Math 241: Homework 5 Solutions

Graded Problems:

Section 3.5: 10, 30	Section 3.7: 23, 78
Section 3.6: 23	Section 3.8: 6

Section 3.5

Problem 10. *Find* $\dfrac{d^2}{dx^2}(\sec x)$.

Solution

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\begin{aligned}\frac{d^2}{dx^2}(\sec x) &= \frac{d}{dx}(\sec x \tan x)\\ &= \left(\frac{d}{dx}(\sec x)\right)\tan x + \sec x\left(\frac{d}{dx}\tan x\right)\\ &= (\sec x \tan x)\tan x + \sec x(\sec^2 x)\\ &= \boxed{\sec x \tan^2 x + \sec^3 x}\end{aligned}$$

□

Problem 26. *Find the derivative of the following function:* $y = \sin x + \dfrac{4\cos x}{x}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \cos x + \frac{x\left(\frac{d}{dx}(4\cos x)\right) - 4\cos x\left(\frac{d}{dx}x\right)}{x^2}\\ &= \cos x + \frac{x(4(-\sin x)) - 4\cos x(1)}{x^2}\\ &= \boxed{\cos x + \frac{-4x\sin x - 4\cos x}{x^2}}\end{aligned}$$

□

Problem 30. *Find the derivative of the following function:* $y = \dfrac{\sin x}{1 + \cos x}$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 + \cos x)\left(\frac{d}{dx}(\sin x)\right) - \sin x\left(\frac{d}{dx}(1 + \cos x)\right)}{(1 + \cos x)^2} \\
&= \frac{(1 + \cos x)(\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} \\
&= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \\
&= \frac{\cos x + 1}{(1 + \cos x)^2} \\
&= \boxed{\frac{1}{1 + \cos x}}
\end{aligned}$$

□

Problem 44. *Find the derivative of the following function:* $y = \sec x \tan x$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \left(\frac{d}{dx}(\sec x)\right)\tan x + \sec x\left(\frac{d}{dx}\tan x\right) \\
&= (\sec x \tan x)\tan x + \sec x(\sec^2 x) \\
&= \boxed{\sec x \tan^2 x + \sec^3 x}
\end{aligned}$$

□

Problem 52. *Verify the following derivative formula using the Quotient Rule.*

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Solution

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{\sin x\left(\frac{d}{dx}(\cos x)\right) - \cos x\left(\frac{d}{dx}(\sin x)\right)}{\sin^2 x} \\
&= \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x \quad \checkmark
\end{aligned}$$

□

Section 3.6

Problem 16. *Suppose the position of an object moving horizontally along a line after t seconds is given by the following functions $s = f(t)$, where s is measured in feet, with $s > 0$ corresponding to positions right of the origin.*

(a) Graph the position function.
(b) Find and graph the velocity function. When is the object stationary, moving to the right, and moving to the left?
(c) Determine the velocity and acceleration of the object at $t = 1$.
(d) Determine the acceleration of the object when its velocity is zero.
(e) On what intervals is the speed increasing?

$$f(t) = -t^2 + 4t - 3; \quad 0 \le t \le 5$$

Solution

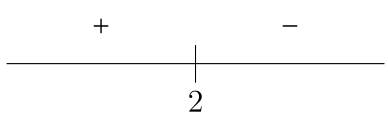
(a) Not assigned
(b) Graphing not assigned

$$v(t) = f'(t) = \boxed{-2t + 4}$$

The object is stationary when $v(t) = 0$, moving to the right when $v(t) > 0$, and moving to the left when $v(t) < 0$.

$$v(t) = 0 \Rightarrow -2t + 4 - 0 \Rightarrow t = 2$$

Testing the intervals we have



This gives that the object is stationary at 2 sec, moving to the right on $(0, 2)$ and moving to the left on $(2, \infty)$.

(c)

$$a(t) = v'(t) = -2$$

$$v'(1) = -2 + 4 = \boxed{2 \text{ ft/s}} \text{ and } a'(1) = \boxed{-2 \text{ ft/s}^2}$$

(d)

$$a'(2) = \boxed{-2 \text{ ft/s}^2}$$

(e) The speed is increasing when velocity and acceleration have the same sign, so $\boxed{(2, \infty)}$

□

Problem 23. *Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 32 ft/s from a height of 48 ft above the ground. The height (in feet) of the stone above the ground t seconds after it is thrown is $s(t) = -16t^2 + 32t + 48$.*

(a) Determine the velocity v of the stone after t seconds.
(b) When does the stone reach its highest point?
(c) What is the height of the stone at the highest point?
(d) When does the stone strike the ground?
(e) With what velocity does the stone strike the ground?
(f) On what intervals is the speed increasing?

