Math 241: Homework 3 Solutions

Graded Problems:
Section 2.4: 22, 34
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Section 2.3

Problem 60. *Find the following limit or state that they do not exist:* $\lim_{x\to 0} \frac{\sin 2x}{\sin x}$

Solution

Method 1

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \sin 2x \cdot \frac{1}{\sin x} \\
&= \lim_{x\to 0} \sin 2x \cdot \frac{2x}{2x} \cdot \frac{1}{\sin x} \cdot \frac{x}{x} \\
&= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{x}{\sin x} \cdot \frac{2x}{x} \\
&= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}} \\
&= 1 \cdot 1 \cdot \lim_{x\to 0} 2 \\
&= \boxed{2}
\end{aligned}$$

Method 2

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}} \\
&= \lim_{x\to 0} 2\cos x \\
&= 2\cos 0 \\
&= \boxed{2}
\end{aligned}$$

□

Section 2.4

Problem 22. *Determine the following limits.*

a) $\displaystyle\lim_{x\to3^+}\frac{2}{(x-3)^3}$ *b)* $\displaystyle\lim_{x\to3^-}\frac{2}{(x-3)^3}$ *c)* $\displaystyle\lim_{x\to3}\frac{2}{(x-3)^3}$

Solution

a) $x\to3^+$ means $x>3$, so $x-3>0$

$$\begin{aligned}\lim_{x\to3^+}\frac{2}{(x-3)^3} &\to \frac{2}{(0^+)^3}\\ &\to \frac{2}{0^+}\\ &\to \boxed{\infty}\end{aligned}$$

b) $x\to3^-$ means $x<3$, so $x-3<0$

$$\begin{aligned}\lim_{x\to3^+}\frac{2}{(x-3)^3} &\to \frac{2}{(0^-)^3}\\ &\to \frac{2}{0^-}\\ &\to \boxed{-\infty}\end{aligned}$$

c) Since the one-sided limits are not equal, the limit $\boxed{\text{DNE}}$

□

Problem 24. *Determine the following limits.*

a) $\displaystyle\lim_{x\to1^+}\frac{x}{|x-1|}$ *b)* $\displaystyle\lim_{x\to1^-}\frac{x}{|x-1|}$ *c)* $\displaystyle\lim_{x\to1}\frac{x}{|x-1|}$

Solution

a) $x\to1^+$ means $x>1$ so $x-1>0$.

$$\begin{aligned}\lim_{x\to1^-}\frac{x}{|x-1|} &= \lim_{x\to1^+}\frac{x}{x-1}\\ &\to \frac{1}{0^+}\\ &\to \boxed{\infty}\end{aligned}$$

b) $x \to 1^-$ means $x < 1$ so $x - 1 < 0$.

$$\lim_{x\to 1^-} \frac{x}{|x-1|} = \lim_{x\to 1^-} \frac{x}{-(x-1)}$$
$$\to \frac{1}{-0^-}$$
$$\to \frac{1}{0^+}$$
$$\to \boxed{\infty}$$

c) $\boxed{\infty}$

□

Problem 34. *Determine the following limit:* $\lim_{t\to 5} \dfrac{4t^2 - 100}{t - 5}$

Solution

$$\begin{aligned}\lim_{t\to 5} \frac{4t^2 - 100}{t-5} &= \lim_{t\to 5} \frac{4(t^2 - 25)}{t-5} \\ &= \lim_{t\to 5} \frac{4\cancel{(t-5)}(t+5)}{\cancel{t-5}} \\ &= \lim_{t\to 5} 4(t+5) \\ &= 4(5+5) \\ &= \boxed{40}\end{aligned}$$

□

Section 2.5

Problem 22. *Determine the following limits:* $\lim_{x \to -\infty} (3x^7 + x^2)$

Solution

$$\begin{aligned} \lim_{x \to -\infty} (3x^7 + x^2) &= \lim_{x \to -\infty} (3x^7) \\ &\to 3(-\infty)^7 \\ &\to 3(-\infty) \\ &\to \boxed{-\infty} \end{aligned}$$

□

Problem 26. *Determine the following limits:* $\lim_{x \to \infty} \frac{9x^3 + x^2 - 5}{3x^4 + 4x^2}$

