the horizontal ground extends 20 ft from the fence.
32. **Circle and square** A piece of wire of length 60 is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to (a) maximize and (b) minimize the combined area of the circle and the square?
33. **Maximum-volume cone** A cone is constructed by cutting a sector from a circular sheet of metal with radius 20. The cut sheet is then folded up and welded (see figure). Find the radius and height of the cone with maximum volume that can be formed in this way.

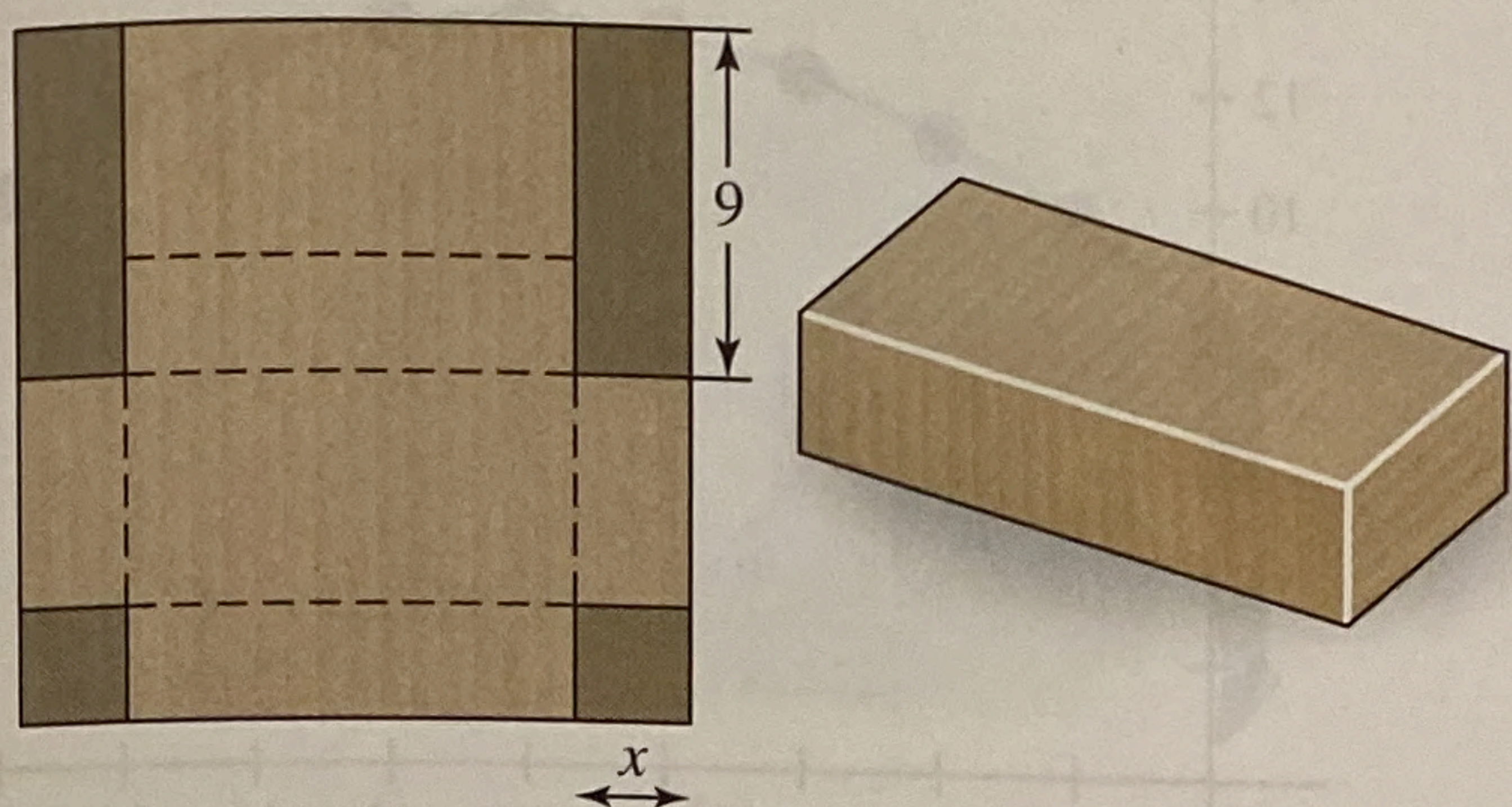


34. **Slant height and cones** Among all right circular cones with a slant height of 3, what are the dimensions (radius and height) that maximize the volume of the cone? The slant height of a cone is the distance from the outer edge of the base to the vertex.
35. **Optimal soda can**
- Classical problem** Find the radius and height of a cylindrical soda can with a volume of 354 cm^3 that minimize the surface area.
 - Real problem** Compare your answer in part (a) to a real soda can, which has a volume of 354 cm^3 , a radius of 3.1 cm, and a height of 12.0 cm, to conclude that real soda cans do not seem to have an optimal design. Then use the fact that real soda cans have a double thickness in their top and bottom surfaces to find the radius and height that minimize the surface area of a real can (the surface areas of the top and bottom are now twice their values in part (a)). Are these dimensions closer to the dimensions of a real soda can?
36. **Covering a marble** Imagine a flat-bottomed cylindrical pot with a circular cross section of radius 4. A marble with radius $0 < r < 4$ is placed in the bottom of the pot. What is the radius of the marble that requires the most water to cover it completely?
37. **Optimal garden** A rectangular flower garden with an area of 30 m^2 is surrounded by a grass border 1 m wide on two sides and 2 m wide on the other two sides (see figure). What dimensions of the garden minimize the combined area of the garden and borders?



38. **Rectangles beneath a line**
- A rectangle is constructed with one side on the positive x -axis, one side on the positive y -axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?
 - Is it possible to construct a rectangle with a greater area than that found in part (a) by placing one side of the rectangle on the line $y = 10 - 2x$ and the two vertices not on that line on the positive x - and y -axes? Find the dimensions of the rectangle of maximum area that can be constructed in this way.
39. **Designing a box** Two squares of length x are cut out of adjacent corners of an $18'' \times 18''$ piece of cardboard and two rectangles of length 9'' and width x are cut out of the other two corners of the cardboard (see figure). The resulting piece of cardboard is then folded along the dashed lines to form an enclosed box.

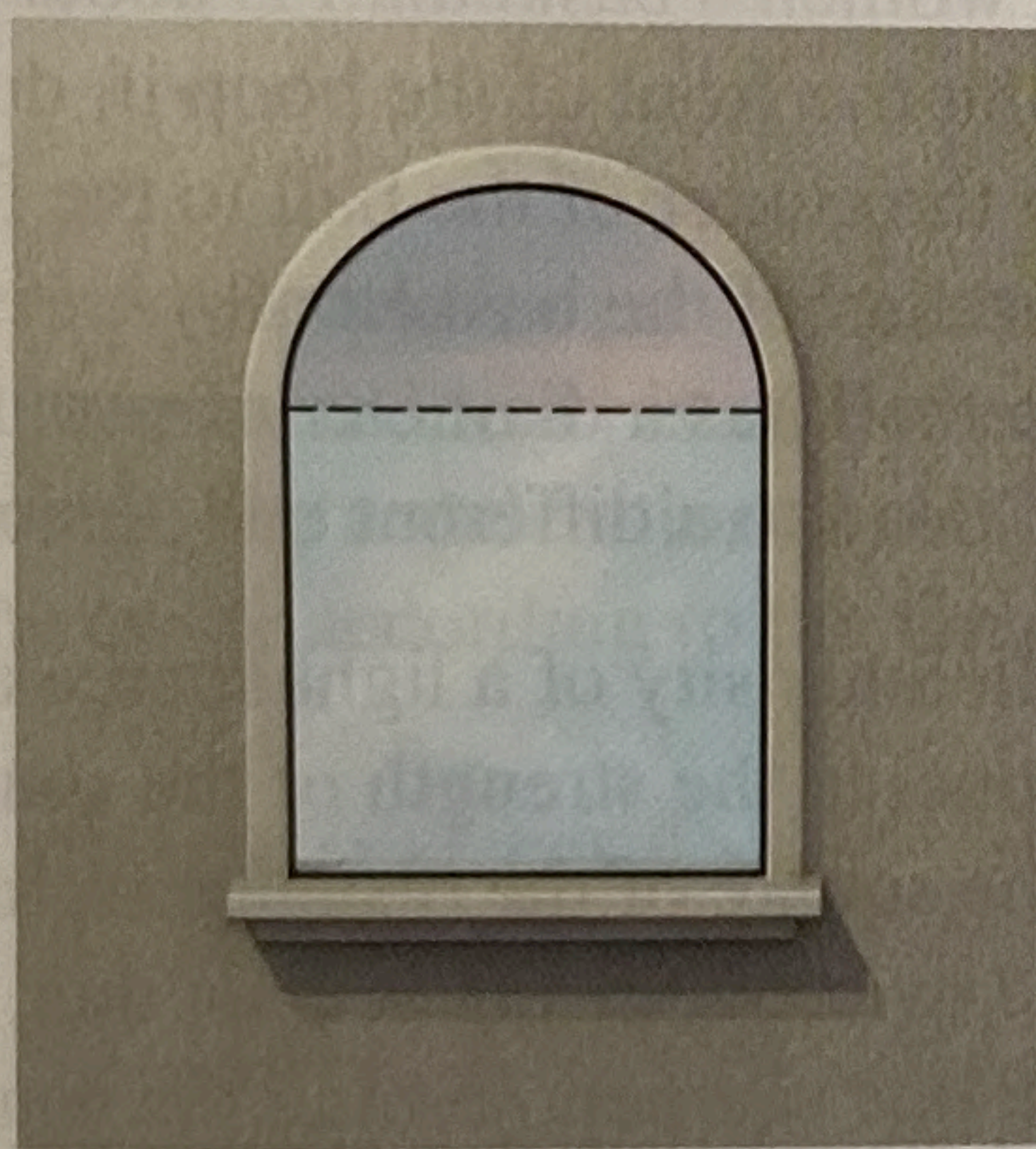
Find the dimensions and volume of the largest box that can be formed in this way.



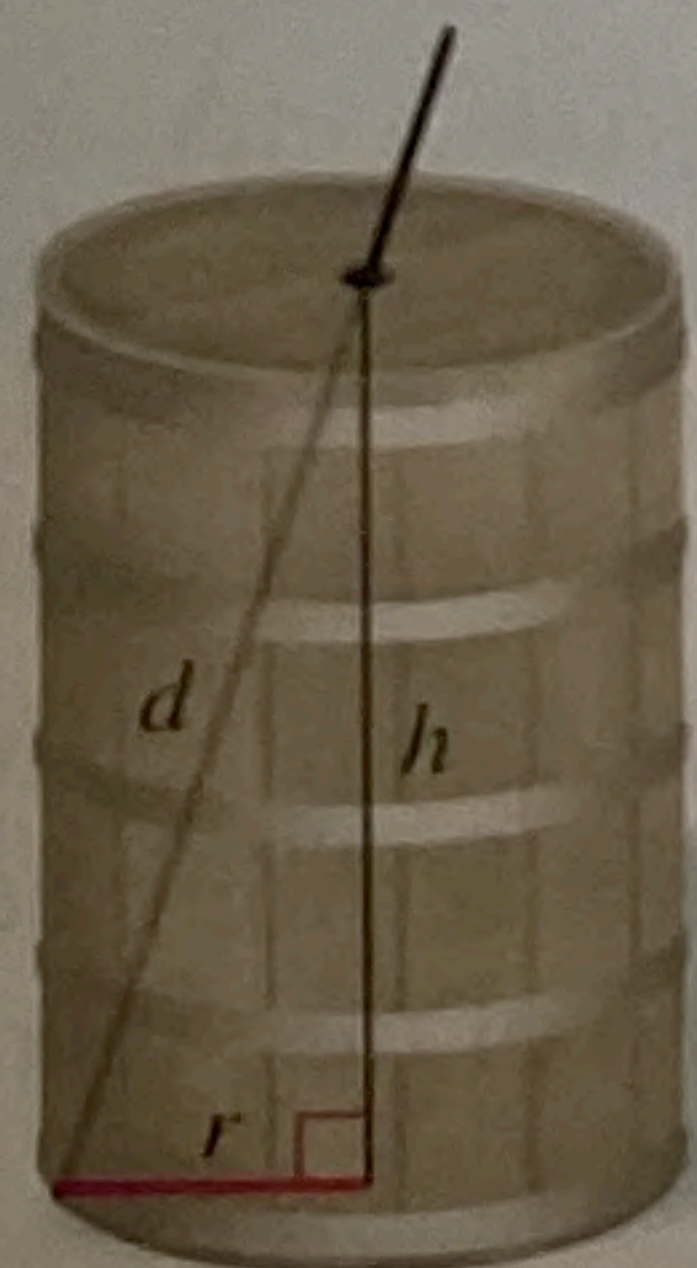
40. Folded boxes

- Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard measuring 5 ft by 8 ft. The resulting piece of cardboard is then folded into a box without a lid. Find the volume of the largest box that can be formed in this way.
- Squares with sides of length x are cut out of each corner of a square piece of cardboard with sides of length ℓ . Find the volume of the largest open box that can be formed in this way.

41. **Norman window** A window consists of a rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

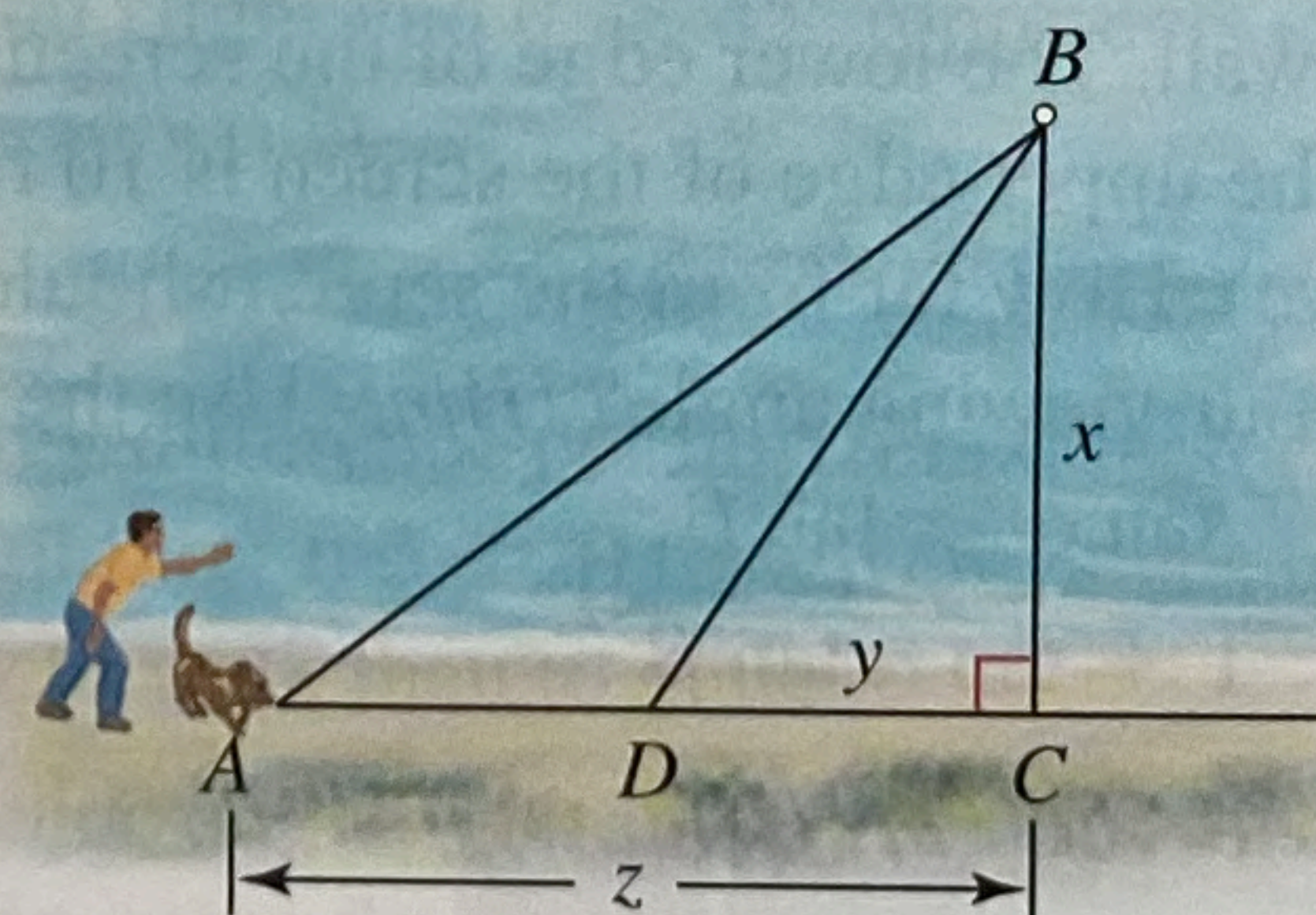


42. **Light transmission** A window consists of a rectangular pane of clear glass surmounted by a semicircular pane of tinted glass. The clear glass transmits twice as much light per unit of surface area as the tinted glass. Of all such windows with a fixed perimeter P , what are the dimensions of the window that transmits the most light?
43. **Kepler's wine barrel** Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one: While shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a centered hole in the top of the barrel to the edge of the base of the barrel (see figure). Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d , the volume varies with the radius r and height h of the barrel. For a fixed value of d , what is the ratio r/h that maximizes the volume of the barrel?



44. **Maximizing profit** Suppose you own a tour bus and you book groups of 20 to 70 people for a day tour. The cost per person is \$30 minus \$0.25 for every ticket sold. If gas and other miscellaneous costs are \$200, how many tickets should you sell to maximize your profit? Treat the number of tickets as a nonnegative real number.
45. **Maximum-volume cylinder in a sphere** Find the dimensions of the right circular cylinder of maximum volume that can be placed inside of a sphere of radius R .
46. **Cone in a sphere** Find the height h , radius r , and volume of a right circular cone with maximum volume that is inscribed in a sphere of radius R .
47. **Cone in a cone** A right circular cone is inscribed inside a larger right circular cone with a volume of 150 cm^3 . The axes of the cones coincide, and the vertex of the inner cone touches the center of the base of the outer cone. Find the ratio of the heights of the cones that maximizes the volume of the inner cone.
48. **Do dogs know calculus?** A mathematician stands on a beach with his dog at point A . He throws a tennis ball so that it hits the water at point B . The dog, wanting to get to the tennis ball as quickly as possible, runs along the straight beach line to point D and then swims from point D to point B to retrieve his ball. Assume C is the point on the edge of the beach closest to the tennis ball (see figure).

- Assume the dog runs at speed r and swims at speed s , where $r > s$ and both are measured in meters per second. Also assume the lengths of BC , CD , and AC are x , y , and z , respectively. Find a function $T(y)$ representing the total time it takes for the dog to get to the ball.