Solution

(a)

$$v(t) = s'(t) = \boxed{-32t + 32}$$

(b) It reaches its highest point when $v(t) = 0$

$$v(t) = 0 \Rightarrow -32t + 32 = 0 \Rightarrow \boxed{t = 1 \text{ sec}}$$

(c)

$$s(1) = -16 + 32 + 48 = \boxed{64 \text{ ft}}$$

(d) The stone strikes the ground when $s(t) = 0$

$$\begin{aligned} s(t) = 0 &\Leftrightarrow -16t^2 + 32t + 48 = 0 \\ &\Leftrightarrow -16(t^2 - 2t + 3) = 0 \\ &\Leftrightarrow -16(t - 3)(t + 1) = 0 \\ &\Rightarrow t = -1, 3 \end{aligned}$$

You can't have negative time so $\boxed{t = 3 \text{ sec}}$

(e)

$$v(3) = -32(3) + 32 = \boxed{-64 \text{ ft/sec}}$$

(f) Not assigned

□

Section 3.7

Problem 23. *Let* $h(x) = f(g(x))$ *and* $p(x) = g(f(x))$*. Use the table to compute the following derivatives.*

a) $h'(3)$ *b)* $h'(2)$ *c)* $p'(4)$ *d)* $p'(2)$ *e)* $h'(5)$

x	1	2	3	4	5
$f(x)$	0	3	5	1	0
$f'(x)$	5	2	−5	−8	−10
$g(x)$	4	5	1	3	2
$g'(x)$	2	10	20	15	20

Solution

(a)

$$h'(3) = f'(g(3))g'(3) = f'(1)(20) = 5(20) = \boxed{100}$$

(b)

$$h'(2) = f'(g(2))g'(2) = f'(5)(10) = (-10)(10) = \boxed{-100}$$

(c)

$$p'(4) = g'(f(4))f'(4) = g'(1)(-8) = (2)(-8) = \boxed{-16}$$

(d)

$$p'(2) = g'(f(2))f'(2) = g'(3)(2) = (20)(2) = \boxed{40}$$

(e)

$$h'(5) = f'(g(5))g'(5) = f'(2)(20) = (2)(20) = \boxed{40}$$

□

Problem 28. *Calculate the derivative of the following function:* $y = \sqrt[3]{x^2 + 9}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2+9)^{1/3} \\ &= \frac{1}{3}(x^2+9)^{-2/3} \cdot \frac{d}{dx}(x^2+9) \\ &= \frac{1}{3}(x^2+9)^{-2/3}(2x) \\ &= \boxed{\frac{2}{3}x(x^2+9)^{-2/3}}\end{aligned}$$

□

Problem 40. *Calculate the derivative of the following function:* $y = \cos^4\theta + \sin^4\theta$

Solution

$$\begin{aligned}\frac{d}{d\theta}(\cos^4\theta + \sin^4\theta) &= 4\cos^3\theta \cdot \frac{d}{d\theta}\cos\theta + 4\sin^3\theta \cdot \frac{d}{d\theta}\sin\theta \\ &= 4\cos^3\theta(-\sin\theta) + 4\sin^3\theta\cos\theta \\ &= \boxed{-4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta}\end{aligned}$$

□

Problem 58. *Calculate the derivative of the following function:* $y = \left(\frac{x-1}{x+1}\right)^8$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\frac{x-1}{x+1}\right)^8 &= 8\left(\frac{x-1}{x+1}\right)^7 \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right) \\ &= 8\left(\frac{x-1}{x+1}\right)^7 \cdot \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} \\ &= 8\left(\frac{x-1}{x+1}\right)^7 \cdot \frac{x+1-x+1}{(x+1)^2} \\ &= 8\left(\frac{x-1}{x+1}\right)^7 \frac{2}{(x+1)^2} \\ &= \boxed{\frac{16}{(x+1)^2}\left(\frac{x-1}{x+1}\right)^7}\end{aligned}$$

□

Problem 78.

