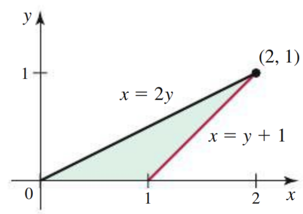
Math 241: Homework 12 Solutions

Graded Problems:
Section 6.2: 6, 10, 16
Section 6.3: 22, 26

Section 6.2

Problem 6. *Find the area of the region (see figure) in two ways.*

(a) By integrating with respect to y
(b) Using geometry



Solution

(a)

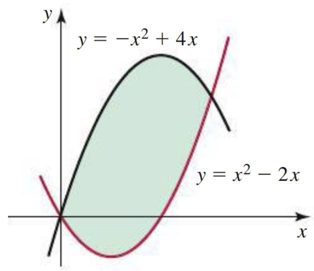
$$\begin{aligned}\text{Area} &= \int_0^1 [(y+1)-(2y)]\, dy \\ &= \int_0^1 (1-y)\, dy \\ &= \left(y - \frac{y^2}{2}\right)\Big|_0^1 \\ &= \left(1-\frac{1}{2}\right) - (0-0) \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

(b) If you draw a line down from $(2,1)$ to the x-axis, you can see that the given triangle is the result of a bigger right triangle (with base 2 and height 1) minus a smaller right triangle (with base 1 and height 1.

$$\text{Area} = \frac{1}{2}(2)(1) - \frac{1}{2}(1)(1) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

□

Problem 10. *Determine the area of the shaded region in the following figure.*



Solution First we have to find where they intersect.

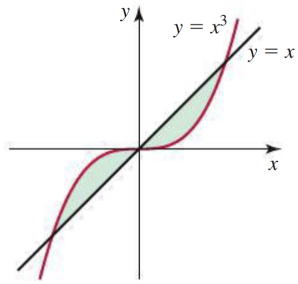
$$\begin{aligned} -x^2 + 4x = x^2 - 2x &\Leftrightarrow 2x^2 - 6x = 0 \\ &\Leftrightarrow 2x(x-3) = 0 \\ &\Leftrightarrow x = 0, 3 \end{aligned}$$

Now we have

$$\begin{aligned} \text{Area} &= \int_0^3 [(-x^2 + 4x) - (x^2 - 2x)]\ dx \\ &= \int_0^3 (-2x^2 + 6x)\ dx \\ &= \left(\frac{-2x^3}{3} + 3x^2 \right)\Big|_0^3 \\ &= \left(-\frac{54}{3} + 27 \right) - (0 - 0) \\ &= -\frac{54}{3} + \frac{81}{3} \\ &= \frac{27}{3} \\ &= \boxed{9} \end{aligned}$$

□

Problem 16. *Determine the area of the shaded region in the following figure.*



Solution First we find where they intersect.

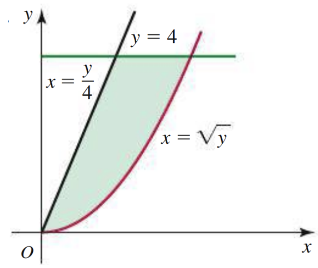
$$\begin{aligned} x^3 = x &\Leftrightarrow x^3 - x = 0 \\ &\Leftrightarrow x(x^2-1) = 0 \\ &\Leftrightarrow x(x-1)(x+1) = 0 \\ &\Leftrightarrow x = -1, 0, 1 \end{aligned}$$

Using this and the diagram we have

$$\begin{aligned} \text{Area} &= \int_{-1}^{0} (x^3 - x)\ dx + \int_{0}^{1} (x - x^3)\ dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{0}^{1} \\ &= \left[(0-0) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - (0-0)\right] \\ &= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

□

Problem 20. *Determine the area of the shaded region in the following figure.*



Solution

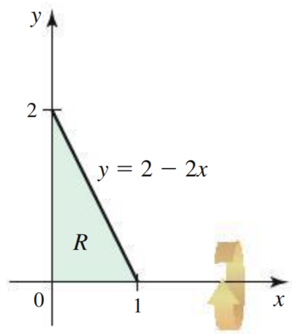
$$\begin{aligned}
\text{Area} &= \int_0^4 \left(\sqrt{y} - \frac{y}{4}\right)\, dy \\
&= \int_0^4 \left(y^{1/2} - \frac{y}{4}\right) \\
&= \left(\frac{2}{3}y^{3/2} - \frac{y^2}{8}\right)\Bigg|_0^4 \\
&= \left(\frac{2}{3}(8) - \frac{16}{8}\right) - (0 - 0) \\
&= \frac{16}{3} - 2 \\
&= \frac{16}{3} - \frac{6}{3} \\
&= \boxed{\frac{10}{3}}
\end{aligned}$$