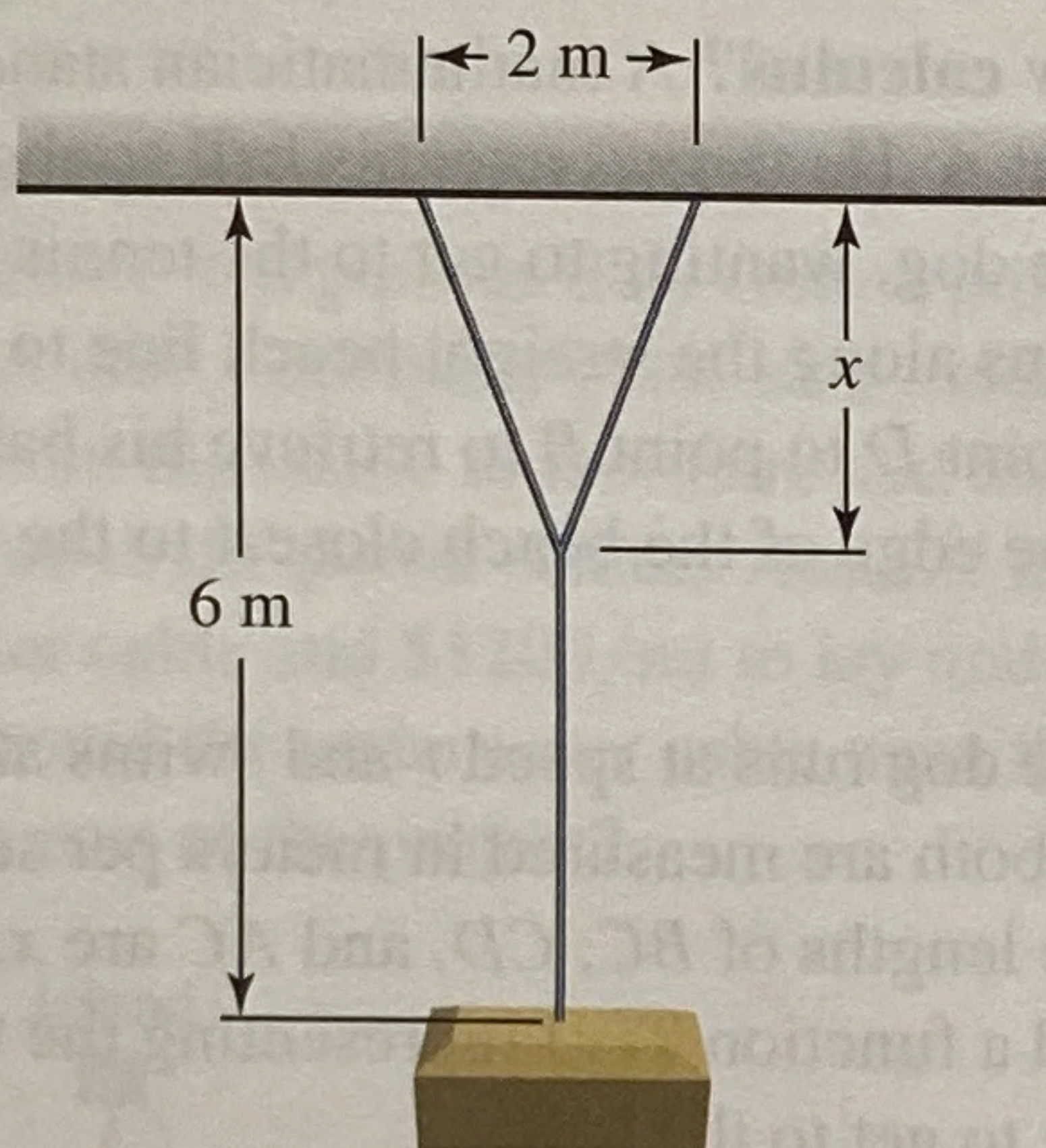
- Verify that the value of y that minimizes the time it takes to retrieve the ball is $y = \frac{x}{\sqrt{r/s + 1}\sqrt{r/s - 1}}$.
- If the dog runs at 8 m/s and swims at 1 m/s, what ratio y/x produces the fastest retrieving time?
- A dog named Elvis who runs at 6.4 m/s and swims at 0.910 m/s was found to use an average ratio of y/x of 0.144 to retrieve his ball. Does Elvis appear to know calculus?

(Source: T. Pennings, *Do Dogs Know Calculus?* *The College Mathematics Journal*, 34, 3, May 2003)

49. **Travel costs** A simple model for travel costs involves the cost of gasoline and the cost of a driver. Specifically, assume gasoline costs $\$p$ /gallon and the vehicle gets g miles per gallon. Also assume the driver earns $\$w$ /hour.
- A plausible function to describe how gas mileage (in mi/gal) varies with speed is $g(v) = v(85 - v)/60$. Evaluate $g(0)$, $g(40)$, and $g(60)$ and explain why these values are reasonable.
 - At what speed does the gas mileage function have its maximum?

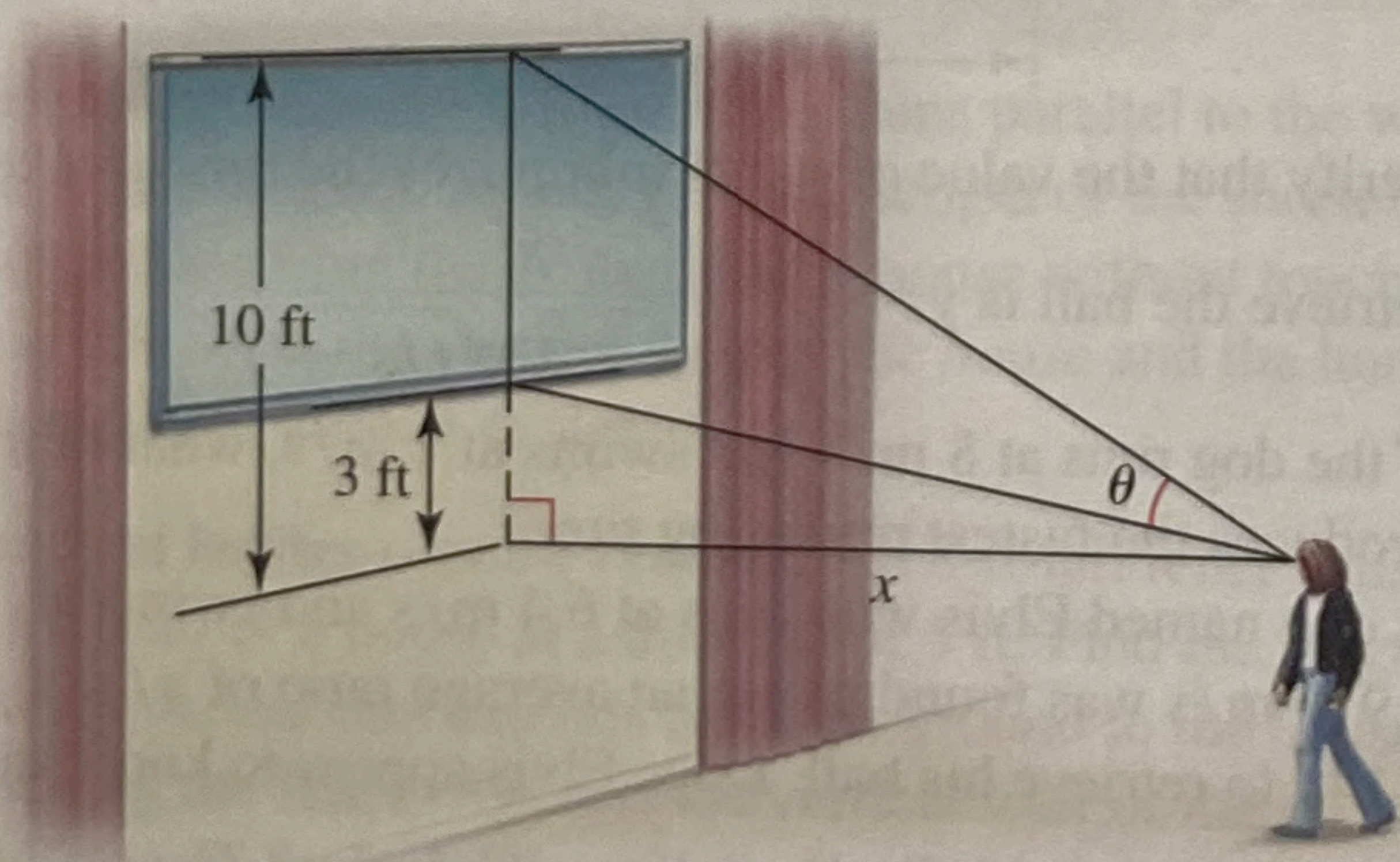
- c. Explain why $C(v) = Lp/g(v) + Lw/v$ gives the cost of the trip in dollars, where L is the length of the trip and v is the constant speed. Show that the dimensions are consistent.
- d. Let $L = 400$ mi, $p = \$4/\text{gal}$, and $w = \$20/\text{hr}$. At what (constant) speed should the vehicle be driven to minimize the cost of the trip?
- e. Should the optimal speed be increased or decreased (compared with part (d)) if L is increased from 400 mi to 500 mi? Explain.
- f. Should the optimal speed be increased or decreased (compared with part (d)) if p is increased from $\$4/\text{gal}$ to $\$4.20/\text{gal}$? Explain.
- g. Should the optimal speed be increased or decreased (compared with part (d)) if w is decreased from $\$20/\text{hr}$ to $\$15/\text{hr}$? Explain.

50. **Suspension system** A load must be suspended 6 m below a high ceiling using cables attached to two supports that are 2 m apart (see figure). How far below the ceiling (x in the figure) should the cables be joined to minimize the total length of cable used?

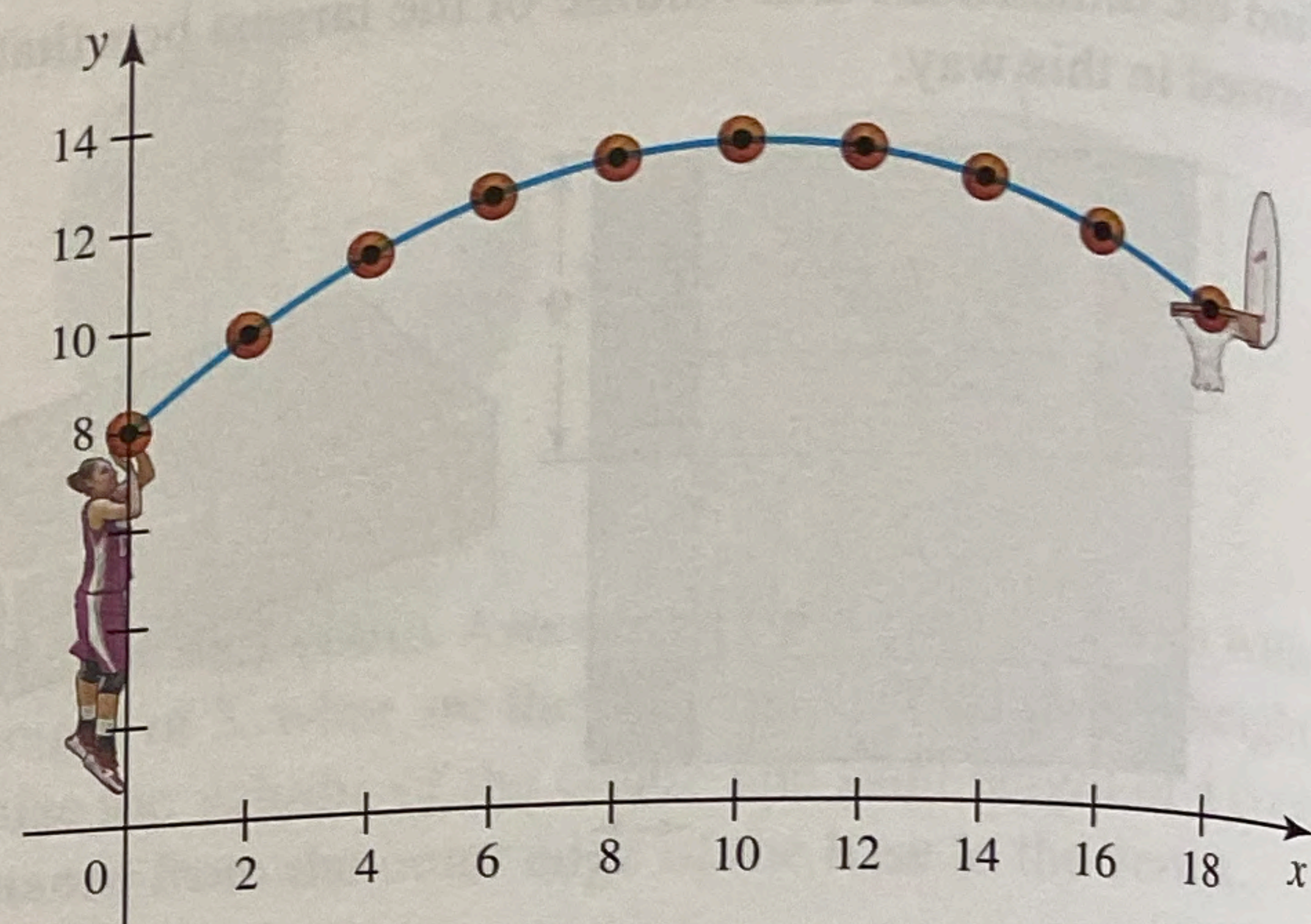


51. **Viewing angles** An auditorium with a flat floor has a large screen on one wall. The lower edge of the screen is 3 ft above eye level and the upper edge of the screen is 10 ft above eye level (see figure). How far from the screen should you stand to maximize your viewing angle? (Hint: Use the identity

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.)$$



52. **Basketball shot** A basketball is shot with an initial velocity of v ft/s at an angle of 45° to the floor. The center of the basketball is 8 ft above the floor at a horizontal distance of 18 feet from the center of the basketball hoop when it is released. The height h (in feet) of the center of the basketball after it has traveled a horizontal distance of x feet is modeled by the function
- $$h(x) = -\frac{32x^2}{v^2} + x + 8 \text{ (see figure).}$$



- a. Find the initial velocity v if the center of the basketball passes the center of the hoop that is located 10 ft above the floor. Assume the ball does not hit the front of the hoop (otherwise it might not pass through the center of the hoop). The validity of this assumption is explored in the remainder of this exercise.
- b. During the flight of the basketball, show that the distance s from the center of the basketball to the front of the hoop is

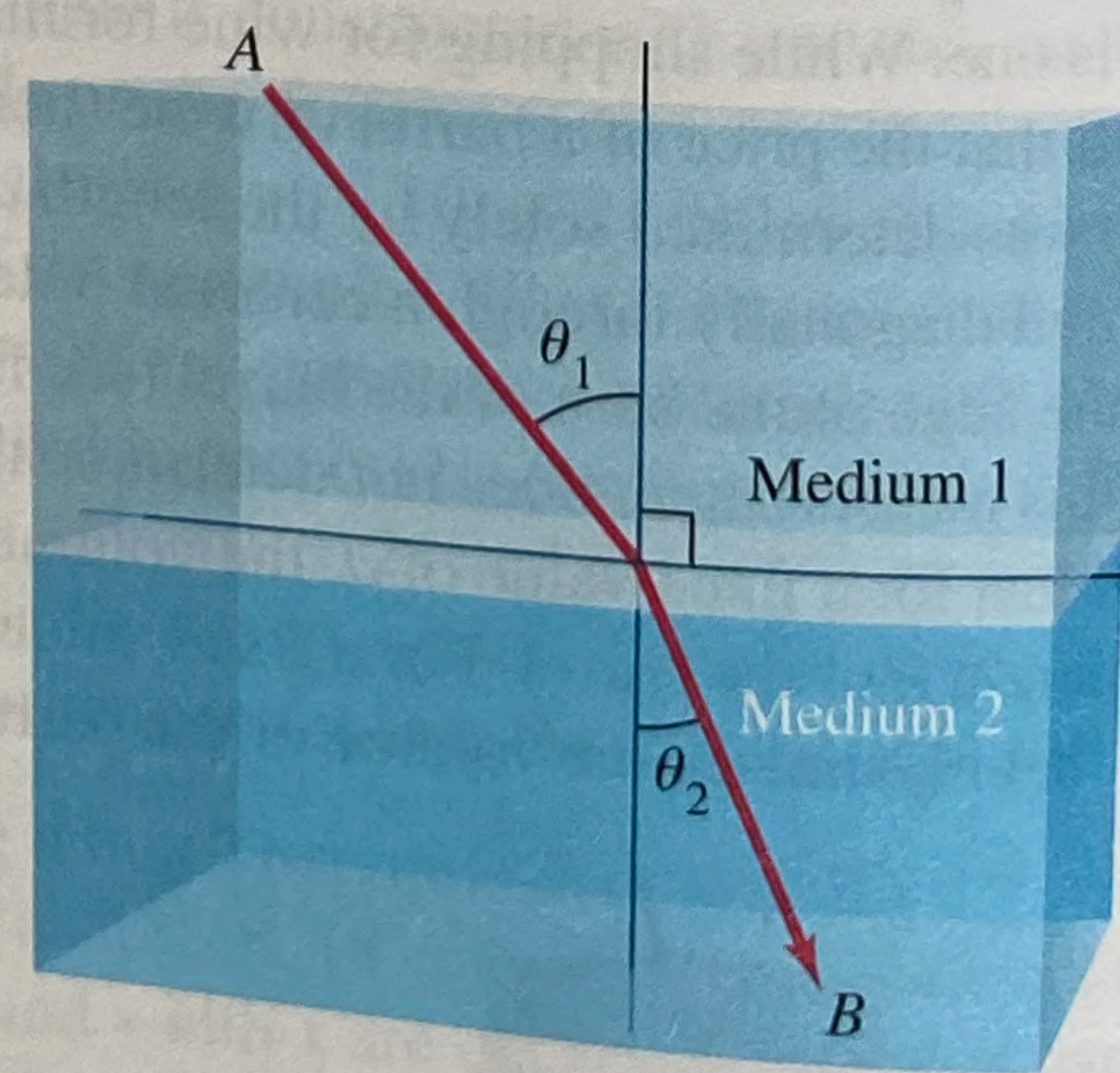
$$s = \sqrt{(x - 17.25)^2 + \left(-\frac{4x^2}{81} + x - 2\right)^2}.$$

(Hint: The diameter of the basketball hoop is 18 inches.)

- c. Determine whether the assumption that the basketball does not hit the front of the hoop in part (a) is valid. Use the fact that the diameter of a women's basketball is about 9.23 inches. (Hint: The ball will hit the front of the hoop if, during its flight, the distance from the center of the ball to the front of the hoop is less than the radius of the basketball.)
- d. A men's basketball has a diameter of about 9.5 inches. Would this larger ball lead to a different conclusion than in part (c)?

53. **Light sources** The intensity of a light source at a distance is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources, one twice as strong as the other, are 12 m apart. At what point on the line segment joining the sources is the intensity the weakest?

