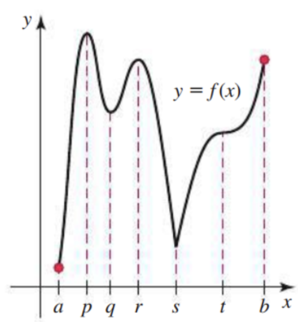
Math 241: Homework 8 Solutions

Graded Problems:

Section 4.1: 44	Section 4.2: 22
Section 4.3: 24, 70	Section 4.4: 18

Section 4.1

Problem 16. *Use the following graphs to identify the points on the interval $[a, b]$ at which local and absolute extreme values occur.*



Solution Local max at p and r. Local min at q and s. Absolute max at p. Absolute min at a.

□

Problem 26. *Find the critical points of the following function:*

$$f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10$$

Solution

$$f'(x) = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x-3)(x+2)$$

f' is never undefined and $f'(x) = 0 \Leftrightarrow x = \boxed{-2, 3}$

□

Problem 30. *Find the critical points of the following function:*

$$f(x) = \frac{(x+1)^2}{x^2+1}$$

Solution

$$\begin{aligned}
f'(x) &= \frac{(x^2+1)(2(x+1)) - (x+1)^2(2x)}{(x^2+1)^2} \\
&= \frac{(x^2+1)(2x+2) - (x^2+2x+1)(2x)}{(x^2+1)^2} \\
&= \frac{(2x^3+2x^2+2x+2) - (2x^3+4x^2+2x)}{(x^2+1)^2} \\
&= \frac{\cancel{2x^3} + 2x^2 + \cancel{2x} + 2 - \cancel{2x^3} - 4x^2 - \cancel{2x}}{(x^2+1)^2} \\
&= \frac{-2x^2+2}{(x^2+1)^2} \\
&= \frac{-2(x^2-1)}{(x^2+1)^2} \\
&= \frac{-2(x-1)(x+1)}{(x^2+1)^2}
\end{aligned}$$

f' is never undefined (because the denominator is never 0) and $f' = 0 \Leftrightarrow x = \boxed{\pm 1}$

□

Problem 44. *Determine the location and value of the absolute extreme values of f on the given interval, if they exist.*

$$f(x) = x^4 - 4x^3 + 4x^2 \text{ on } [-1, 3]$$

Solution

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$$

Therefore, the critical numbers are $x = 0, 1, 2$

$$\begin{aligned} f(-1) &= (-1)^4 - 4(-1)^3 + 4(-1)^2 \\ &= 1 + 4 + 4 \\ &= 9 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^3 + 4(0)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(1) &= (1)^4 - 4(1)^3 + 4(1)^2 \\ &= 1 - 4 + 4 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^4 - 4(2)^3 + 4(2)^2 \\ &= 16 - 32 + 16 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(3) &= (3)^4 - 4(3)^3 + 4(3)^2 \\ &= 81 - 108 + 36 \\ &= 9 \end{aligned}$$

So the absolute max occurs at $x = -1$ and $x = 3$ and has a value of 9. The absolute min occurs at $x = 0$ and $x = 2$ and has a value of 0.

□

Section 4.2

Problem 8. *At what points c does the conclusion of the Mean Value Theorem hold for $f(x) = x^3$ on the interval $[-10, 10]$?*

Solution

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 = \frac{f(10) - f(-10)}{10 - (-10)}$$

$$\Leftrightarrow 3c^2 = \frac{1000 - (-1000)}{20}$$

$$\Leftrightarrow 3c^2 = \frac{2000}{20}$$

$$\Leftrightarrow 3c^2 = 100$$

$$\Leftrightarrow c^2 = \frac{100}{3}$$

$$\Rightarrow \boxed{c = \pm\sqrt{\frac{100}{3}}}$$

□

Problem 16. *Determine whether Rolle's Theorem applies to the following functions on the given interval. If so, find the point(s) guaranteed to exist by Rolle's Theorem.*

$$f(x) = x^3 - 3x^2 - 8x; \quad [-2, 4]$$

Solution $f(x)$ is a polynomial so it is continuous and differentiable everywhere but $f(-2) = -4$ and $f(4) = -16$ so Rolle's Theorem does not apply.

□

Problem 22. *Consider the following function on the interval* $[a, b]$.

(a) Determine whether the Mean Value Theorem applies to the following function on the given interval $[a, b]$.

(b) If so, find the point(s) that are guaranteed to exist by the Mean Value Theorem.

$$f(x) = x^3 - 2x^2; \; [0, 1]$$

Solution

(a) $f(x)$ is a polynomial so it is continuous and differentiable everywhere so MVT does apply.

