Math 241: Homework 4 Solutions

Graded Problems:

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Section 3.1

Problem 38.

(a) For the following functions and values of a, find $f'(a)$.

(b) Determine an equation of the line tangent to the graph of f at the point $(a, f(a))$ for the given value of a.

$$f(x) = \frac{1}{x^2}; \quad a = 1$$

Solution

(a) (read across)

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} &&= \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} &&= \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} \cdot \frac{(1+h)^2}{(1+h)^2} \\
&= \lim_{h\to 0} \frac{1 - (1+h)^2}{h(1+h)^2} &&= \lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{1} - \cancel{1} - 2h - h^2}{h(1+h)^2} &&= \lim_{h\to 0} \frac{-2h - h^2}{h(1+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2-h)}{\cancel{h}(1+h)^2} &&= \lim_{h\to 0} \frac{-2-h}{(1+h)^2} \\
&= \frac{-2-0}{(1+0)^2} &&= \frac{-2}{1} \\
&= \boxed{-2}
\end{aligned}$$

(b) The slope is $m = -2$ and the point is $(1, f(1)) = (1, 1/1^2) = (1, 1)$. Using point-slope form we have

$$\boxed{y - 1 = -2(x - 1)}$$

□

Problem 44. *Evaluate the derivative of the following functions at the given point:*

$$f(t) = t - t^2; \quad a = 2$$

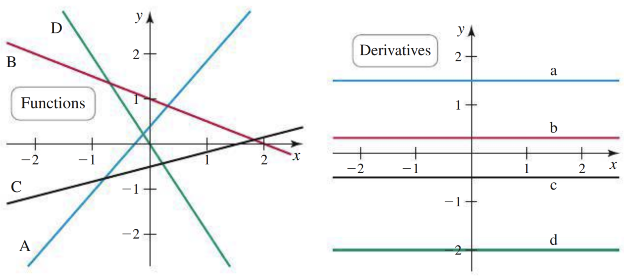
Solution

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{[(2+h) - (2+h)^2] - [2 - 2^2]}{h} \\
&= \lim_{h\to 0} \frac{[2 + h - (4 + 4h + h^2)] - (2 - 4)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{2} + h - \cancel{4} - 4h - h^2 - \cancel{2} + \cancel{4}}{h} \\
&= \lim_{h\to 0} \frac{h - 4h - h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(1 - 4 - h)}{\cancel{h}} \\
&= \lim_{h\to 0} (1 - 4 - h) \\
&= 1 - 4 - 0 \\
&= \boxed{-3}
\end{aligned}$$

□

Section 3.2

Problem 16. *Match graphs a-d of derivative functions with possible graphs A-D of corresponding functions.*



Solution A-a, C-b, D-d, B-c

□

Problem 22.

(a) *Use limits to find the derivative function f' for the following function f.*

(b) *Evaluate $f'(a)$ for the given values of a.*

$$f(x) = 7; \quad a = -1, 2$$

Solution

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{7-7}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= \boxed{0} \end{aligned}$$

(b) $\boxed{f(-1) = 0, f(2) = 0}$

□

Problem 24.

(a) Use limits to find the derivative function f' for the following functions f.

(b) Evaluate $f'(a)$ for the given values of a.

$$f(x) = x^2 + 3x; \quad a = -1, 4$$

Solution

(a)

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[(x+h)^2 + 3(x+h)] - (x^2 + 3x)}{h} \\
&= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} \\
&= \lim_{h \to 0} (2x + h + 3) \\
&= 2x + 0 + 3 \\
&= \boxed{2x + 3}
\end{aligned}$$

(b)

$$\begin{aligned}
f'(-1) &= 2(-1) + 3 = \boxed{1} \\
f'(4) &= 2(4) + 3 = \boxed{11}
\end{aligned}$$

□

Problem 38.

(a) Find the derivative function f' for the following function f.