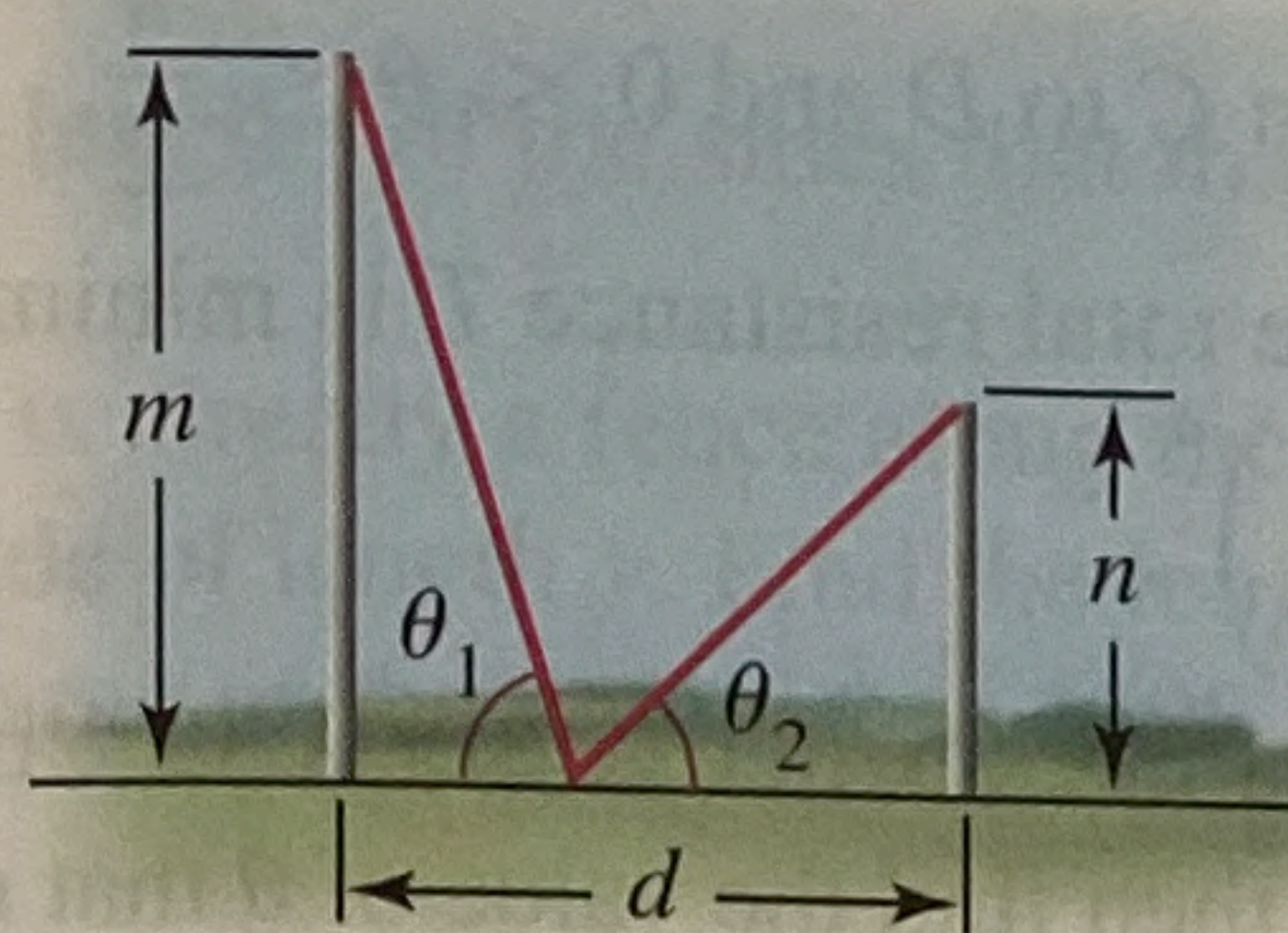
54. **Snell's Law** Suppose a light source at A is in a medium in which light travels at a speed v_1 and that point B is in a medium in which light travels at a speed v_2 (see figure). Using Fermat's Principle, which states that light travels along the path that requires the minimum travel time (Exercise 55), show that the path taken between points A and B satisfies $(\sin \theta_1)/v_1 = (\sin \theta_2)/v_2$.



55. **Fermat's Principle**

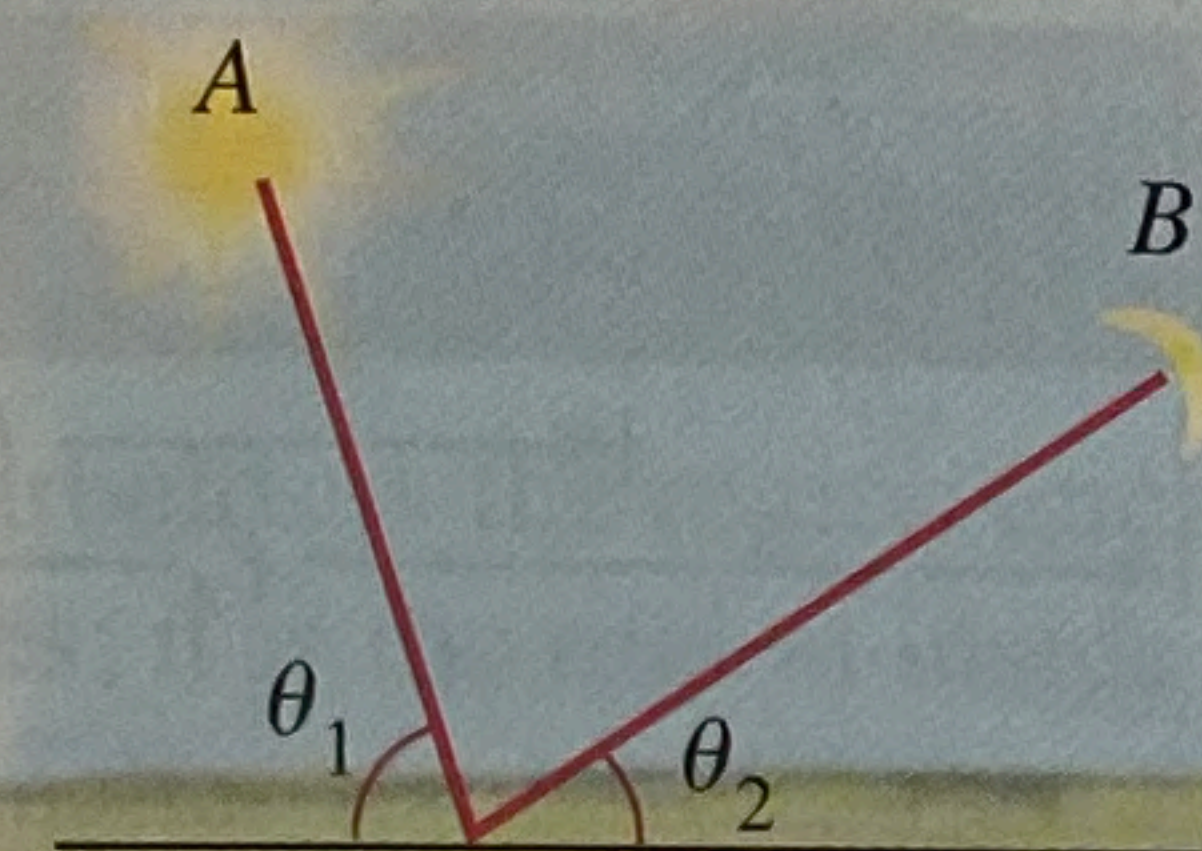
- a. Two poles of heights m and n are separated by a horizontal distance d . A rope is stretched from the top of one pole to the ground and then to the top of the other pole. Show that the

configuration that requires the least amount of rope occurs when $\theta_1 = \theta_2$ (see figure).



(a)

- b. Fermat's Principle states that when light travels between two points in the same medium (at a constant speed), it travels on the path that minimizes the travel time. Show that when light from a source A reflects off a surface and is received at point B , the angle of incidence equals the angle of reflection, or $\theta_1 = \theta_2$ (see figure).

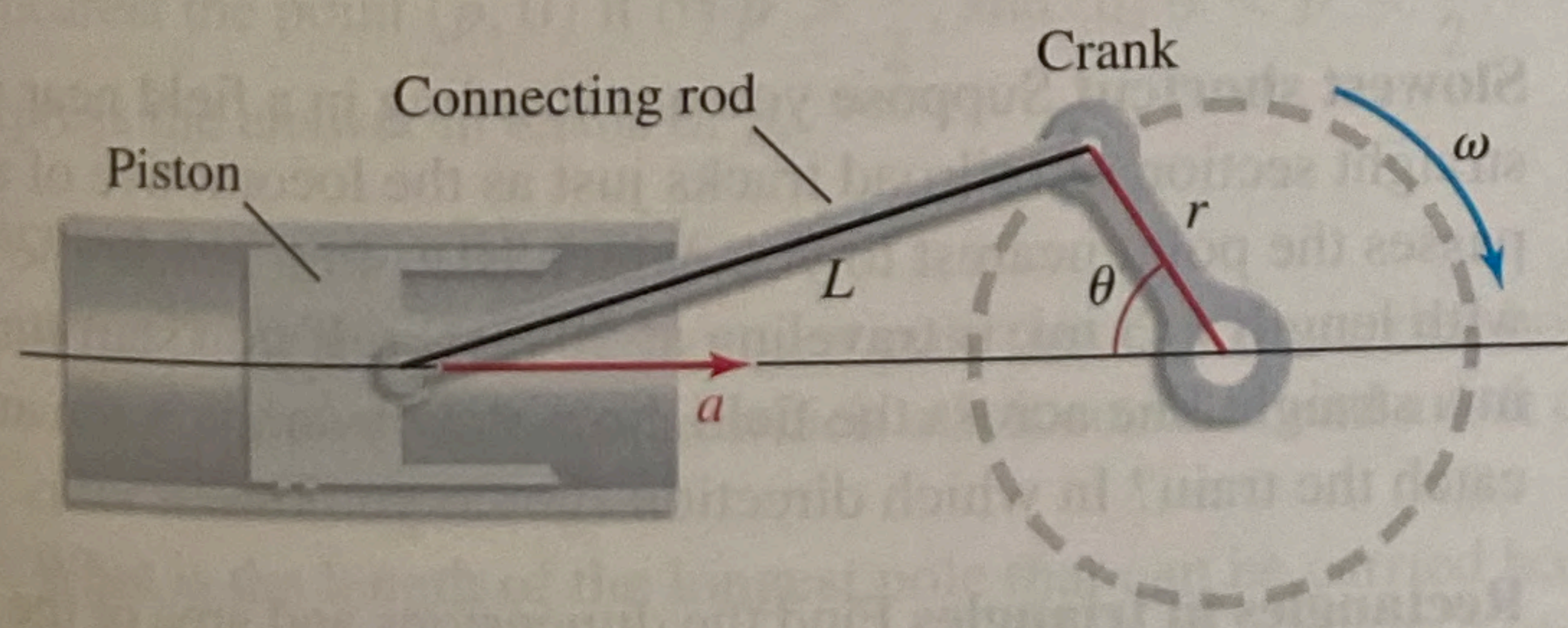


(b)

56. **Crankshaft** A crank of radius r rotates with an angular frequency ω . It is connected to a piston by a connecting rod of length L (see figure). The acceleration of the piston varies with the position of the crank according to the function

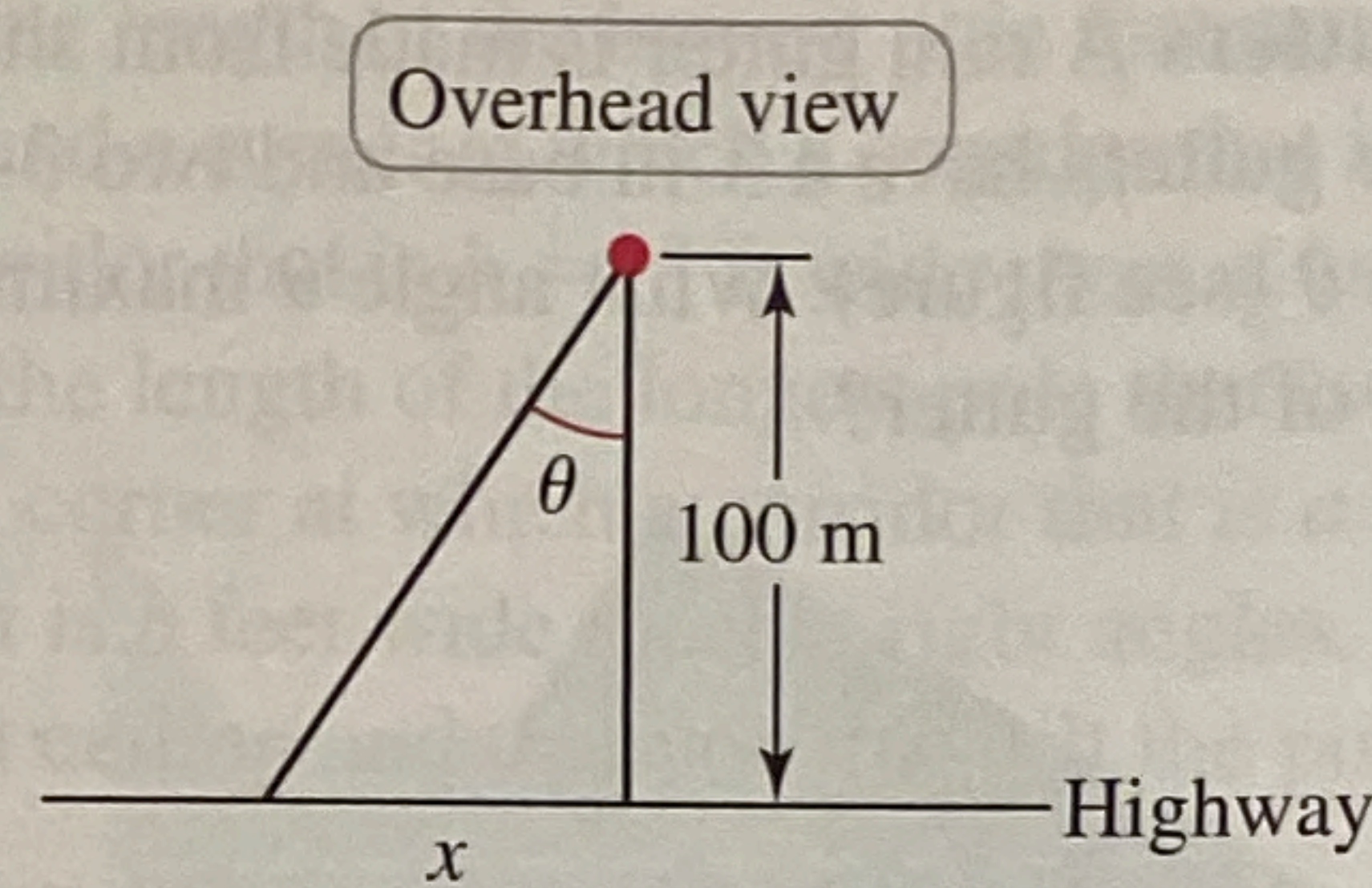
$$a(\theta) = \omega^2 r \left(\cos \theta + \frac{r \cos 2\theta}{L} \right).$$

For $\omega = 1$, $L = 2$, and $r = 1$, find the values of θ , with $0 \leq \theta \leq 2\pi$, for which the acceleration of the piston is a maximum and minimum.



57. **Making silos** A grain silo consists of a cylindrical concrete tower surmounted by a metal hemispherical dome. The metal in the dome costs 1.5 times as much as the concrete (per unit of surface area). If the volume of the silo is 750 m^3 , what are the dimensions of the silo (radius and height of the cylindrical tower) that minimize the cost of the materials? Assume the silo has no floor and no flat ceiling under the dome.

58. **Searchlight problem—narrow beam** A searchlight is 100 m from the nearest point on a straight highway (see figure). As it rotates, the searchlight casts a horizontal beam that intersects the highway in a point. If the light revolves at a rate of $\pi/6 \text{ rad/s}$, find the rate at which the beam sweeps along the highway as a function of θ . For what value of θ is this rate maximized?



59. **Minimizing related functions** Complete each of the following parts.

- What value of x minimizes $f(x) = (x - a_1)^2 + (x - a_2)^2$, for constants a_1 and a_2 ?
- What value of x minimizes $f(x) = (x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2$, for constants a_1, a_2 , and a_3 ?
- Based on the answers to parts (a) and (b), make a conjecture about the values of x that minimize $f(x) = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2$, for a positive integer n and constants a_1, a_2, \dots, a_n . Use calculus to verify your conjecture.

(Source: T. Apostol, *Calculus*, Vol. 1, John Wiley and Sons, 1967)

Explorations and Challenges

60. **Gliding mammals** Many species of small mammals (such as flying squirrels and marsupial gliders) have the ability to walk and glide. Recent research suggests that these animals choose the most energy-efficient means of travel. According to one empirical model, the energy required for a glider with body mass m to walk a horizontal distance D is $8.46Dm^{2/3}$ (where m is measured in grams, D is measured in meters, and energy is measured in microliters of oxygen consumed in respiration). The energy cost of climbing to a height $D \tan \theta$ and gliding a horizontal distance D at an angle of θ is modeled by $1.36mD \tan \theta$ (where $\theta = 0$ represents horizontal flight and $\theta > 45^\circ$ represents controlled falling). Therefore, the function

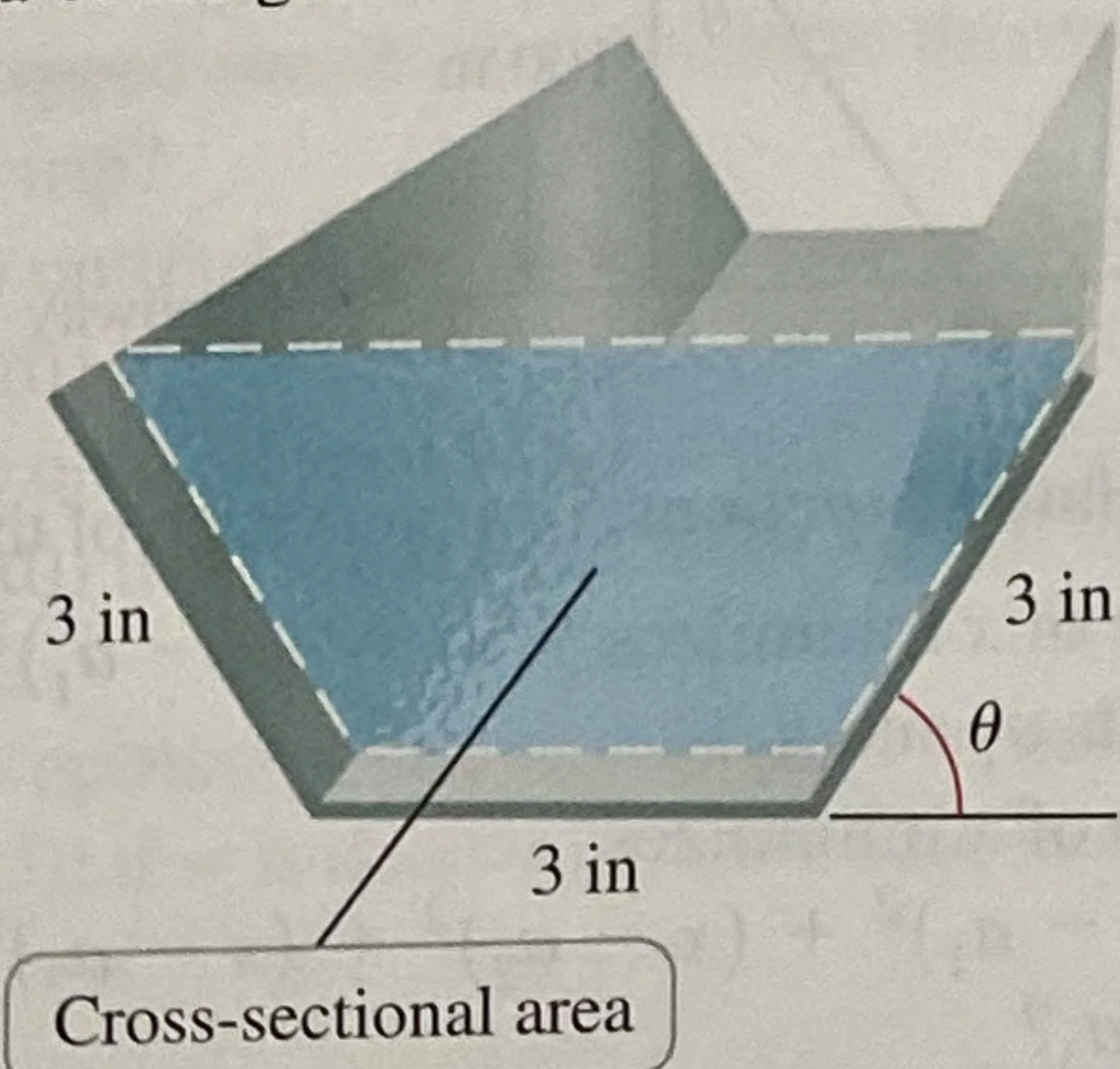
$$S(m, \theta) = 8.46m^{2/3} - 1.36m \tan \theta$$

gives the energy difference per horizontal meter traveled between walking and gliding: If $S > 0$ for given values of m and θ , then it is more costly to walk than to glide.