□

Section 6.3

Problem 18. *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given axis.*

$y = 2 - 2x$, $y = 0$, *and* $x = 0$; *about the* x*-axis. (Verify that your answer agrees with the volume formula for a cone.)*



Solution

$$\begin{aligned}\text{Volume} &= \int_0^1 \pi(2-2x)^2\ dx \\ &= \pi \int_0^1 (4 - 8x + 4x^2)\ dx \\ &= \pi\left(4x - 4x^2 + \frac{4x^3}{3}\right)\Big|_0^1 \\ &\pi\left(4 - 4 + \frac{4}{3}\right) - \pi(0 - 0 + 0) \\ &= \frac{4}{3}\pi\end{aligned}$$

The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

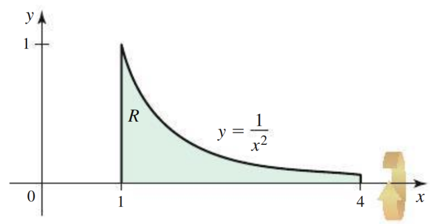
The cone in this problem has radius 2 and height 1 so the volume is

$$V = \frac{1}{3}\pi(2)^2(1) = \frac{4}{3}\pi\checkmark$$

□

Problem 20. *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given axis.*

$y = \frac{1}{x^2}$, $y = 0$, $x = 1$, *and* $x = 4$; *about the* x*-axis*



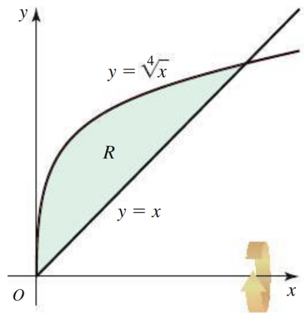
Solution

$$\begin{aligned}
\text{Volume} &= \int_1^4 \pi \left(\frac{1}{x^2}\right)^2 \, dx \\
&= \pi \int_1^4 (x^{-4}) \, dx \\
\pi \left(-\frac{x^{-3}}{3}\right)\Bigg|_1^4 \\
&= \pi \left(-\frac{1}{3x^3}\right)\Bigg|_1^4 \\
&= -\frac{\pi}{192} + \frac{\pi}{3} \\
&= -\frac{\pi}{162} + \frac{64\pi}{162} \\
&= \frac{63\pi}{192} \\
&= \boxed{\frac{12\pi}{64}}
\end{aligned}$$

□

Problem 22. *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given axis.*

$y = x$ *and* $y = \sqrt[4]{x}$*; about the x-axis*



Solution First we find where they intersect:

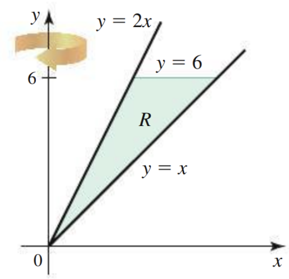
$$
\begin{aligned}
\sqrt[4]{x} = x &\Leftrightarrow x = x^4 \\
&\Leftrightarrow x^4 - x = 0 \\
&\Leftrightarrow x(x^3 - 1) = 0 \\
&\Leftrightarrow x = 0, 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Volume} &= \int_0^1 \pi\left(\sqrt[4]{x}^2 - x^2\right)\, dx \\
&= \pi \int_0^1 (x^{1/2} - x^2)\, dx \\
&= \pi \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1 \\
&= \pi\left(\frac{2}{3} - \frac{1}{3}\right) - \pi(0 - 0) \\
&= \boxed{\frac{\pi}{3}}
\end{aligned}
$$

□

Problem 26. *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given axis.*

$y = x$, $y = 2x$, *and* $y = 6$; *about the y-axis*



Solution First we need to solve each equation for x:

$$y = x \Leftrightarrow x = y \text{ and } y = 2x \Leftrightarrow x = \frac{y}{2}$$

$$\begin{aligned}
\text{Volume} &= \int_0^6 \pi \left(y^2 - \left(\frac{y}{2}\right)^2 \right) \, dy \\
&= \pi \int_0^6 \left(y^2 - \frac{y^2}{4} \right) \, dy \\
&= \pi \int_0^6 \left(\frac{3y^2}{4} \right) \, dy \\
&= \pi \left(\frac{y^3}{4} \right) \Big|_0^6 \\
&= \frac{216\pi}{4} - 0 \\
&= \boxed{54\pi}
\end{aligned}$$

□