(b) Find an equation of the line tangent to the graph of f at $(a, f(a))$ for the given value of a.

$$f(x) = \sqrt{x+2} \quad a = 7$$

Solution

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{\cancel{x} + h + \cancel{2} - \cancel{x} - \cancel{2}}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+2} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \\
&= \boxed{\frac{1}{2\sqrt{x+2}}}
\end{aligned}$$

(b) Slope is $f'(7) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$ and the point is $(7, f(7)) = (7, \sqrt{9}) = (7, 3)$. Using point slope form we have

$$\boxed{y - 3 = \frac{1}{6}(x - 7)}$$

□

Section 3.3

Problem 24. *Find the derivative of the following function:* $g(w) = \frac{5}{6}w^{12}$

Solution

$$\begin{aligned} g'(w) &= \frac{d}{dw}\left(\frac{5}{6}w^{12}\right) \\ &= \frac{5}{6}\left(\frac{d}{dw}(w^{12})\right) \\ &= \frac{5}{6}(12w^{11}) \\ &= \boxed{10w^{11}} \end{aligned}$$

□

Problem 32. *Find the derivative of the following function:* $g(x) = 6x^5 - \frac{5}{2}x^2 + x + 5$

Solution

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(6x^5 - \frac{5}{2}x^2 + x + 5\right) \\ &= 6\left(\frac{d}{dx}(x^5)\right) - \frac{5}{2}\left(\frac{d}{dx}(x^2)\right) + \frac{d}{dx}(x) + \frac{d}{dx}(5) \\ &= 6(5x^{5-5}) - \frac{5}{2}(2x^{2-2}) + 1 + 0 \\ &= \boxed{30x^4 - 5x + 1} \end{aligned}$$

□

Problem 34. *Find the derivative of the following function:* $f(t) = 6\sqrt{t} - 4t^3 + 9$

Solution

$$\begin{aligned} f'(t) &= \frac{d}{dt}(6t^{1/2} - 4t^3 + 9) \\ &= 6\left(\frac{d}{dt}(t^{1/2})\right) - 4\left(\frac{d}{dt}(t^3)\right) + \frac{d}{dt}(9) \\ &= 6\left(\frac{1}{2}t^{1/2-1}\right) - 4(3t^{3-1}) + 0 \\ &= \boxed{3t^{-1/2} - 12t^2} \end{aligned}$$

□

Problem 68. *Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following function:* $f(x) = \frac{1}{8}x^4 - 3x^2 + 1$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{8}x^4 - 3x^2 + 1\right) \\ &= \frac{1}{8}\left(\frac{d}{dx}(x^4)\right) - 3\left(\frac{d}{dx}(x^2)\right) + \frac{d}{dx}(1) \\ &= \frac{1}{8}(4x^3) - 3(2x) + 0 \\ &= \boxed{\frac{1}{2}x^3 - 6x} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(\frac{1}{2}x^3 - 6x\right) \\ &= \frac{1}{2}\left(\frac{d}{dx}(x^3)\right) - 6\left(\frac{d}{dx}(x)\right) \\ &= \frac{1}{2}(3x^2) - 6(1) \\ &= \boxed{\frac{3}{2}x^2 - 6} \end{aligned}$$

□

Problem 82. *For the constant function $f(x) = c$, use the definition of the derivative to show that $f'(x) = 0$.*

Solution

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \quad \checkmark \end{aligned}$$

□

Section 3.4

Problem 8. *Find the derivative the following ways:*

(a) Using the Product Rule or the Quotient Rule. Simplify your result.

(b) By expanding the product first or by simplifying the quotient first. Verify that your answer agrees with part (a).

$$g(t) = (t + 1)(t^2 - t + 1)$$

Solution

(a)

$$\begin{aligned} g'(t) &= \left(\frac{d}{dt}(t+1)\right)(t^2 - t + 1) + (t+1)\left(\frac{d}{dt}(t^2 - t + 1)\right) \\ &= (1)(t^2 - t + 1) + (t + 1)(2t - 1) \\ &= t^2 - t + 1 + 2t^2 - t + 2t - 1 \\ &= \boxed{3t^2} \end{aligned}$$

(b)

$$(t + 1)(t^2 - t + 1) = t^3 - t^2 + t + t^2 - t + 1 = t^3 + 1$$

$$\frac{d}{dt}(t^3 + 1) = \boxed{3t^2}$$

□

Problem 12. *Find the derivative the following ways:*

(a) Using the Product Rule or the Quotient Rule. Simplify your result.