Solution

$$\begin{aligned} \lim_{x \to \infty} \frac{9x^3 + x^2 - 5}{3x^4 + 4x^2} &= \lim_{x \to \infty} \frac{\frac{9x^3}{x^4} + \frac{x^2}{x^4} - \frac{5}{x^4}}{\frac{3x^4}{x^4} + \frac{4x^2}{x^4}} \\ &= \lim_{x \to \infty} \frac{\frac{9}{x} + \frac{1}{x} - \frac{5}{x^4}}{3 + \frac{4}{x^2}} \\ &= \frac{0 + 0 - 0}{3 + 0} \\ &= \boxed{0} \end{aligned}$$

□

Section 2.6

Problem 22. *Determine whether the following functions are continuous at a. Use the continuity checklist to justify your answer.*

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases} \quad ; \quad a = 3$$

Solution

$$\begin{aligned} \lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{x^2-4x+3}{x-3} \\ &= \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{x-3}} \\ &= \lim_{x\to 3}(x-1) \\ &= 3-1 \\ &= 2 \end{aligned}$$

$$f(3) = 2$$

$\boxed{\text{Since } \lim_{x\to 3} f(x) = f(3), \text{ the function is continuous at } 3}$

□

Problem 28. *Determine the interval(s) on which the following functions are continuous:*

$$s(x) = \frac{x^2-4x+3}{x^2-1}$$

Solution $s(x)$ is a rational function so it is continuous on its domain.

$$\begin{aligned} x^2 - 1 \neq 0 &\Rightarrow (x-1)(x+1) \neq 0 \\ &\Rightarrow x \neq \pm 1 \end{aligned}$$

In interval notation, $\boxed{(-\infty,-1)\cup(-1,1)\cup(1,\infty)}$

□

Problem 40. *Complete the following steps for each functions*

a) *Use the continuity checklist to show that f is not continuous at the given value of a.*

b) *Determine whether f is continuous from the left or the right at a.*

c) *State the interval(s) of continuity.*

$$f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0 \end{cases} \quad ; \quad a = 0$$

Solution

a)

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x^3 + 4x + 1) \\ &= (0)^3 + 4(0) + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (2x^3) \\ &= 2(0)^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^3 + 4(0) + 1 \\ &= 1 \end{aligned}$$

Since the limit DNE, it is not continuous at 0.

b) Since $\lim_{x \to 0^-} f(x) = f(0)$, it is continuous from the left at 0.

c) $(-\infty, 0], (0, \infty)$

□

Problem 65a. *Use the Intermediate Value Theorem to show that the following equations have a solution on the given interval.*

$$2x^3 + x - 2 = 0; (-1, 1)$$

Solution Let $f(x) = 2x^3 + x - 2$.

$$\begin{aligned} f(-1) &= 2(-1)^3 + (-1) - 2 \\ &= 2(-1) - 1 - 2 \\ &= -2 - 1 - 2 \\ &= -5 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 + (1) - 2 \\ &= 2 + 1 - 2 \\ &= 1 \end{aligned}$$

Since f is continuous on $(-1, 1)$ with $f(-1) > 0$ and $f(1) < 0$, by the Intermediate Value Theorem, $f(x) = 0$ has at least one solution on the interval $(-1, 1)$.

$\square$

Problem 82. *Determine the value of the constant a for which the function*

$$f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & \text{if } x \neq -1 \\ a & \text{if } x = -1 \end{cases}$$

Solution We want $f(-1) = \lim_{x \to -1} f(x)$.

$$\begin{aligned} \lim_{x \to -1} f(x) &= \lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} \\ &= \lim_{x \to -1} \frac{\cancel{(x+1)}(x+2)}{\cancel{x+1}} \\ &= \lim_{x \to -1} (x + 2) \\ &= -1 + 2 \\ &= 1 \end{aligned}$$

$$f(-1) = a$$

Therefore, we want $\boxed{a = 1}$

$\square$

Common Mistakes

- When you're doing infinite limits, make sure you show your work as to why the answer is ∞ or $-\infty$.
- You need to show all of your work to get full credit. Correct answers with no work will receive barely any points (if any).