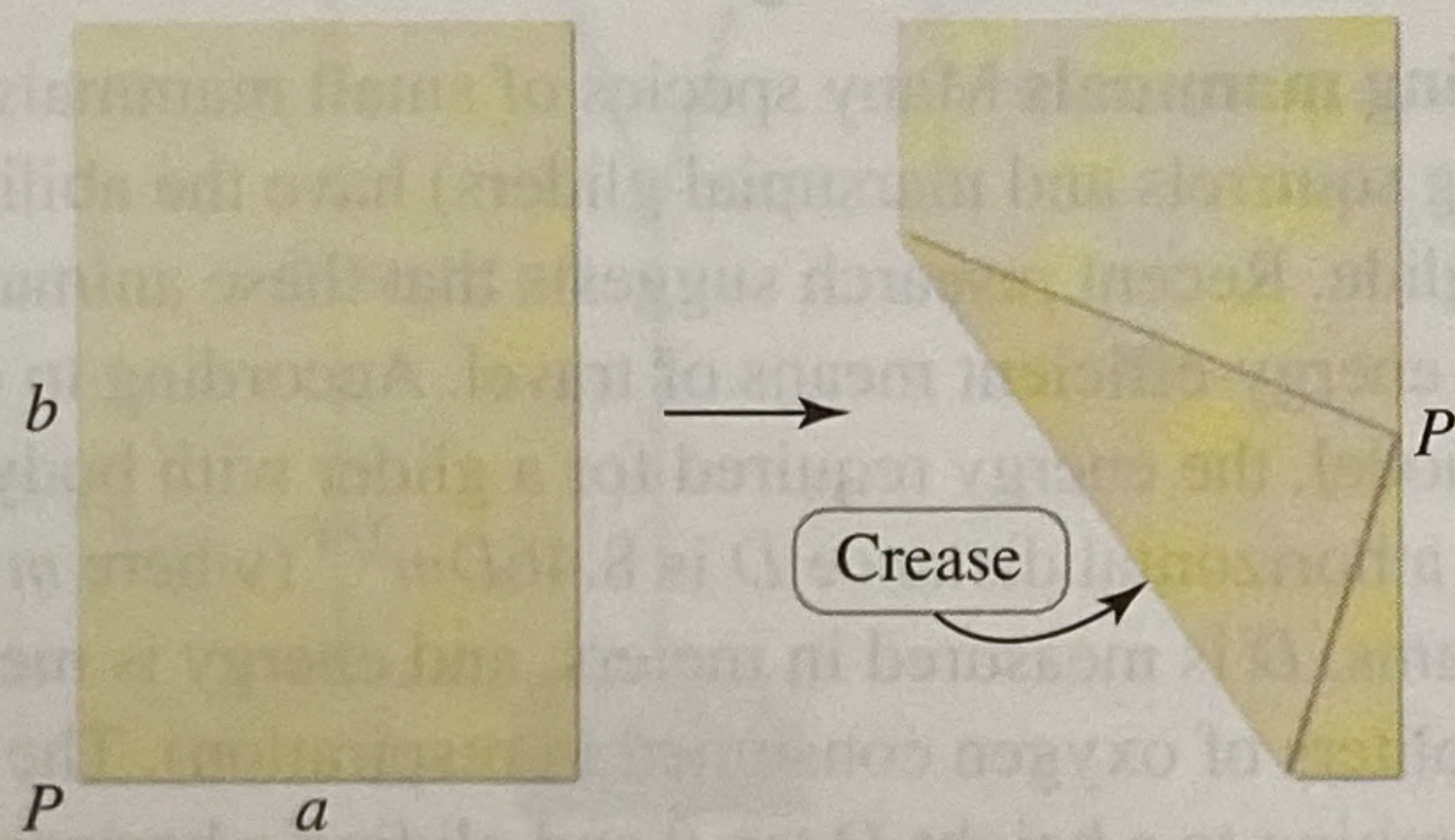
- For what glide angles is it more efficient for a 200-gram animal to glide rather than walk?
- Find an equation that relates θ to m in the case that walking and gliding are equally efficient. Does the angle θ increase or decrease as the mass m increases?
- To make gliding more efficient than walking, do larger gliders have a larger or smaller selection of glide angles than smaller gliders?
- Let $\theta = 25^\circ$ (a typical glide angle). Graph S as a function of m , for $0 \leq m \leq 3000$. For what values of m is gliding more efficient?
- For $\theta = 25^\circ$, what value of m (call it m^*) maximizes S ?
- Does m^* , as defined in part (e), increase or decrease with increasing θ ? That is, as a glider reduces its glide angle, does its optimal size become larger or smaller?
- Assuming Dumbo is a gliding elephant whose weight is 1 metric ton (10^6 g), what glide angle would Dumbo use to be more efficient at gliding than walking?

(Source: R. Dial, *Energetic savings and the body size distribution of gliding mammals*, *Evolutionary Ecology Research*, 5, 2003)

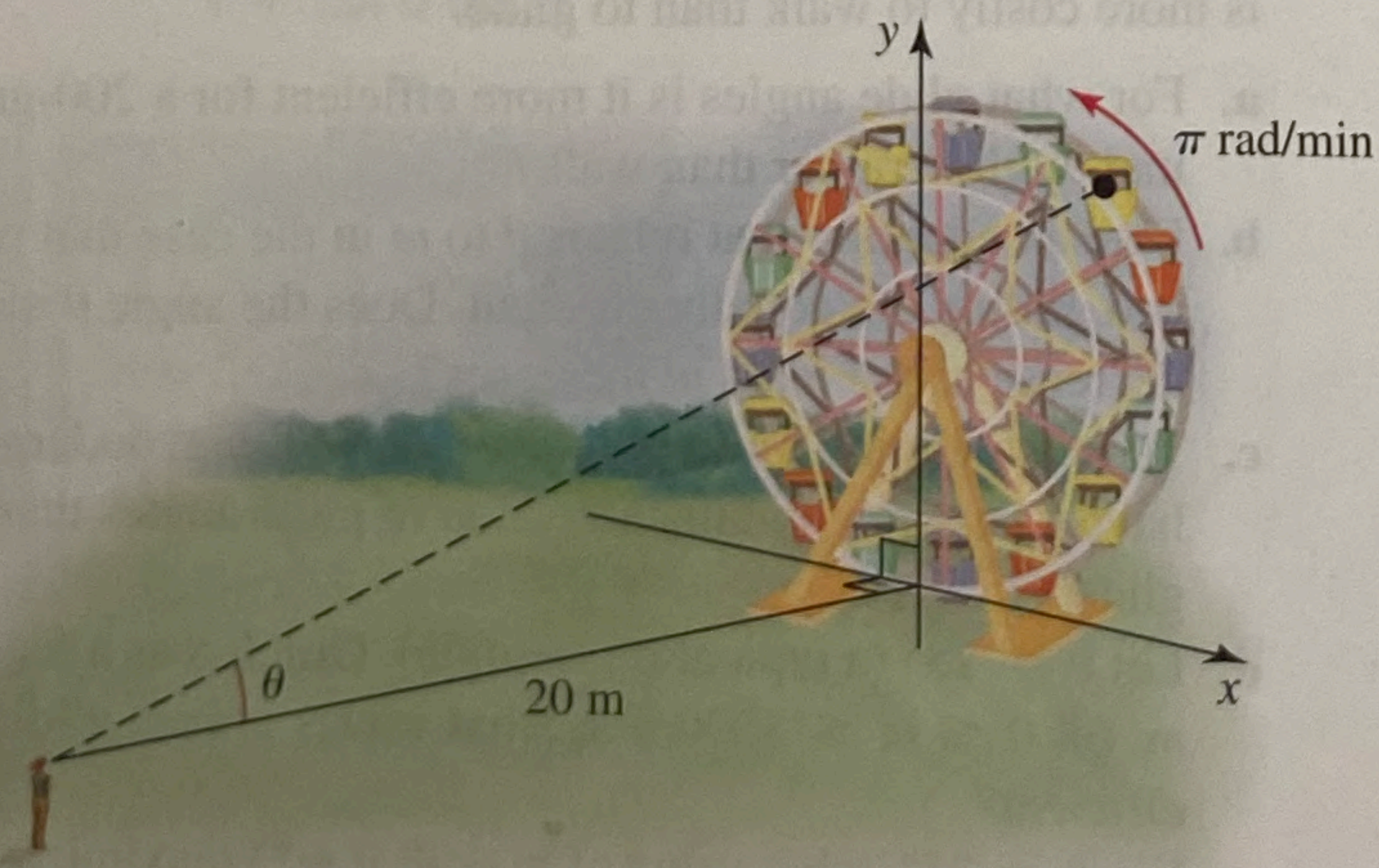
61. **Metal rain gutters** A rain gutter is made from sheets of metal 9 in wide. The gutters have a 3-in base and two 3-in sides, folded up at an angle θ (see figure). What angle θ maximizes the cross-sectional area of the gutter?



62. **Crease-length problem** A rectangular sheet of paper of width a and length b , where $0 < a < b$, is folded by taking one corner of the sheet and placing it at a point P on the opposite long side of the sheet (see figure). The fold is then flattened to form a crease across the sheet. Assuming that the fold is made so that there is no flap extending beyond the original sheet, find the point P that produces the crease of minimum length. What is the length of that crease?



63. **Watching a Ferris wheel** An observer stands 20 m from the bottom of a Ferris wheel on a line that is perpendicular to the face of the wheel, with her eyes at the level of the bottom of the wheel. The wheel revolves at a rate of π rad/min, and the observer's line of sight to a specific seat on the Ferris wheel makes an angle θ with the horizontal (see figure). At what time during a full revolution is θ changing most rapidly?

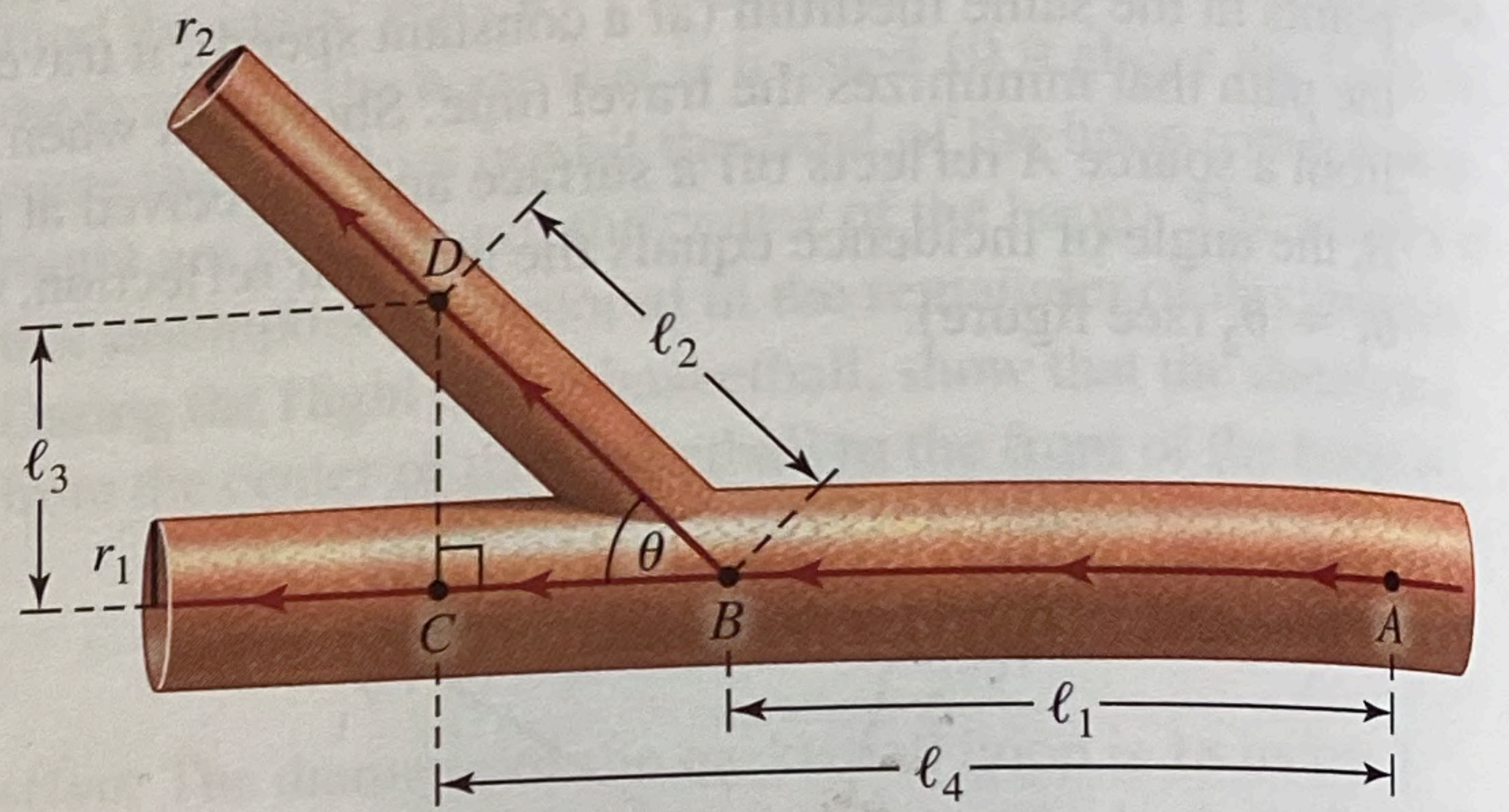


64. **Blood flow** The resistance to blood flow in the circulatory system is one measure of how hard the heart works to pump blood through blood vessels. Lower resistance may correspond to a healthier, higher-efficiency circulatory system. Consider a smaller straight blood vessel of radius r_2 that branches off a larger straight blood vessel of radius r_1 at an angle θ (see figure with given lengths ℓ_1 , ℓ_2 , ℓ_3 , and ℓ_4). Using Poiseuille's Law, it can be shown that the total resistance T to blood flowing along the path from points A to B to D is $T = k\left(\frac{\ell_1}{r_1^4} + \frac{\ell_2}{r_2^4}\right)$, where $k > 0$ is a constant.

- a. Show that $T = k\left(\frac{\ell_4 - \ell_3 \cot \theta}{r_1^4} + \frac{\ell_3 \csc \theta}{r_2^4}\right)$, assuming that the line segment from A to C is perpendicular to the line segment from C to D and $0 < \theta < \frac{\pi}{2}$.

- b. Show that the total resistance T is minimized when $\cos \theta = \left(\frac{r_2}{r_1}\right)^4$.

- c. If the radius of the smaller vessel is 85% of the radius of the larger vessel, then find the value of θ that minimizes T . State your answer in degrees. (Source: *Blood Vessel Branching: Beyond the Standard Calculus Problem, Mathematics Magazine*, 84, 2011)



65. **Sum of isosceles distances**

- a. An isosceles triangle has a base of length 4 and two sides of length $2\sqrt{2}$. Let P be a point on the perpendicular bisector of the base. Find the location P that minimizes the sum of the distances between P and the three vertices.
- b. Assume in part (a) that the height of the isosceles triangle is $h > 0$ and its base has length 4. Show that the location of P that gives a minimum solution is independent of h for $h \geq \frac{2}{\sqrt{3}}$.

66. **Cylinder and cones (Putnam Exam 1938)** Right circular cones of height h and radius r are attached to each end of a right circular cylinder of height h and radius r , forming a double-pointed object. For a given surface area A , what are the dimensions r and h that maximize the volume of the object?

67. **Slowest shortcut** Suppose you are standing in a field near a straight section of railroad tracks just as the locomotive of a train passes the point nearest to you, which is $1/4$ mi away. The train, with length $1/3$ mi, is traveling at 20 mi/hr. If you start running in a straight line across the field, how slowly can you run and still catch the train? In which direction should you run?

68. **Rectangles in triangles** Find the dimensions and area of the rectangle of maximum area that can be inscribed in the following figures.

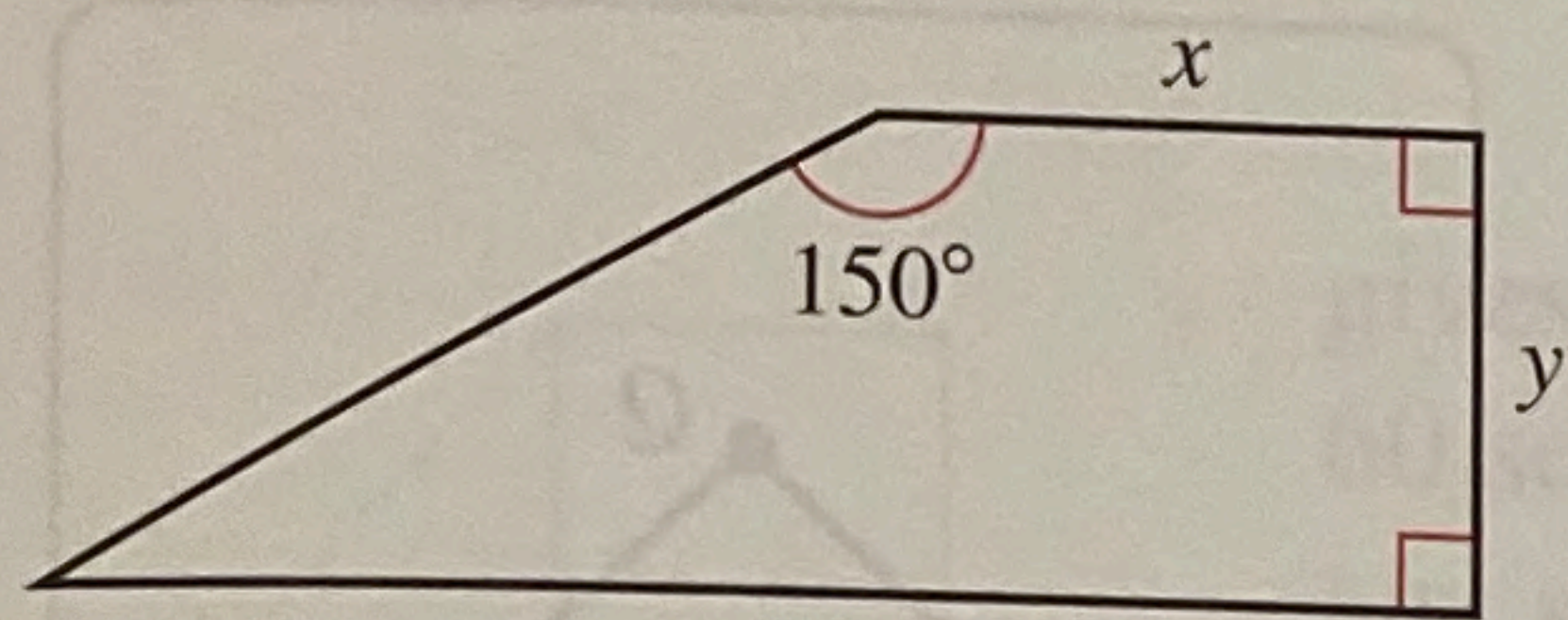
- a. A right triangle with a given hypotenuse length L
- b. An equilateral triangle with a given side length L
- c. A right triangle with a given area A
- d. An arbitrary triangle with a given area A (The result applies to any triangle, but first consider triangles for which all the angles are less than or equal to 90° .)