(b)

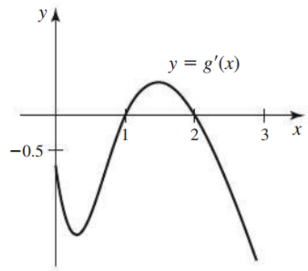
$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} &\Rightarrow 3c^2 - 4c = \frac{f(1) - f(0)}{1 - 0} \\ &\Rightarrow 3c^2 - 4c = \frac{-1 - 0}{1} \\ &\Rightarrow 3c^2 - 4c = -1 \\ &\Rightarrow 3c^2 - 4c + 1 = 0 \\ &\Rightarrow (3c - 1)(c - 1) = 0 \\ &\Rightarrow \boxed{c = \frac{1}{3}, 1} \end{aligned}$$

□

Section 4.3

Problem 6. *The following graph of the derivative g' has exactly two roots.*

(a) Find the critical points of g.
(b) For what values of x in $(0,3)$ is g increasing? Decreasing?
(c) For what values of x in $(0,3)$ does g have a local maximum? A local minimum?



Solution

(a) These happen when $g' = 0$ so $x = 1, 2$
(b) g is increasing when $g' > 0$ and decreasing when $g' < 0$ so g is increasing on $(1,2)$ and decreasing on $(0,1), (2,3)$.
(c) There is a local max when g' switches from positive to negative and a local min when g' switches from negative to positive. This there is a local max at $x = 2$ and a local min at $x = 1$

□

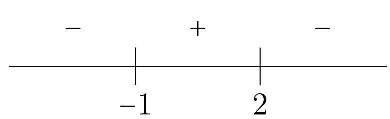
Problem 24. *Find the intervals on which f is increasing and the intervals on which it is decreasing.*

$$f(x) = -\frac{x^3}{3} + \frac{x^2}{2} + 2x$$

Solution

$$f'(x) = -x^2 + x + 2 = -(x^2 - x - 2) = -(x-2)(x+1)$$

Thus the critical numbers are $x = -1, 2$. Plotting these and testing we have the following number line for f'



So f is increasing on $(-1, 2)$ and decreasing on $(-\infty, -1), (2, \infty)$

□

Problem 44.

(a) *Locate the critical points of f.*
(b) *Use the First Derivative Test to locate the local maximum and minimum values.*
(c) *Identify the absolute maximum and minimum values of the function on the given interval (when they exist).*

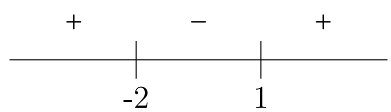
$$f(x) = 2x^3 + 3x^2 - 12x + 1 \text{ on } [-2, 4]$$

Solution

(a)
$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1)$$

Thus the critical numbers are $x = -2, 1$

(b) Testing the intervals, we get



Thus there is a local maximum at $x = -2$ and a local minimum at $x = 1$. Plugging into the original function to get the y-values, we have

$$\begin{aligned} f(-2) &= 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 \\ &= 2(-8) + 3(4) - 12(-2) + 1 \\ &= -16 + 12 + 24 + 1 \\ &= 21 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 + 3(1)^2 - 12(1) + 1 \\ &= 2 + 3 - 12 + 1 \\ &= -6 \end{aligned}$$

So the local minimum is $(1, -6)$ and the local maximum is $(-2, 21)$.

(c)
$$\begin{aligned} f(4) &= 2(4)^3 + 3(4)^2 - 12(4) + 1 \\ &= 2(64) + 3(16) - 24 + 1 \\ &= 153 \end{aligned}$$

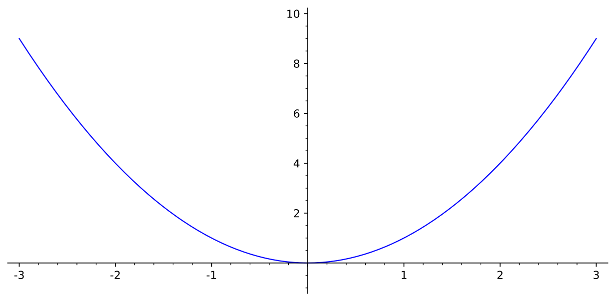
So the absolute minimum is $f(1) = -6$ and the absolute maximum is $f(4) = 153$

□

Problem 54. *Sketch a graph of a function f that is continuous on $(-\infty, \infty)$ and has the following properties:*

$$f'(x) < 0 \text{ and } f''(x) > 0 \text{ on } (-\infty, 0); f'(x) > 0 \text{ and } f''(x) > 0 \text{ on } (0, \infty)$$