(b) By expanding the product first or by simplifying the quotient first. Verify that your answer agrees with part (a).

$$g(s) = \frac{6s^3 - 15s^2 + 9s}{3s}$$

Solution

(a)

$$\begin{aligned}
g'(s) &= \frac{(3s)\left(\frac{d}{ds}(6s^3 - 15s^2 + 9s)\right) - (6s^3 - 15s^2 + 9s)\left(\frac{d}{ds}(3s)\right)}{(3s)^2} \\
&= \frac{(3s)(18s^2 - 30s + 9) - (6s^3 - 15s^2 + 9s)(3)}{9s^2} \\
&= \frac{54s^3 - 90s^2 + 27s - (18s^3 - 45s^2 + 27s)}{9s^2} \\
&= \frac{54s^3 - 90s^2 + \cancel{27s} - 18s^3 + 45s^2 - \cancel{27s}}{9s^2} \\
&= \frac{36s^3 - 45s^2}{9s^2} \\
&= \frac{36s^3}{9s^2} - \frac{45s^2}{9s^2} \\
&= \boxed{4s - 5}
\end{aligned}$$

(b)

$$\frac{6s^3 - 15s^2 + 9s}{3s} = \frac{6s^3}{3s} - \frac{15s^2}{3s} + \frac{9s}{3s} = 2s^2 - 5s + 3$$

$$\frac{d}{ds}(2s^2 - 5s + 3) = \boxed{4s - 5}$$

□

Problem 32. *Find and simplify the derivative of the following function:*

$$f(x) = \frac{2x+1}{x-1}$$

Solution

$$\begin{aligned} f'(x) &= \frac{(x-1)\left(\frac{d}{dx}(2x+1)\right) - (2x+1)\left(\frac{d}{dx}(x-1)\right)}{(x-1)^2} \\ &= \frac{(x-1)(2) - (2x+1)(1)}{(x-1)^2} \\ &= \frac{\cancel{2x} - 2 - \cancel{2x} - 1}{(x-1)^2} \\ &= \boxed{\frac{-3}{(x-1)^2}} \end{aligned}$$

□

Problem 60.

(a) *Find an equation of the line tangent to the given curve at a.*

$$y = \frac{x-2}{x+1}; \quad a = 1$$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x+1)\left(\frac{d}{dx}(x-2)\right) - (x-2)\left(\frac{d}{dx}(x+1)\right)}{(x+1)^2} \\ &= \frac{(x+1)(1) - (x-2)(1)}{(x+1)^2} \\ &= \frac{\cancel{x} + 1 - \cancel{x} + 2}{(x+1)^2} \\ &= \frac{3}{(x+1)^2} \end{aligned}$$

Plugging in $a = 1$ we have

$$m = \frac{3}{(1+1)^2} = \frac{3}{4} \quad \text{and} \quad f(1) = \frac{1-2}{1+1} = -\frac{1}{2}$$

Using point-slope form, we have

$$\boxed{y + \frac{1}{2} = \frac{3}{4}(x-1)}$$

□

Problem 68. *Find $f'(x)$ and $f''(x)$.*

$$f(x) = \frac{x}{x+2}$$

Solution

$$\begin{aligned} f'(x) &= \frac{(x+2)\left(\frac{d}{dx}(x)\right) - (x)\left(\frac{d}{dx}(x+2)\right)}{(x+2)^2} \\ &= \frac{(x+2)(1) - (x)(1)}{(x+2)^2} \\ &= \frac{\cancel{x} + 2 - \cancel{x}}{(x+2)^2} \\ &= \frac{2}{(x+2)^2} \\ &= \frac{2}{x^2+4x+4} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(x+2)^2\left(\frac{d}{dx}(2)\right) - 2\left(\frac{d}{dx}(x^2+4x+4)\right)}{((x+2)^2)^2} \\ &= \frac{0 - 2(2x+4)}{(x+2)^4} \\ &= \frac{-4(x+2)}{(x+2)^4} \\ &= \boxed{\frac{-4}{(x+2)^3}} \end{aligned}$$

□

Common Mistakes

- Read instructions carefully.
- Remember in class we said that for sections 3.1-3.2, you had to used the limit definition of a derivative. The properties of derivatives are introduced in section 3.3.