69. **Cylinder in a cone** A right circular cylinder is placed inside a cone of radius R and height H so that the base of the cylinder lies on the base of the cone.

- a. Find the dimensions of the cylinder with maximum volume. Specifically, show that the volume of the maximum-volume cylinder is $4/9$ the volume of the cone.
- b. Find the dimensions of the cylinder with maximum lateral surface area (area of the curved surface).

70. **Another pen problem** A rancher is building a horse pen on the corner of her property using 1000 ft of fencing. Because of the unusual shape of her property, the pen must be built in the shape of a trapezoid (see figure).

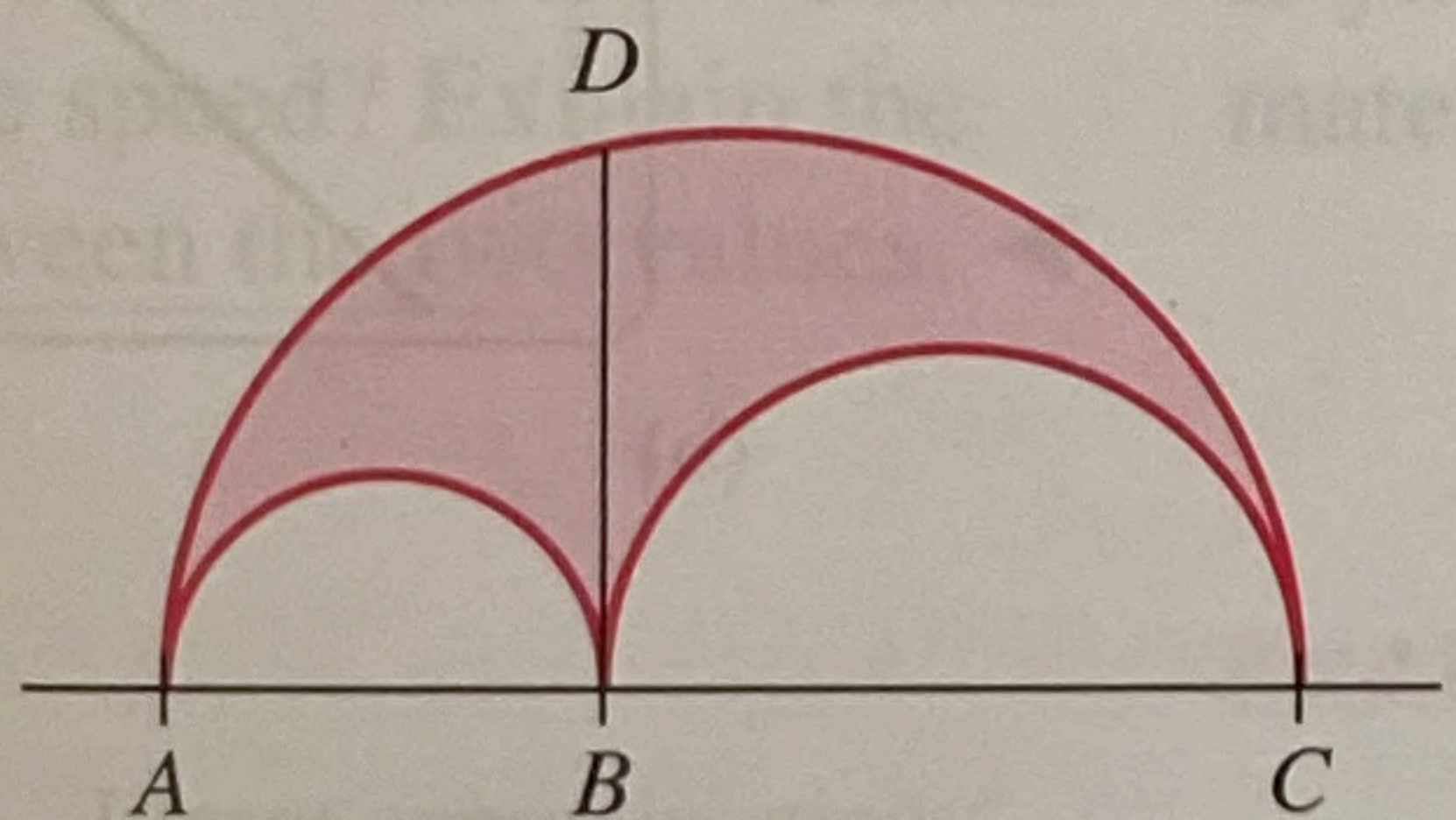
- Determine the lengths of the sides that maximize the area of the pen.
- Suppose there is already a fence along the side of the property opposite the side of length y . Find the lengths of the sides that maximize the area of the pen, using 1000 ft of fencing.



71. **Minimum-length roads** A house is located at each corner of a square with side lengths of 1 mi. What is the length of the shortest road system with straight roads that connects all of the houses by roads (that is, a road system that allows one to drive from any house to any other house)? (*Hint*: Place two points inside the square at which roads meet.) (*Source*: Paul Halmos, *Problems for Mathematicians Young and Old*, MAA, 1991)

72. **The arbelos** An arbelos is the region enclosed by three mutually tangent semicircles; it is the region inside the larger semicircle and outside the two smaller semicircles (see figure).

- Given an arbelos in which the diameter of the largest circle is 1, what positions of point B maximize the area of the arbelos?
- Show that the area of the arbelos is the area of a circle whose diameter is the distance BD in the figure.



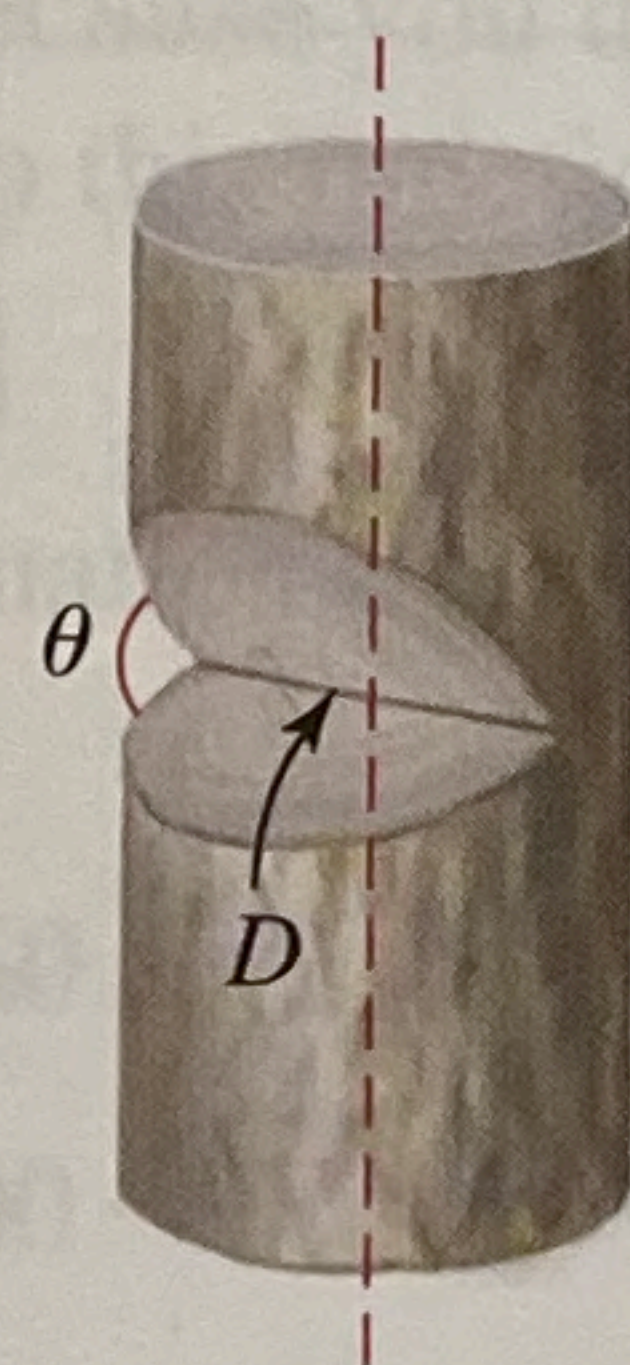
73. **Proximity questions** Find the point on the graph of $y = \sqrt{x}$ that is nearest the point $(p, 0)$ if (i) $p > \frac{1}{2}$; and (ii) $0 < p < \frac{1}{2}$. Express the answer in terms of p .

74. **Turning a corner with a pole**

- What is the length of the longest pole that can be carried horizontally around a corner at which a 3-ft corridor and a 4-ft corridor meet at right angles?
- What is the length of the longest pole that can be carried horizontally around a corner at which a corridor that is a ft wide and a corridor that is b ft wide meet at right angles?

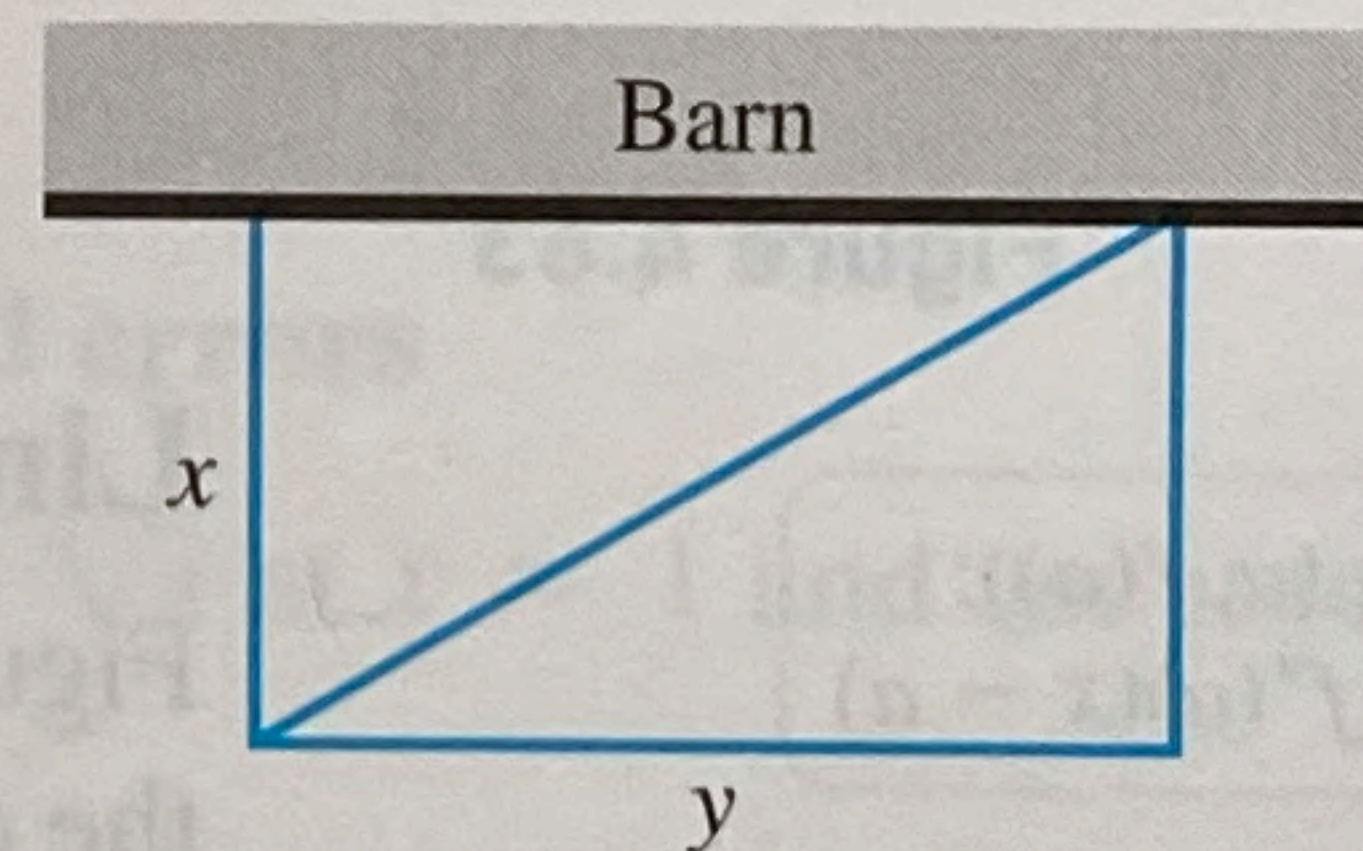
- What is the length of the longest pole that can be carried horizontally around a corner at which a corridor that is $a = 5$ ft wide and a corridor that is $b = 5$ ft wide meet at an angle of 120° ?
- What is the length of the longest pole that can be carried around a corner at which a corridor that is a ft wide and a corridor that is b feet wide meet at right angles, assuming there is an 8-foot ceiling and that you may tilt the pole at any angle?

75. **Tree notch (Putnam Exam 1938, rephrased)** A notch is cut in a cylindrical vertical tree trunk (see figure). The notch penetrates to the axis of the cylinder and is bounded by two half-planes that intersect on a diameter D of the tree. The angle between the two half-planes is θ . Prove that for a given tree and fixed angle θ , the volume of the notch is minimized by taking the bounding planes at equal angles to the horizontal plane that also passes through D .



76. **Circle in a triangle** What are the radius and area of the circle of maximum area that can be inscribed in an isosceles triangle whose two equal sides have length 1?

77. **A challenging pen problem** A farmer uses 200 meters of fencing to build two triangular pens against a barn (see figure); the pens are constructed with three sides and a diagonal dividing fence. What dimensions maximize the area of the pen?



78. **Folded boxes** Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard with sides of length ℓ and L . The resulting piece of cardboard is then folded into a box without a lid. Holding ℓ fixed, find the size of the corner squares x that maximizes the volume of the box as $L \rightarrow \infty$. (*Source*: *Mathematics Teacher*, Nov 2002)

QUICK CHECK ANSWERS

- (i) $A = 400x - 2x^2$ (ii) $A = 400x - 3x^2$
- $V = 108w^2 - 2w^3$ ◀

4.6 Linear Approximation and Differentials

Imagine plotting a smooth curve with a graphing utility. Now pick a point P on the curve, draw the line tangent to the curve at P , and zoom in on it several times. As you successively enlarge the curve near P , it looks more and more like the tangent line (Figure 4.63a). This fundamental observation—that smooth curves appear straighter on smaller scales—is called *local linearity*; it is the basis of many important mathematical ideas, one of which is *linear approximation*.

Now consider a curve with a corner or cusp at a point Q (Figure 4.63b). No amount of magnification “straightens out” the curve or removes the corner at Q . The different behavior at P and Q is related to the idea of differentiability: The function in Figure 4.63a is differentiable at P , whereas the function in Figure 4.63b is not differentiable at Q . One of the requirements for the techniques presented in this section is that the function be differentiable at the point in question.