Solution



□

Problem 60. *Determine the intervals on which the following functions are concave up or concave down. Identify any inflection points.*

$$f(x) = \frac{1}{1+x^2}$$

Solution

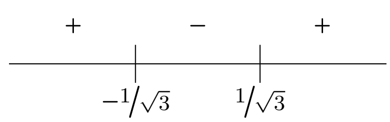
$$f(x) = (1+x^2)^{-1} \Rightarrow f'(x) = -(1+x^2)^{-2}(2x) = \frac{-2x}{(1+x^2)^2}$$

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^2(-2) - (-2x)(2(1+x^2)(2x))}{((1+x^2)^2)^2} \\ &= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \\ &= \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3} \\ &= \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} \\ &= \frac{6x^2 - 2}{(1+x^2)^3} \end{aligned}$$

$f''(x)$ is never undefined and

$$f''(x) = 0 \Rightarrow 6x^2 - 2 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

Testing our intervals we have



So f is concave up on $(-\infty, -1/\sqrt{3}), (1/\sqrt{3}, \infty)$ and concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$. There are inflection points at $x = \pm 1/\sqrt{3}$. Plugging into our original function we get that the inflection points are

$$\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$$

□

Problem 70. *Locate the critical points of the following functions. Then use the Second Derivative Test to determine (if possible) whether they correspond to local maxima or local minima.*

$$f(x) = 6x^2 - x^3$$

Solution

$$f'(x) = 12x - 3x^2 = -3x(x - 4)$$

So the critical numbers are $x = 0, 4$.

$$f''(x) = 12 - 6x$$

$$f''(0) = 12 \text{ and } f''(4) = -12$$

Using the Second Derivative Test, we have that there is a local maximum at $x = 4$ and a local minimum at $x = 0$.

□

Section 4.4

Problem 18. *Use the guidelines of this section to make a complete graph of f: $f(x) = 3x - x^3$*

Solution

Domain

f is a polynomial so it has domain $(-\infty, \infty)$

Intercepts

$$f(0) = 3(0) - (0)^3 = 0$$

$$0 = 3x - x^3 \Rightarrow x(3 - x^2) = 0 \Rightarrow x = 0, \pm\sqrt{3}$$

Therefore the x-intercepts are 0, $\sqrt{3}$, and $-\sqrt{3}$. The y-intercept is 0.

Symmetry

$$f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -(3x - x^3) = -f(x)$$

So there is origin symmetry.

Asymptotes

f is never undefined so there are no vertical asymptotes.

$$\lim_{x\to\infty}(3x - x^3) = \lim_{x\to\infty}(-x^3) = -\infty$$

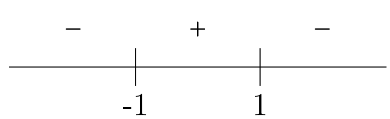
$$\lim_{x\to-\infty}(3x - x^3) = \lim_{x\to-\infty}(-x^3) = \infty$$

So there are also no horizontal asymptotes.

Increase/Decrease

$$f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1 - x)(1 + x)$$

Our critical numbers are $x = -1, 1$. Plotting these on a number line we have



So f is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1), (1, \infty)$.

Local Extrema

From the previous step, there is a relative maximum at $x = 1$ and a relative minimum at $x = -1$. Plugging these into our original function we have

$$\begin{aligned} f(-1) &= 3(-1) - (-1)^3 \\ &= -3 + 1 \\ &= -2 \end{aligned}$$

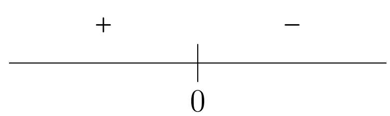
$$\begin{aligned} f(1) &= 3(1) - (1)^3 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

So we have a relative min at the point $(-1, -2)$ and a relative max at the point $(1, 2)$

Concavity/Inflection Points

$$f''(x) = -6x$$

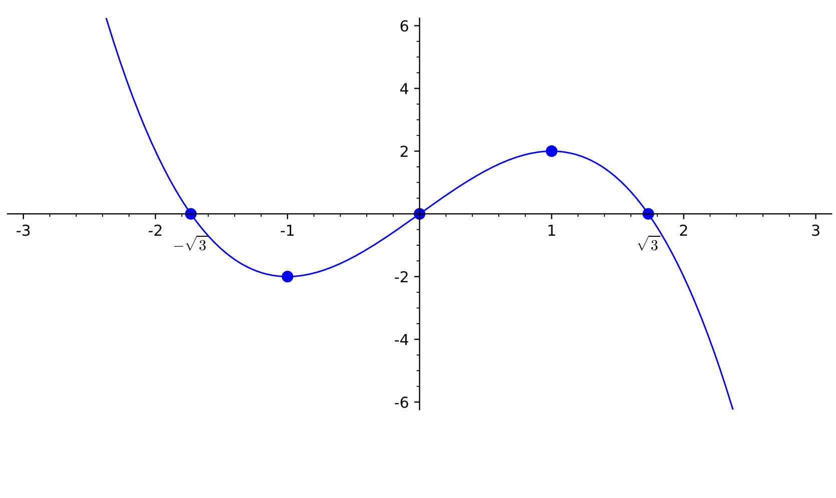
So the number we need to plot is $x = 0$. Testing the intervals we have



So f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. There is an inflection point at $x = 0$ so the inflection point is $(0, 0)$.