(a) *Calculate* $\frac{d}{dx}(x^2 + x)^2$ *using the Chain Rule. Simplify your answer.*

(b) *Expand* $(x^2+x)^2$ *first then calculate the derivative. Verify that your answer agrees with part (a).*

Solution

(a)

$$\begin{aligned}\frac{d}{dx}(x^2 + x)^2 &= 2(x^2 + x) \cdot \frac{d}{dx}(x^2 + x)\\ &= 2(x^2 + x)(2x + 1)\\ &= 2(2x^3 + x^2 + 2x^2 + x)\\ &= \boxed{4x^3 + 6x^2 + 2x}\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}(x^2 + x)^2 &= \frac{d}{dx}(x^4 + 2x^3 + x^2)\\ &= \boxed{4x^3 + 6x^2 + 2x} \quad \checkmark\end{aligned}$$

$\square$

Section 3.8

Problem 4. *Identify and correct the error in the following argument:*

Suppose $y^2 + 2y = 2x^3 - 7$*. Differentiating both sides with respect to* x *to find* $\frac{dy}{dx}$*, we have* $2y + 2\frac{dy}{dx} = 6x^2$*, which implies that* $\frac{dy}{dx} = 3x^2 - y$*.*

Solution Differentiating both sides should give

$$2y\frac{dy}{dx} + 2\frac{dy}{dx} = 6x^2$$

Which implies

$$\frac{dy}{dx} = \frac{6x^2}{2y+2} = \frac{3x^2}{y+1}$$

□

Problem 6. *Calculate* $\frac{dy}{dx}$ *using implicit differentiation.*

$$3x + 4y^3 = 7$$

Solution

$$3x + 4y^3 = 7 \Rightarrow 3 + 12y^2\frac{dy}{dx} = 0$$

$$\Leftrightarrow 12y^2\frac{dy}{dx} = -3$$

$$\Leftrightarrow \frac{dy}{dx} = -\frac{3}{12y^2}$$

$$\Leftrightarrow \frac{dy}{dx} = \boxed{-\frac{1}{4y^2}}$$

□

Problem 32. *Use implicit differentiation to find* $\dfrac{dy}{dx}$.

$$\tan(x+y)=2y$$

Solution

$$\begin{aligned}
\tan(x+y)=2y &\Rightarrow \sec^2(x+y)\frac{d}{dx}(x+y)=2\frac{dy}{dx}\\
&\Rightarrow \sec^2(x+y)\left(1+\frac{dy}{dx}\right)=2\frac{dy}{dx}\\
&\Leftrightarrow \sec^2(x+y)+\sec^2(x+y)\frac{dy}{dx}=2\frac{dy}{dx}\\
&\Leftrightarrow \sec^2(x+y)=2\frac{dy}{dx}-\sec^2(x+y)\frac{dy}{dx}\\
&\Leftrightarrow \sec^2(x+y)=\frac{dy}{dx}(2-\sec^2(x+y))\\
&\Leftrightarrow \frac{dy}{dx}=\frac{\sec^2(x+y)}{2-\sec^2(x+y)}
\end{aligned}$$

□

Problem 38. *Use implicit differentiation to find* $\dfrac{dy}{dx}$.

$$\sin x\cos y=\sin x+\cos y$$

Solution

$$\begin{aligned}
\sin x\cos y=\sin x+\cos y &\Rightarrow \cos x\cos y+\sin x\left(-\sin y\frac{dy}{dx}\right)=\cos x-\sin y\frac{dy}{dx}\\
&\Leftrightarrow \cos x\cos y-\sin x\sin y\frac{dy}{dx}=\cos x-\sin y\frac{dy}{dx}\\
&\Leftrightarrow \cos x\cos y-\cos x=\sin x\sin y\frac{dy}{dx}-\sin y\frac{dy}{dx}\\
&\Leftrightarrow \cos x\cos y-\cos x=\frac{dy}{dx}(\sin x\sin y-\sin y)\\
&\Leftrightarrow \frac{dy}{dx}=\boxed{\frac{\sin x\sin y-\sin y}{\cos x\cos y-\cos x}}
\end{aligned}$$

□

Common Mistakes

- Read instructions carefully.
- Make sure all work is clear and readable.