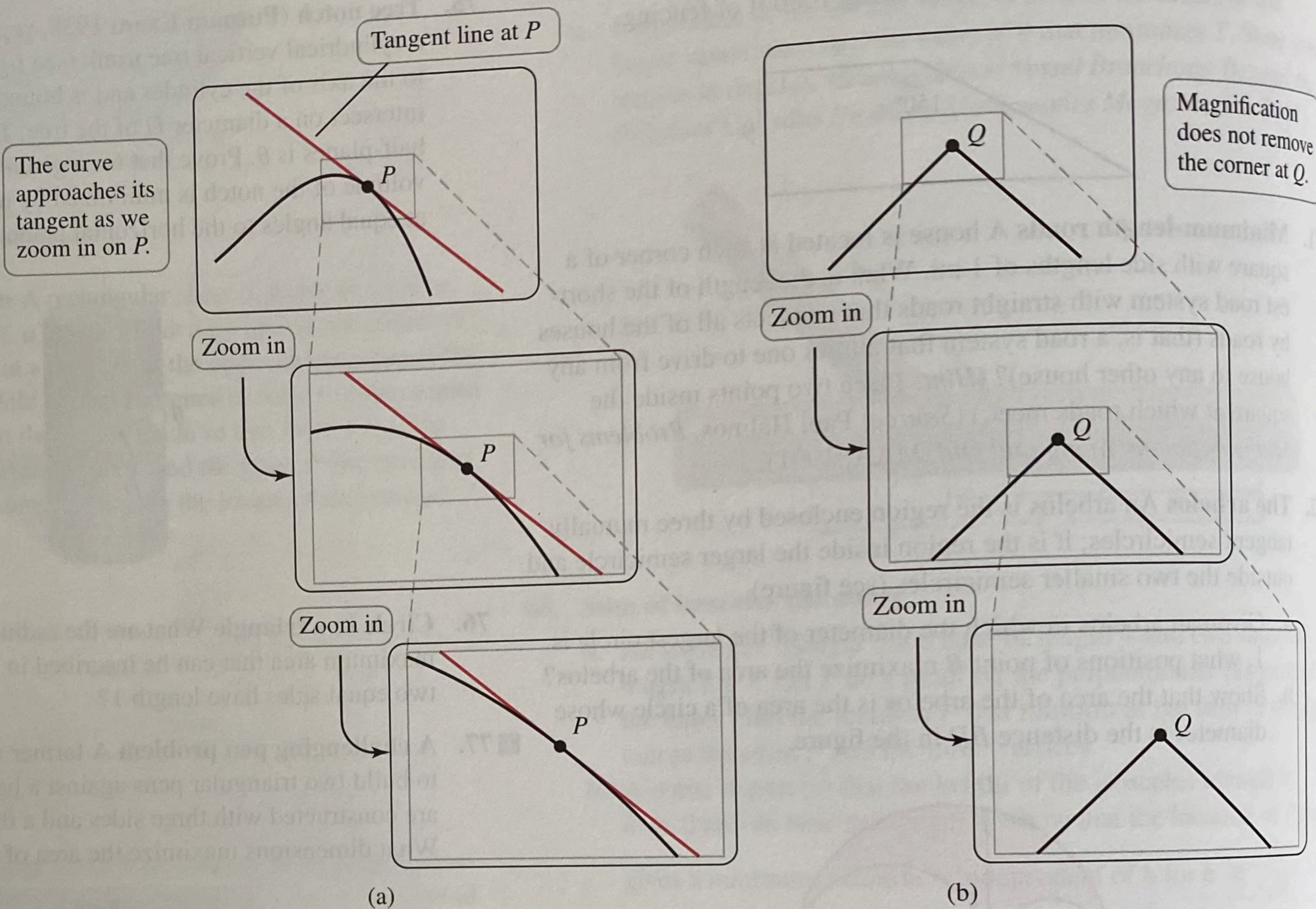


Figure 4.63

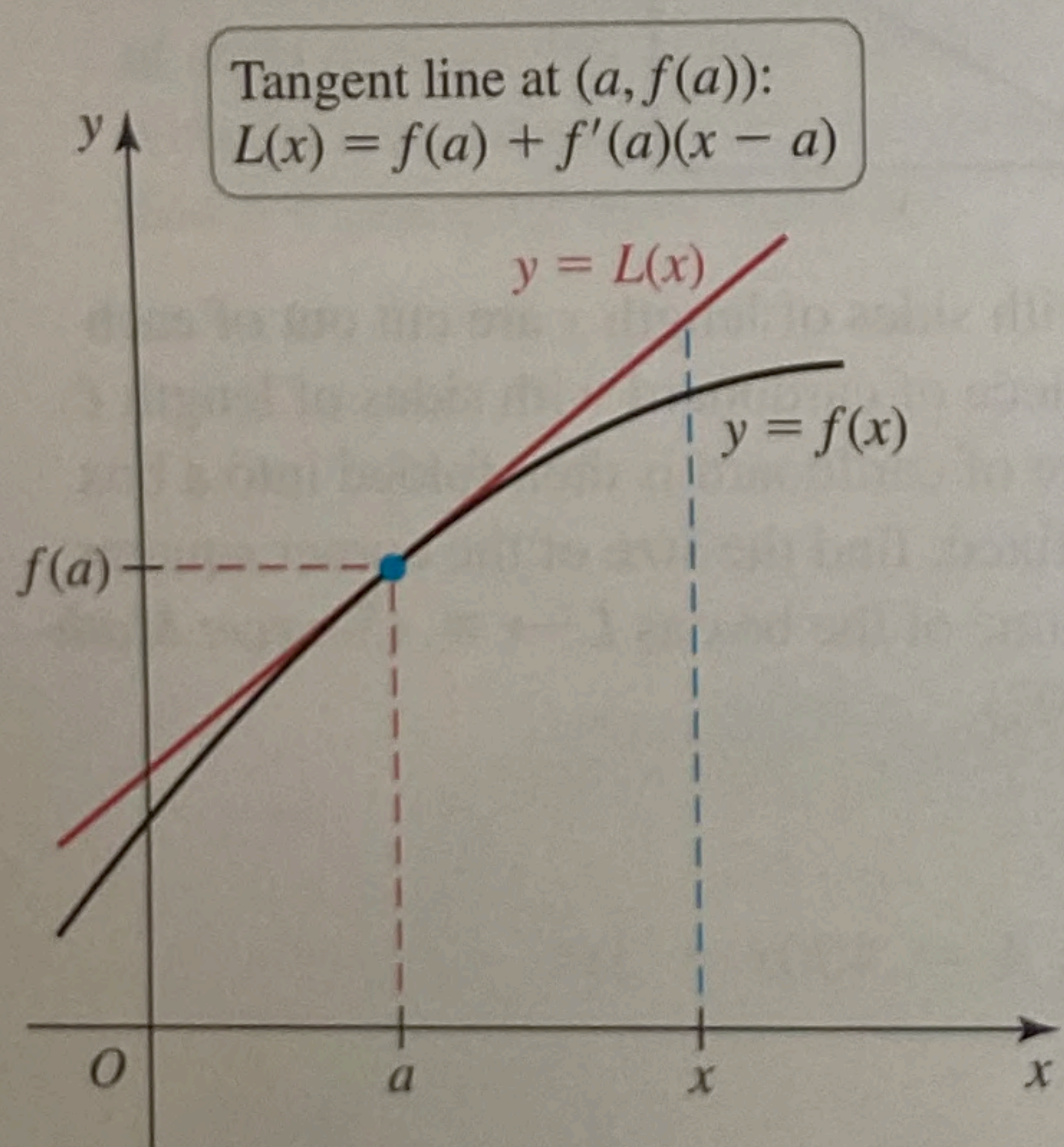


Figure 4.64

Linear Approximation

Figure 4.63a suggests that when we zoom in on the graph of a smooth function at a point P , the curve approaches its tangent line at P . This fact is the key to understanding linear approximation. The idea is to use the line tangent to the curve at P to approximate the value of the function at points near P . Here’s how it works.

Assume f is differentiable on an interval containing the point a . The slope of the line tangent to the curve at the point $(a, f(a))$ is $f'(a)$. Therefore, an equation of the tangent line is

$$y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = \underbrace{f(a) + f'(a)(x - a)}_{L(x)}$$

This tangent line represents a new function L that we call the *linear approximation* to f at the point a (Figure 4.64). If f and f' are easy to evaluate at a , then the value of f at points near a is easily approximated using the linear approximation L . That is,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

This approximation improves as x approaches a .

DEFINITION Linear Approximation to f at a

Suppose f is differentiable on an interval I containing the point a . The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

QUICK CHECK 1 Sketch the graph of a function f that is concave up on an interval containing the point a . Sketch the linear approximation to f at a . Is the graph of the linear approximation above or below the graph of f ? ◀

► In Example 1, notice that when x is positive, you are driving slower than 60 mi/hr; when x is negative, you are driving faster than 60 mi/hr.

EXAMPLE 1 Useful driving math Suppose you are driving along a highway at a constant speed and you record the number of seconds it takes to travel between two consecutive mile markers. If it takes 60 seconds to travel one mile, then your average speed is 1 mi/60 s or 60 mi/hr. Now suppose you travel one mile in $60 + x$ seconds; for example, if it takes 62 seconds, then $x = 2$, and if it takes 57 seconds, then $x = -3$. In this case, your average speed over one mile is 1 mi/(60 + x) s. Because there are 3600 s in 1 hr, the function

$$s(x) = \frac{3600}{60 + x} = 3600(60 + x)^{-1}$$

gives your average speed in mi/hr if you travel one mile in x seconds more or less than 60 seconds. For example, if you travel one mile in 62 seconds, then $x = 2$ and your average speed is $s(2) \approx 58.06$ mi/hr. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is $s(-3) \approx 63.16$ mi/hr. Because you don't want to use a calculator while driving, you need an easy approximation to this function. Use linear approximation to derive such a formula.

SOLUTION The idea is to find the linear approximation to s at the point 0. We first use the Chain Rule to compute

$$s'(x) = -3600(60 + x)^{-2},$$

and then note that $s(0) = 60$ and $s'(0) = -3600 \cdot 60^{-2} = -1$. Using the linear approximation formula, we find that

$$s(x) \approx L(x) = s(0) + s'(0)(x - 0) = 60 - x.$$

For example, if you travel one mile in 62 seconds, then $x = 2$ and your average speed is approximately $L(2) = 58$ mi/hr, which is very close to the exact value given previously. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is approximately $L(-3) = 63$ mi/hr, which again is close to the exact value.

Related Exercises 13–14 ◀

QUICK CHECK 2 In Example 1, suppose you travel one mile in 75 seconds. What is the average speed given by the linear approximation formula? What is the exact average speed? Explain the discrepancy between the two values. ◀

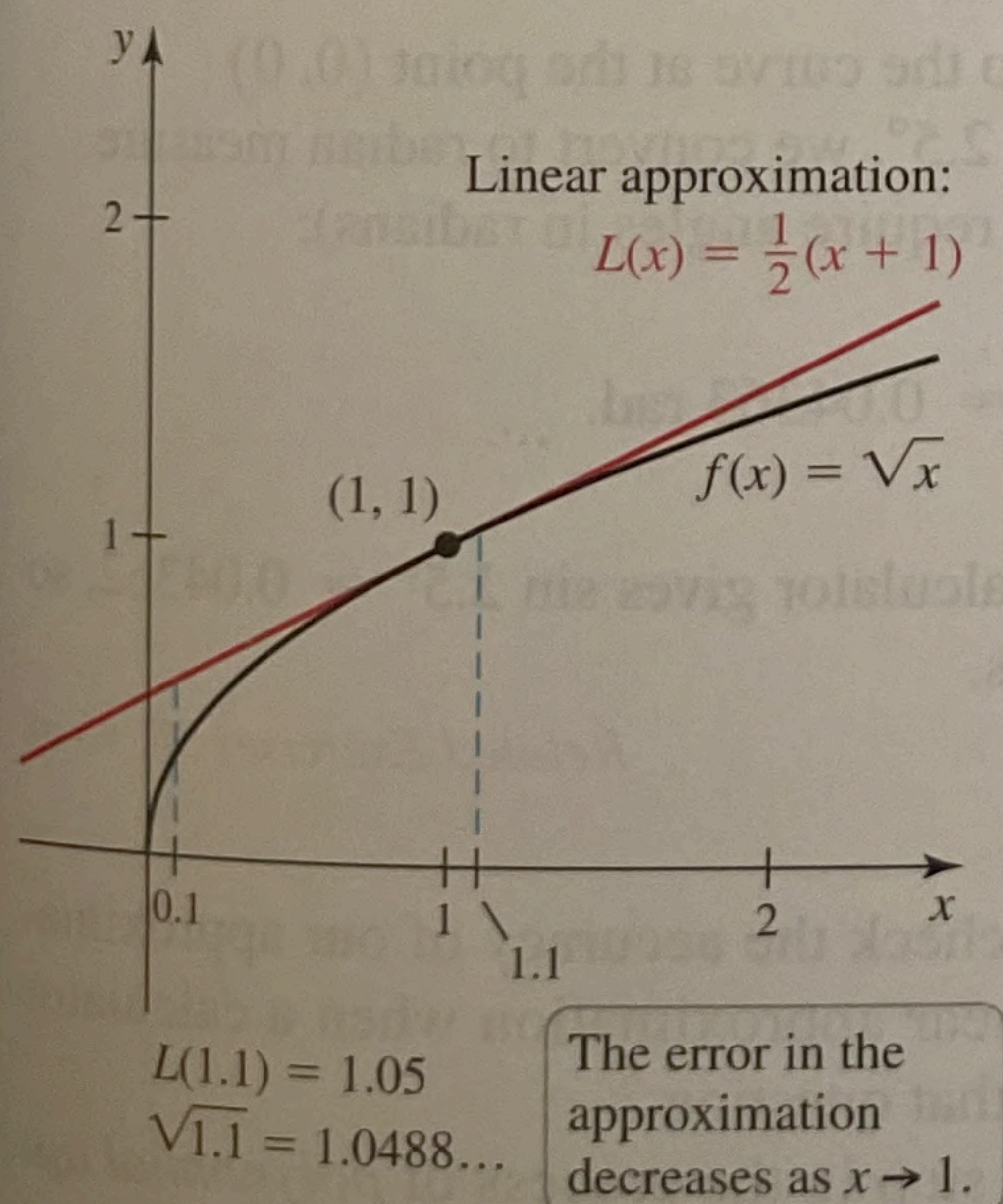


Figure 4.65

Table 4.4

x	$L(x)$	Exact \sqrt{x}	Error
1.2	1.1	1.0954...	4.6×10^{-3}
1.1	1.05	1.0488...	1.2×10^{-3}
1.01	1.005	1.0049...	1.2×10^{-5}
1.001	1.0005	1.0005...	1.2×10^{-7}

EXAMPLE 2 Linear approximations and errors

- Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$.
- Use linear approximation to estimate the value of $\sqrt{0.1}$.

SOLUTION

- We construct the linear approximation

$$L(x) = f(a) + f'(a)(x - a),$$

where $f(x) = \sqrt{x}$, $f'(x) = 1/(2\sqrt{x})$, and $a = 1$. Noting that $f(a) = f(1) = 1$ and $f'(a) = f'(1) = \frac{1}{2}$, we have

$$L(x) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}(x + 1),$$

which describes the line tangent to the curve at the point $(1, 1)$ (Figure 4.65). Because $x = 1.1$ is near $x = 1$, we approximate $\sqrt{1.1}$ by $L(1.1)$:

$$\sqrt{1.1} \approx L(1.1) = \frac{1}{2}(1.1 + 1) = 1.05.$$

The exact value is $f(1.1) = \sqrt{1.1} = 1.0488\dots$; therefore, the linear approximation has an error of about 0.0012. Furthermore, our approximation is an *overestimate* because the tangent line lies above the graph of f . In Table 4.4, we see several approximations to \sqrt{x} for x near 1 and the associated errors $|L(x) - \sqrt{x}|$. Clearly, the errors decrease as x approaches 1.

b. If the linear approximation $L(x) = \frac{1}{2}(x + 1)$ obtained in part (a) is used to approximate $\sqrt{0.1}$, we have

$$\sqrt{0.1} \approx L(0.1) = \frac{1}{2}(0.1 + 1) = 0.55.$$

A calculator gives $\sqrt{0.1} = 0.3162\dots$, which shows that the approximation is well off the mark. The error arises because the tangent line through $(1, 1)$ is not close to the curve at $x = 0.1$ (Figure 4.65). For this reason, we seek a different value of a , with the requirement that it is near $x = 0.1$, and both $f(a)$ and $f'(a)$ are easily computed. It is tempting to try $a = 0$, but $f'(0)$ is undefined. One choice that works well is $a = \frac{9}{100} = 0.09$. Using the linear approximation $L(x) = f(a) + f'(a)(x - a)$, we have

$$\begin{aligned} \sqrt{0.1} \approx L(0.1) &= \frac{f(a)}{\sqrt{\frac{9}{100}}} + \frac{f'(a)}{2\sqrt{9/100}} \left(\frac{1}{10} - \frac{9}{100} \right) \\ &= \frac{3}{10} + \frac{10}{6} \left(\frac{1}{100} \right) \\ &= \frac{19}{60} \approx 0.3167. \end{aligned}$$

► We choose $a = \frac{9}{100}$ because it is close to 0.1 and its square root is easy to evaluate.

QUICK CHECK 3 Suppose you want to use linear approximation to estimate $\sqrt{0.18}$. What is a good choice for a ? ◀

This approximation agrees with the exact value to three decimal places.

Related Exercises 31–32 ◀

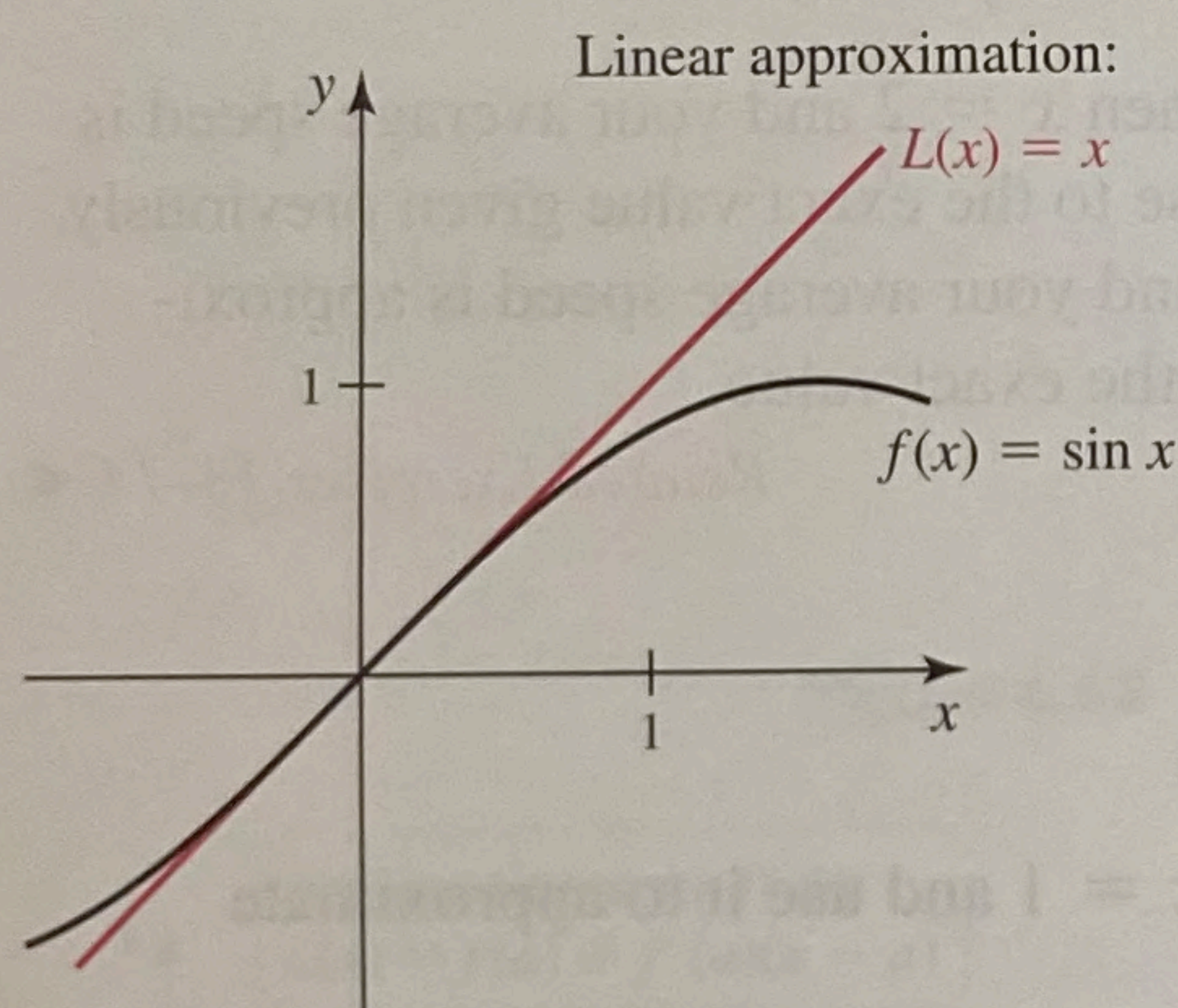


Figure 4.66

EXAMPLE 3 Linear approximation for the sine function Find the linear approximation to $f(x) = \sin x$ at $x = 0$ and use it to approximate $\sin 2.5^\circ$.

SOLUTION We first construct a linear approximation $L(x) = f(a) + f'(a)(x - a)$, where $f(x) = \sin x$ and $a = 0$. Noting that $f(0) = 0$ and $f'(0) = \cos(0) = 1$, we have

$$L(x) = 0 + 1(x - 0) = x.$$

Again, the linear approximation is the line tangent to the curve at the point $(0, 0)$ (Figure 4.66). Before using $L(x)$ to approximate $\sin 2.5^\circ$, we convert to radian measure (the derivative formulas for trigonometric functions require angles in radians):

$$2.5^\circ = 2.5^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{\pi}{72} \approx 0.04363 \text{ rad.}$$

Therefore, $\sin 2.5^\circ \approx L(0.04363) = 0.04363$. A calculator gives $\sin 2.5^\circ \approx 0.04362$, so the approximation is accurate to four decimal places.

Related Exercises 38, 46 ◀

In Examples 2 and 3, we used a calculator to check the accuracy of our approximations. This raises a question: Why bother with linear approximation when a calculator does a better job? There are some good answers to that question.

Linear approximation is actually just the first step in the process of *polynomial approximation*. While linear approximation does a decent job of estimating function values when x is near a , we can generally do better with higher-degree polynomials. These ideas are explored further in Chapter 11.

Linear approximation also allows us to discover simple approximations to complicated functions. In Example 3, we found the *small-angle approximation to the sine function*: $\sin x \approx x$ for x near 0.

Linear Approximation and Concavity

Additional insight into linear approximation is gained by bringing concavity into the picture. Figure 4.67a shows the graph of a function f and its linear approximation (tangent line) at the point $(a, f(a))$. In this particular case, f is concave up on an interval containing a , and the

QUICK CHECK 4 Explain why the linear approximation to $f(x) = \cos x$ at $x = 0$ is $L(x) = 1$. ◀

graph of L lies below the graph of f near a . As a result, the linear approximation evaluated at a point near a is less than the exact value of f at that point. In other words, the linear approximation *underestimates* values of f near a .

The contrasting case is shown in **Figure 4.67b**, where we see graphs of f and L when f is concave down on an interval containing a . Now the graph of L lies above the graph of f , which means the linear approximation *overestimates* values of f near a .

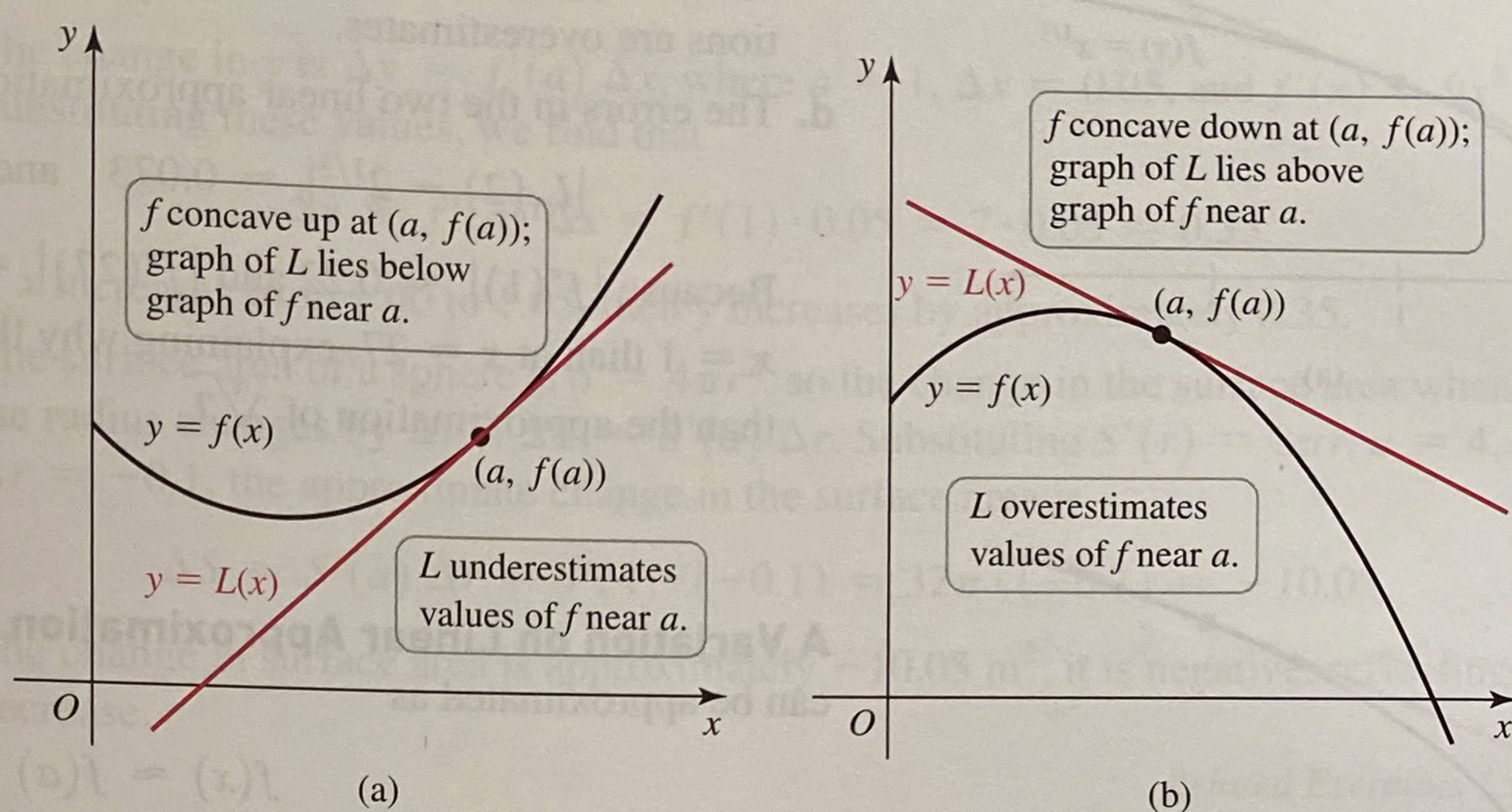


Figure 4.67

We can make another observation related to the degree of concavity (also called *curvature*). A large value of $|f''(a)|$ (large curvature) means that near $(a, f(a))$, the slope of the curve changes rapidly and the graph of f separates quickly from the tangent line. A small value of $|f''(a)|$ (small curvature) means the slope of the curve changes slowly and the curve is relatively flat near $(a, f(a))$; therefore, the curve remains close to the tangent line. As a result, absolute errors in linear approximation are larger when $|f''(a)|$ is large.

EXAMPLE 4 Linear approximation and concavity

- Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.
- Use the linear approximations of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$.
- Are the approximations in part (b) overestimates or underestimates?
- Compute the error in each approximation of part (b). Which error is greater? Explain.

SOLUTION

- a. Note that

$$f(1) = 1, \quad f(27) = 3, \quad f'(x) = \frac{1}{3x^{2/3}}, \quad f'(1) = \frac{1}{3}, \quad \text{and} \quad f'(27) = \frac{1}{27}.$$

Therefore, the linear approximation at $x = 1$ is

$$L_1(x) = 1 + \frac{1}{3}(x - 1) = \frac{1}{3}x + \frac{2}{3},$$

and the linear approximation at $x = 27$ is

$$L_2(x) = 3 + \frac{1}{27}(x - 27) = \frac{1}{27}x + 2.$$

- b. Using the results of part (a), we find that

$$\sqrt[3]{2} \approx L_1(2) = \frac{1}{3} \cdot 2 + \frac{2}{3} = \frac{4}{3} \approx 1.333$$

and

$$\sqrt[3]{26} \approx L_2(26) = \frac{1}{27} \cdot 26 + 2 \approx 2.963.$$

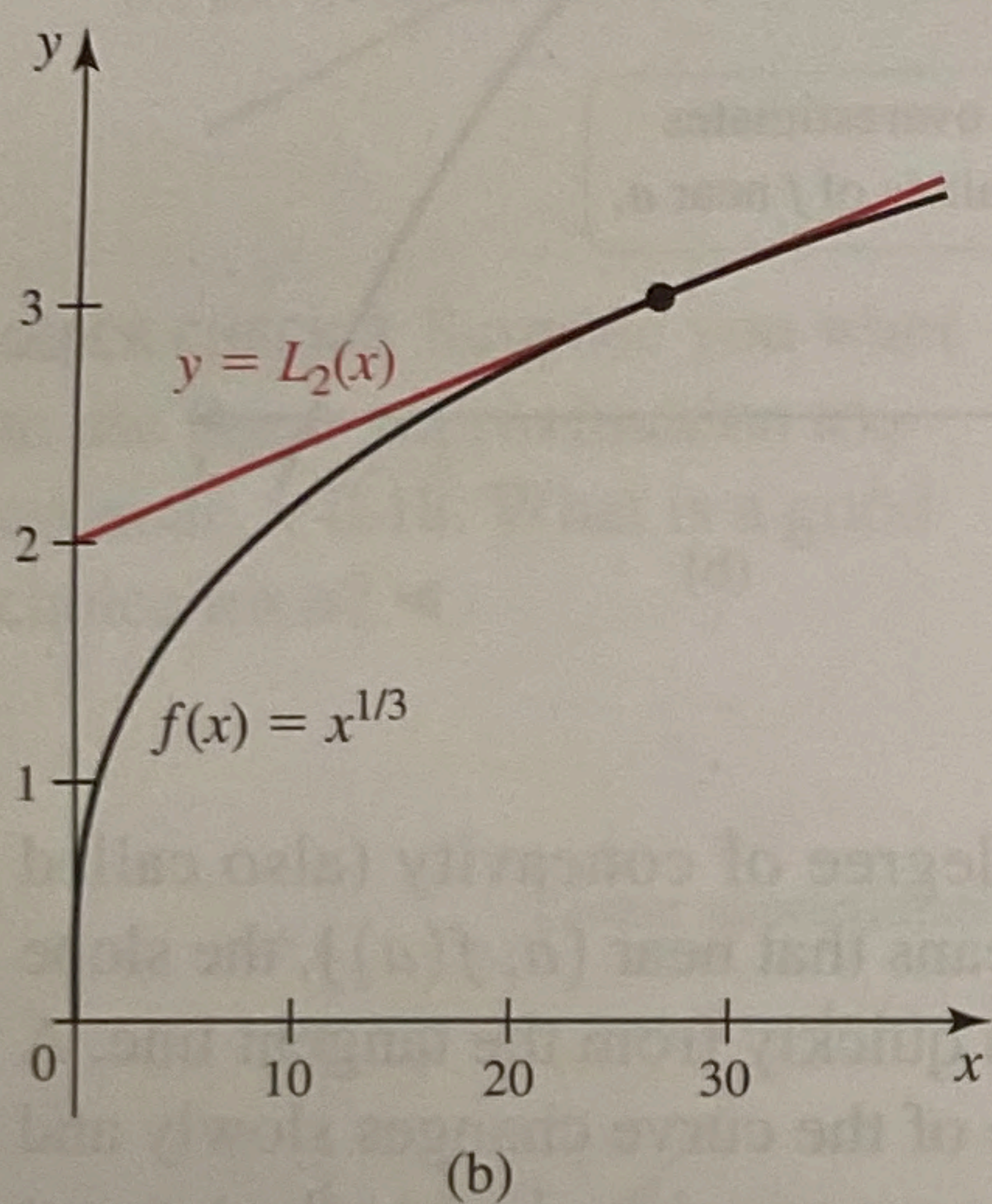
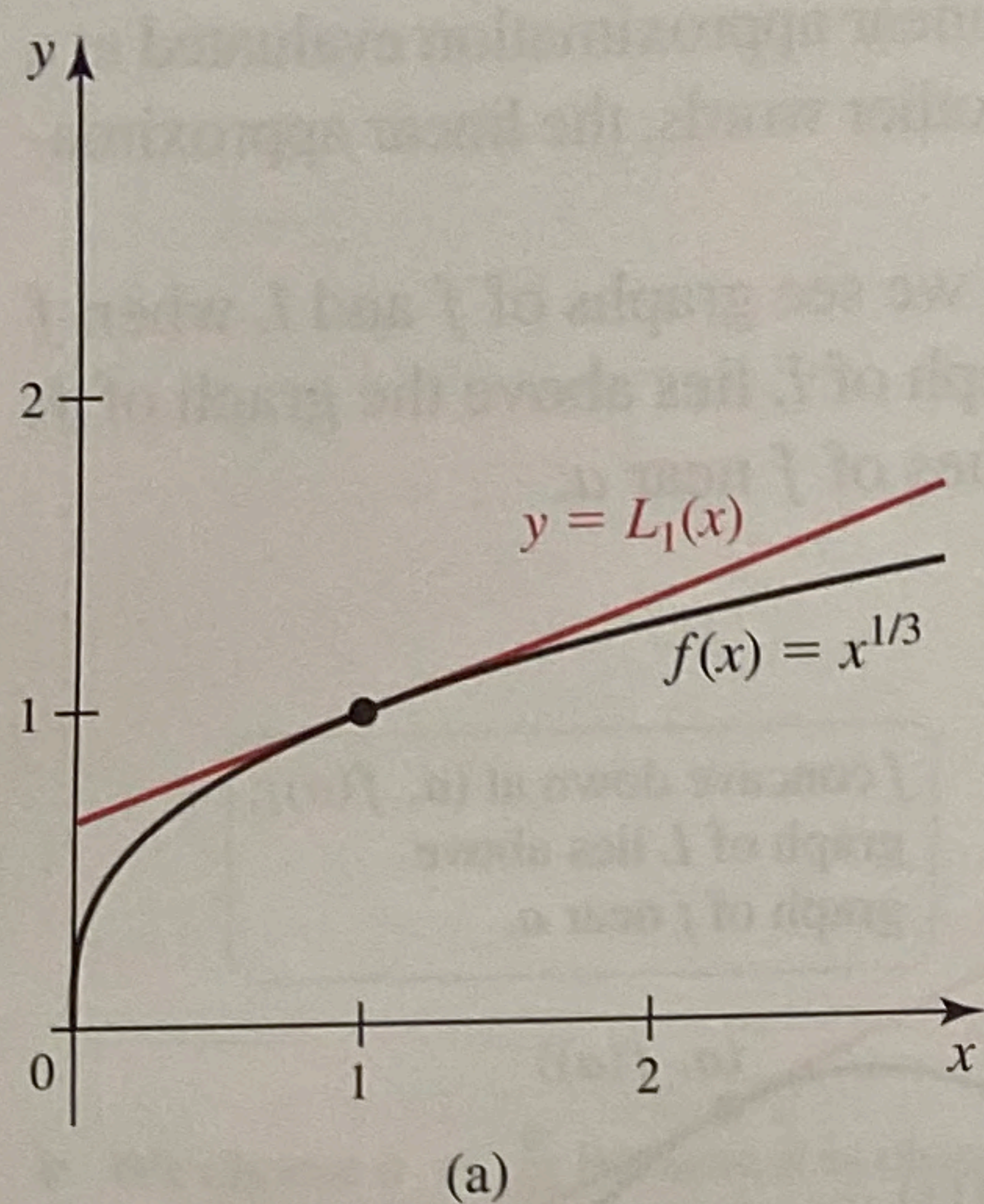


Figure 4.68

- c. Figure 4.68 shows the graphs of f and the linear approximations L_1 and L_2 at $x = 1$ and $x = 27$, respectively (note the different scales on the x -axes). We see that f is concave down at both points, which is confirmed by the fact that

$$f''(x) = -\frac{2}{9}x^{-5/3} < 0, \text{ for } x > 0.$$

As a result, the linear approximations lie above the graph of f and both approximations are overestimates.

- d. The errors in the two linear approximations are

$$|L_1(2) - 2^{1/3}| \approx 0.073 \quad \text{and} \quad |L_2(26) - 26^{1/3}| \approx 0.00047.$$

Because $|f''(1)| \approx 0.22$ and $|f''(27)| \approx 0.00091$, the curvature of f is greater at $x = 1$ than at $x = 27$, explaining why the approximation of $\sqrt[3]{26}$ is more accurate than the approximation of $\sqrt[3]{2}$.

Related Exercises 47–50 ◀

A Variation on Linear Approximation Linear approximation says that a function f can be approximated as

$$f(x) \approx f(a) + f'(a)(x - a),$$

where a is fixed and x is a nearby point. We first rewrite this expression as

$$\underbrace{f(x) - f(a)}_{\Delta y} \approx f'(a) \underbrace{(x - a)}_{\Delta x}.$$

It is customary to use the notation Δ (capital Greek delta) to denote a change. The factor $x - a$ is the change in the x -coordinate between a and a nearby point x . Similarly, $f(x) - f(a)$ is the corresponding change in the y -coordinate (Figure 4.69). So we write this approximation as

$$\Delta y \approx f'(a) \Delta x.$$

In other words, a change in y (the function value) can be approximated by the corresponding change in x magnified or diminished by a factor of $f'(a)$. This interpretation states the familiar fact that $f'(a)$ is the rate of change of y with respect to x .

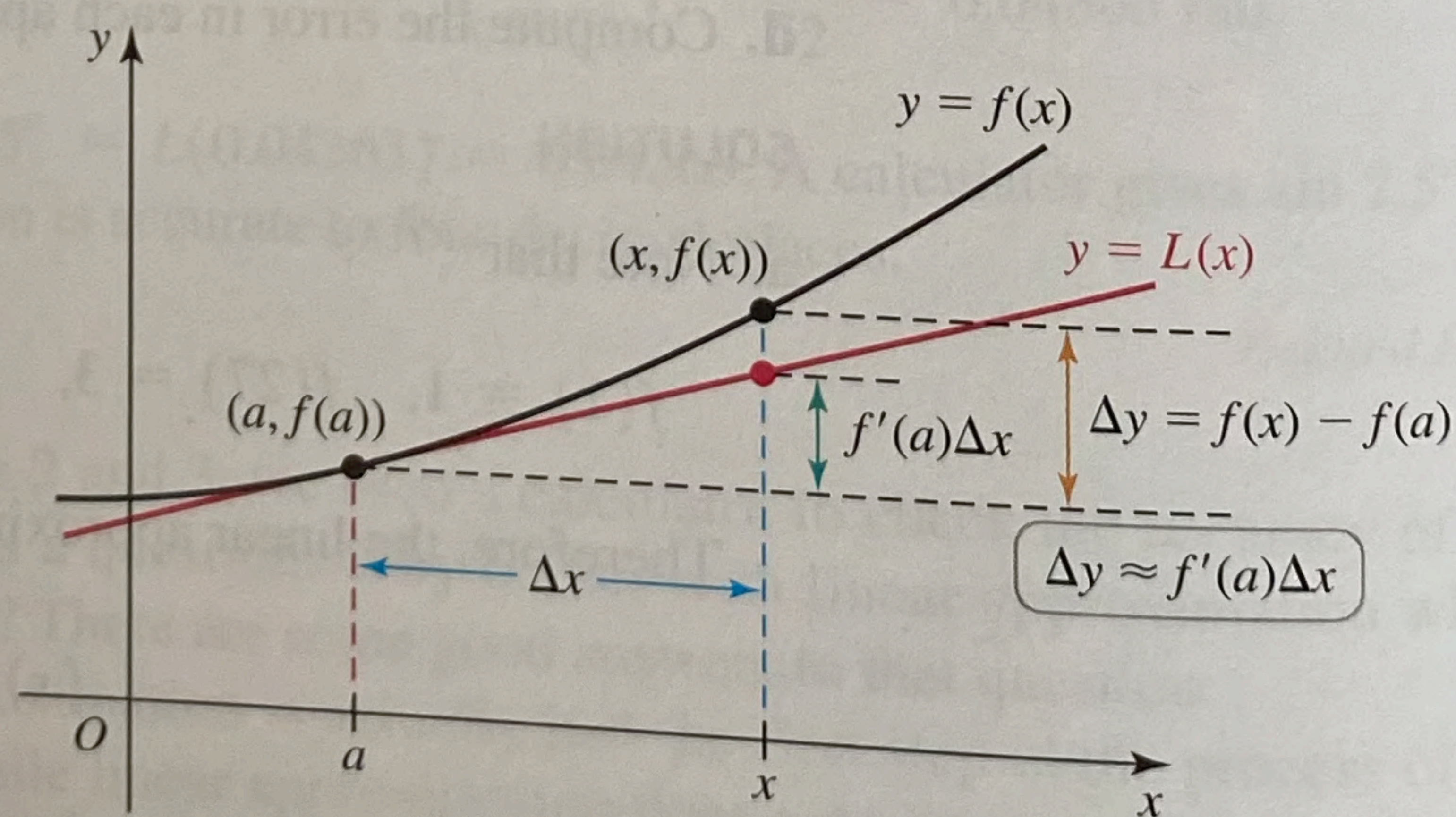


Figure 4.69

Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a . The change in the value of f between two points a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a) \Delta x,$$

where $a + \Delta x$ is in I .

EXAMPLE 5 Estimating changes with linear approximations

- a. Approximate the change in $y = f(x) = x^9 - 2x + 1$ when x changes from 1.00 to 1.05.
- b. Approximate the change in the surface area of a spherical hot-air balloon when the radius decreases from 4 m to 3.9 m.

SOLUTION

- a. The change in y is $\Delta y \approx f'(a) \Delta x$, where $a = 1$, $\Delta x = 0.05$, and $f'(x) = 9x^8 - 2$. Substituting these values, we find that

$$\Delta y \approx f'(a) \Delta x = f'(1) \cdot 0.05 = 7 \cdot 0.05 = 0.35.$$

If x increases from 1.00 to 1.05, then y increases by approximately 0.35.

- b. The surface area of a sphere is $S = 4\pi r^2$, so the change in the surface area when the radius changes by Δr is $\Delta S \approx S'(a) \Delta r$. Substituting $S'(r) = 8\pi r$, $a = 4$, and $\Delta r = -0.1$, the approximate change in the surface area is

$$\Delta S \approx S'(a) \Delta r = S'(4) \cdot (-0.1) = 32\pi \cdot (-0.1) \approx -10.05.$$

The change in surface area is approximately -10.05 m^2 ; it is negative, reflecting a decrease.

Related Exercises 55, 57 ◀

► Notice that the units in these calculations are consistent. If r has units of meters (m), S' has units of $\text{m}^2/\text{m} = \text{m}$, so ΔS has units of m^2 , as it should.

QUICK CHECK 5 Given that the volume of a sphere is $V = 4\pi r^3/3$, find an expression for the approximate change in the volume when the radius changes from a to $a + \Delta r$. ◀

SUMMARY Uses of Linear Approximation

- To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

- To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a) \Delta x.$$

Differentials

We now introduce an important concept that allows us to distinguish two related quantities:

- the change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy , as before), and
- the change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (which we call the *differential* dy).

Consider a function $y = f(x)$ differentiable on an interval containing a . If the x -coordinate changes from a to $a + \Delta x$, then the corresponding change in the function is *exactly*

$$\Delta y = f(a + \Delta x) - f(a).$$

Using the linear approximation $L(x) = f(a) + f'(a)(x - a)$, the change in L as x changes from a to $a + \Delta x$ is

$$\begin{aligned} \Delta L &= L(a + \Delta x) - L(a) \\ &= \underbrace{(f(a) + f'(a)(a + \Delta x - a))}_{L(a + \Delta x)} - \underbrace{(f(a) + f'(a)(a - a))}_{L(a)} \\ &= f'(a) \Delta x. \end{aligned}$$

To distinguish Δy and ΔL , we define two new variables called *differentials*. The differential dx is simply Δx ; the differential dy is the change in the linear approximation, which is $\Delta L = f'(a) \Delta x$. Using this notation,

$$\Delta L = \underbrace{dy}_{\text{same as } \Delta L} = f'(a) \Delta x = f'(a) \underbrace{dx}_{\text{same as } \Delta x}.$$

Therefore, at the point a , we have $dy = f'(a) dx$. More generally, we replace the fixed point a with a variable point x and write

$$dy = f'(x) dx.$$

DEFINITION Differentials

Let f be differentiable on an interval containing x . A small change in x is denoted by the **differential** dx . The corresponding change in f is approximated by the **differential** $dy = f'(x) dx$; that is,

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x) dx.$$

► As one of the two co-inventors of calculus, Gottfried Leibniz relied on the idea of differentials in his development of calculus. Leibniz's notation for differentials is essentially the same as the notation we use today. An Irish philosopher of the day, Bishop Berkeley, called differentials "the ghost of departed quantities."

Figure 4.70 shows that if $\Delta x = dx$ is small, then the change in f , which is Δy , is well approximated by the change in the linear approximation, which is dy . Furthermore, the approximation $\Delta y \approx dy$ improves as dx approaches 0. The notation for differentials is consistent with the notation for the derivative: If we divide both sides of $dy = f'(x) dx$ by dx , we have (symbolically)

$$\frac{dy}{dx} = \frac{f'(x) dx}{dx} = f'(x).$$

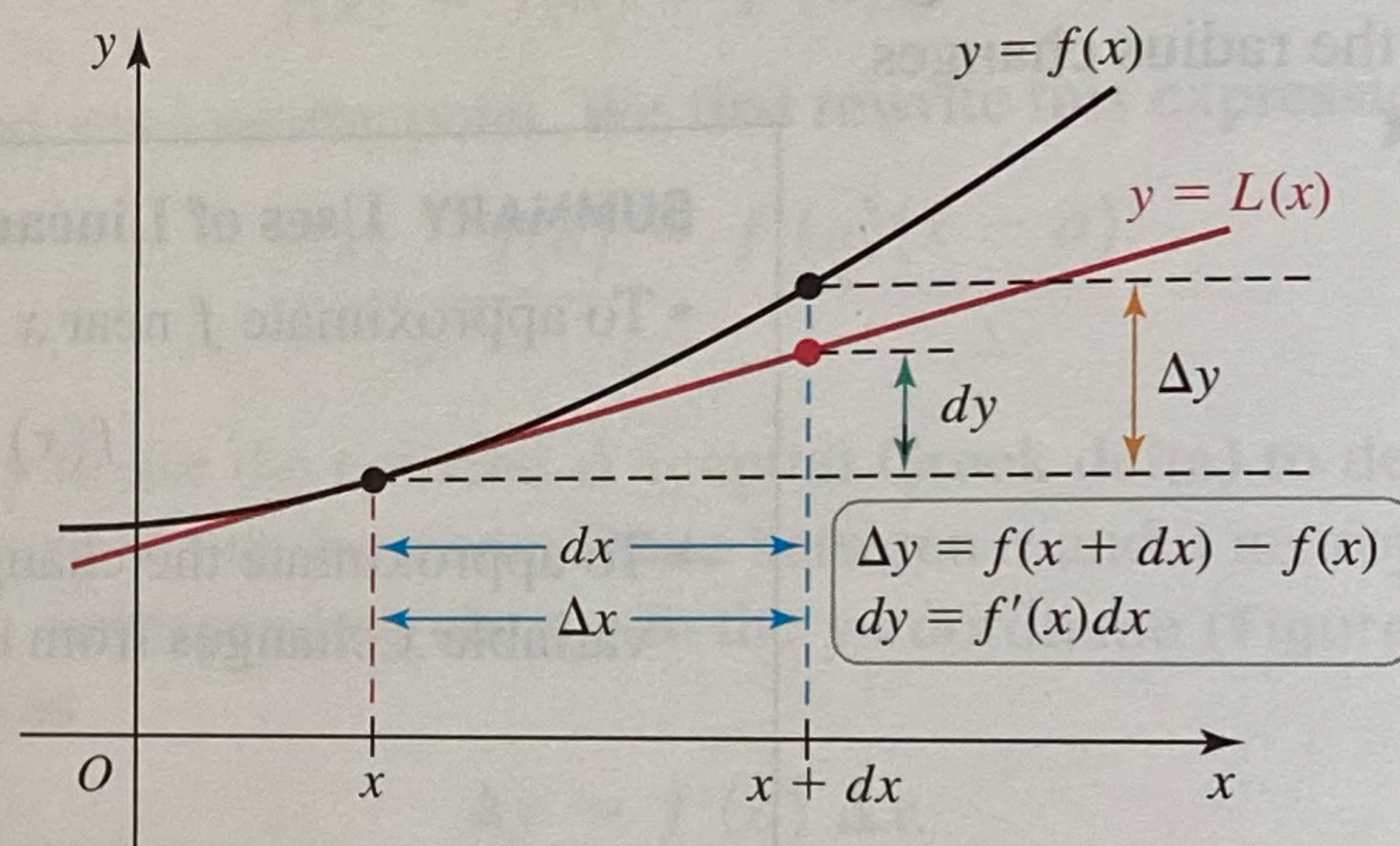


Figure 4.70

EXAMPLE 6 Differentials as change Use the notation of differentials to write the approximate change in $f(x) = 3 \cos^2 x$ given a small change dx .

► Recall that $\sin 2x = 2 \sin x \cos x$.

SOLUTION With $f(x) = 3 \cos^2 x$, we have $f'(x) = -6 \cos x \sin x = -3 \sin 2x$. Therefore,

$$dy = f'(x) dx = -3 \sin 2x dx.$$

The interpretation is that a small change dx in the independent variable x produces an approximate change in the dependent variable of $dy = -3 \sin 2x dx$. For example, if x increases from $x = \pi/4$ to $x = \pi/4 + 0.1$, then $dx = 0.1$ and

$$dy = -3 \sin(\pi/2)(0.1) = -0.3.$$

The approximate change in the function is -0.3 , which means a decrease of approximately 0.3.

Related Exercises 60, 66 ◀

SECTION 4.6 EXERCISES

Getting Started

- Sketch the graph of a smooth function f and label a point $P(a, f(a))$ on the curve. Draw the line that represents the linear approximation to f at P .
- Suppose you find the linear approximation to a differentiable function at a local maximum of that function. Describe the graph of the linear approximation.
- How is linear approximation used to approximate the value of a function f near a point at which f and f' are easily evaluated?
- How can linear approximation be used to approximate the change in $y = f(x)$ given a change in x ?
- Suppose f is differentiable on $(-\infty, \infty)$, $f(1) = 2$, and $f'(1) = 3$. Find the linear approximation to f at $x = 1$ and use it to approximate $f(1.1)$.

25–36. Lin
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6. Suppose f is differentiable on $(-\infty, \infty)$ and the equation of the line tangent to the graph of f at $x = 2$ is $y = 5x - 3$. Use the linear approximation to f at $x = 2$ to approximate $f(2.01)$.
7. Use linear approximation to estimate $f(3.85)$ given that $f(4) = 3$ and $f'(4) = 2$.
8. Use linear approximation to estimate $f(5.1)$ given that $f(5) = 10$ and $f'(5) = -2$.
9. Given a function f that is differentiable on its domain, write and explain the relationship between the differentials dx and dy .
10. Does the differential dy represent the change in f or the change in the linear approximation to f ? Explain.
11. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.01) - f(5) = 0.25$. Use linear approximation to estimate the value of $f'(5)$.
12. Suppose f is differentiable on $(-\infty, \infty)$, $f(5.99) = 7$, and $f(6) = 7.002$. Use linear approximation to estimate the value of $f'(6)$.

Practice Exercises

13–14. Estimating speed Use the linear approximation given in Example 1 to answer the following questions.

13. If you travel one mile in 59 seconds, what is your approximate average speed? What is your exact speed?
14. If you travel one mile in 63 seconds, what is your approximate average speed? What is your exact speed?

15–18. Estimating time Suppose you want to travel D miles at a constant speed of $(60 + x)$ mi/hr, where x could be positive or negative. The time in minutes required to travel D miles is $T(x) = 60D(60 + x)^{-1}$.

15. Show that the linear approximation to T at the point $x = 0$ is
- $$T(x) \approx L(x) = D \left(1 - \frac{x}{60} \right).$$
16. Use the result of Exercise 15 to approximate the amount of time it takes to drive 45 miles at 62 mi/hr. What is the exact time required?
17. Use the result of Exercise 15 to approximate the amount of time it takes to drive 80 miles at 57 mi/hr. What is the exact time required?
18. Use the result of Exercise 15 to approximate the amount of time it takes to drive 93 miles at 63 mi/hr. What is the exact time required?

19–24. Linear approximation Find the linear approximation to the following functions at the given point a .

19. $f(x) = 4x^2 + x; a = 1$ 20. $f(x) = x^3 - 5x + 3; a = 2$
 21. $g(t) = \sqrt{2t + 9}; a = -4$ 22. $h(w) = \sqrt{5w - 1}; a = 1$
 23. $f(x) = \sqrt[3]{x}; a = 8$ 24. $f(x) = 9(4x + 11)^{2/3}; a = 4$

25–36. Linear approximation

- a. Write the equation of the line that represents the linear approximation to the following functions at the given point a .
 b. Use the linear approximation to estimate the given quantity.
 c. Compute the percent error in your approximation, $100 \left| \frac{\text{approximation} - \text{exact}}{\text{exact}} \right|$, where the exact value is given by a calculator.

25. $f(x) = 12 - x^2; a = 2; f(2.1)$

26. $f(x) = \sin x; a = \pi/4; f(0.75)$

27. $f(x) = 1/(1 + x); a = 0; f(-0.1)$

28. $f(x) = x/(x + 1); a = 1; f(1.1)$

29. $f(x) = \cos x; a = 0; f(-0.01)$

30. $f(x) = x^{-3}; a = 1; f(1.05)$

31. $f(x) = (8 + x)^{-1/3}; a = 0; f(-0.1)$

32. $f(x) = \sqrt[4]{x}; a = 81; f(85)$

33. $f(x) = 1/(x + 1); a = 0; 1/1.1$

34. $f(x) = \cos x; a = \pi/4; \cos 0.8$

35. $f(x) = \sqrt[3]{64 + x}; a = 0; \sqrt[3]{62.5}$

36. $f(x) = \tan x; a = 0; \tan 3^\circ$

37–46. Estimations with linear approximation Use linear approximations to estimate the following quantities. Choose a value of a to produce a small error.

37. $1/203$ 38. $\tan(-2^\circ)$ 39. $\sqrt{146}$ 40. $\sqrt[3]{65}$

41. $1/1.05$ 42. $\sqrt{5/29}$ 43. $\sin(\pi/4 + 0.1)$

44. $1/\sqrt{119}$ 45. $1/\sqrt[3]{510}$ 46. $\cos 31^\circ$

47–50. Linear approximation and concavity Carry out the following steps for the given functions f and points a .

- a. Find the linear approximation L to the function f at the point a .
 b. Graph f and L on the same set of axes.
 c. Based on the graphs in part (b), state whether linear approximations to f near $x = a$ are underestimates or overestimates.
 d. Compute $f''(a)$ to confirm your conclusions in part (c).

47. $f(x) = \frac{2}{x}; a = 1$ 48. $f(x) = 5 - x^2; a = 2$

49. $f(x) = 1/\sqrt{x}; a = 1$ 50. $f(x) = \sqrt{2} \cos x; a = \frac{\pi}{4}$

T 51. Error in driving speed Consider again the average speed $s(x)$ and its linear approximation $L(x)$ discussed in Example 1. The error in using $L(x)$ to approximate $s(x)$ is given by $E(x) = |L(x) - s(x)|$. Use a graphing utility to determine the (approximate) values of x for which $E(x) \leq 1$. What does your answer say about the accuracy of the average speeds estimated by $L(x)$ over this interval?

52. Ideal Gas Law The pressure P , temperature T , and volume V of an ideal gas are related by $PV = nRT$, where n is the number of moles of the gas and R is the universal gas constant. For the purposes of this exercise, let $nR = 1$; therefore, $P = T/V$.

- a. Suppose the volume is held constant and the temperature increases by $\Delta T = 0.05$. What is the approximate change in the pressure? Does the pressure increase or decrease?
 b. Suppose the temperature is held constant and the volume increases by $\Delta V = 0.1$. What is the approximate change in the pressure? Does the pressure increase or decrease?
 c. Suppose the pressure is held constant and the volume increases by $\Delta V = 0.1$. What is the approximate change in the temperature? Does the temperature increase or decrease?

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- The linear approximation to $f(x) = x^2$ at $x = 0$ is $L(x) = 0$.
- Linear approximation at $x = 0$ provides a good approximation to $f(x) = |x|$.
- If $f(x) = mx + b$, then the linear approximation to f at any point is $L(x) = f(x)$.

54. Time function Show that the function $T(x) = 60D(60 + x)^{-1}$ gives the time in minutes required to drive D miles at $60 + x$ miles per hour.

55–58. Approximating changes

55. Approximate the change in the volume of a sphere when its radius changes from $r = 5$ ft to $r = 5.1$ ft ($V(r) = \frac{4}{3}\pi r^3$).

56. Approximate the change in the volume of a right circular cone of fixed height $h = 4$ m when its radius increases from $r = 3$ m to $r = 3.05$ m ($V(r) = \frac{1}{3}\pi r^2 h$).

57. Approximate the change in the lateral surface area (excluding the area of the base) of a right circular cone of fixed height $h = 6$ m when its radius decreases from $r = 10$ m to $r = 9.9$ m ($S = \pi r\sqrt{r^2 + h^2}$).

58. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20$ m to $r = 21$ m ($F(r) = 0.01/r^2$).

59–66. Differentials Consider the following functions and express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)dx$.

59. $f(x) = 2x + 1$

60. $f(x) = \sin^2 x$

61. $f(x) = \frac{1}{x^3}$

62. $f(x) = \sqrt{x^2 + 1}$

63. $f(x) = 2 - a \cos x$, a constant

64. $f(x) = \frac{x + 4}{4 - x}$

65. $f(x) = 3x^3 - 4x$

66. $f(x) = \tan x$

Explorations and Challenges

67. Errors in approximations Suppose $f(x) = \sqrt[3]{x}$ is to be approximated near $x = 8$. Find the linear approximation to f at 8. Then complete the following table, showing the errors in various approximations. Use a calculator to obtain the exact values. The percent error is $100|\text{approximation} - \text{exact}|/|\text{exact}|$. Comment on the behavior of the errors as x approaches 8.

x	Linear approx.	Exact value	Percent error
8.1			
8.01			
8.001			
8.0001			
7.9999			
7.999			
7.99			
7.9			

68. Errors in approximations Suppose $f(x) = 1/(x + 1)$ is to be approximated near $x = 0$. Find the linear approximation to f at 0. Then complete the following table, showing the errors in various approximations. Use a calculator to obtain the exact values. The percent error is $100|\text{approximation} - \text{exact}|/|\text{exact}|$. Comment on the behavior of the errors as x approaches 0.

x	Linear approx.	Exact value	Percent error
0.1			
0.01			
0.001			
0.0001			
-0.0001			
-0.001			
-0.01			
-0.1			

69. Linear approximation and the second derivative Draw the graph of a function f such that $f(1) = f'(1) = f''(1) = 1$. Draw the linear approximation to the function at the point $(1, 1)$. Now draw the graph of another function g such that $g(1) = g'(1) = 1$ and $g''(1) = 10$. (It is not possible to represent the second derivative exactly, but your graphs should reflect the fact that $f''(1)$ is relatively small compared to $g''(1)$.) Now suppose linear approximations are used to approximate $f(1.1)$ and $g(1.1)$.

- Which function has the more accurate linear approximation near $x = 1$ and why?
- Explain why the error in the linear approximation to f near a point a is proportional to the magnitude of $f''(a)$.

QUICK CHECK ANSWERS

- The linear approximation lies below the graph of f for x near a .
- $L(15) = 45$, $s(15) = 48$; $x = 15$ is not close to 0.
- $a = 0.16$
- Note that $f(0) = 1$ and $f'(0) = 0$, so $L(x) = 1$ (this is the line tangent to $y = \cos x$ at $(0, 1)$).
- $\Delta V \approx 4\pi a^2 \Delta r$