Graph

Putting everything together we have



□

Problem 34. *Use the guidelines of this section to make a complete graph of f:* $f(x) = \dfrac{4x}{x^2+3}$

Solution

<u>Domain</u>

Since $x^2 + 3$ can never equal 0, f has domain $(-\infty, \infty)$

<u>Intercepts</u>

$$f(0) = \frac{4(0)}{(0)^2+3} = 0 \quad \text{and} \quad 0 = \frac{4x}{x^2+3} \Rightarrow 0 = 4x \Rightarrow x = 0$$

So the x and y intercepts are both 0.

<u>Symmetry</u>

$$f(-x) = \frac{4(-x)}{(-x)^2+3} = \frac{-4x}{x^2+3} = -f(x)$$

So f has origin symmetry.

<u>Asymptotes</u>

f is never undefined so there are no vertical asymptotes.

$$\lim_{x\to\pm\infty} \frac{4x}{x^2+3} = 0$$

So the horizontal asymptote is $y = 0$

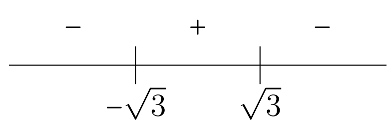
<u>Increase/Decrease</u>

$$\begin{aligned} f'(x) &= \frac{(x^2+3)(4) - 4x(2x)}{(x^2+3)^2} \\ &= \frac{4x^2 + 12 - 8x^2}{(x^2+3)^2} \\ &= \frac{-4x^2+12}{(x^2+3)^2} \\ &= \frac{-4(x^2-3)}{(x^2+3)^2} \end{aligned}$$

f' is never undefined and

$$f'(x) = 0 \Rightarrow (x^2-3) = 0 \Rightarrow x = \pm\sqrt{3}$$

so our critical numbers are $x = -\sqrt{3}$ and $x = \sqrt{3}$. Plotting and testing the intervals we have



So f is increasing on $(-\sqrt{3}, \sqrt{3})$ and decreasing on $(-\infty, -\sqrt{3}), (\sqrt{3}, \infty)$.

Local Extrema

Using our previous step we have that there is a relative max at $x = \sqrt{3}$ and a relative min at $x = -\sqrt{3}$. Plugging these in we have

$$\begin{aligned} f(-\sqrt{3}) &= \frac{4(-\sqrt{3})}{(-\sqrt{3})^2+3} \\ &= \frac{-4\sqrt{3}}{6} \\ &= \frac{-2\sqrt{3}}{3} \end{aligned}$$

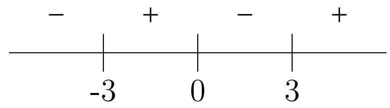
$$\begin{aligned} f(\sqrt{3}) &= \frac{4(\sqrt{3})}{(\sqrt{3})^2+3} \\ &= \frac{4\sqrt{3}}{6} \\ &= \frac{2\sqrt{3}}{3} \end{aligned}$$

So there is a relative max at the point $\left(\sqrt{3}, \frac{2\sqrt{3}}{3}\right)$ and a relative min at the point $\left(-\sqrt{3}, -\frac{2\sqrt{3}}{3}\right)$.

Concavity/Inflection Points

$$\begin{aligned} f''(x) &= \frac{(x^2+3)^2(-8x) - (-4x^2+12)(2(x^2+3)(2x))}{((x^2+3)^2)^2} \\ &= \frac{-8x(x^2+3)^2 - 4x(-4x^2+12)(x^2+3)}{(x^2+3)^4} \\ &= \frac{-8x(x^2+3) - 4x(-4x^2+12)}{(x^2+3)^3} \\ &= \frac{-8x^3 - 24x + 16x^3 - 48x}{(x^2+3)^3} \\ &= \frac{8x^3 - 72x}{(x^2+3)^3} \\ &= \frac{8x(x^2-9)}{(x^2+3)^3} \\ &= \frac{8x(x-3)(x+3)}{(x^2+3)^3} \end{aligned}$$

So we have to plot $x = -3, 0, 3$. Testing the intervals we have.



So f is concave down on $(-\infty, -3), (0, 3)$ and concave up at $(-3, 0)(3, \infty)$. There are inflection points at $x = -3, 0, 3$. Plugging into our original function we have

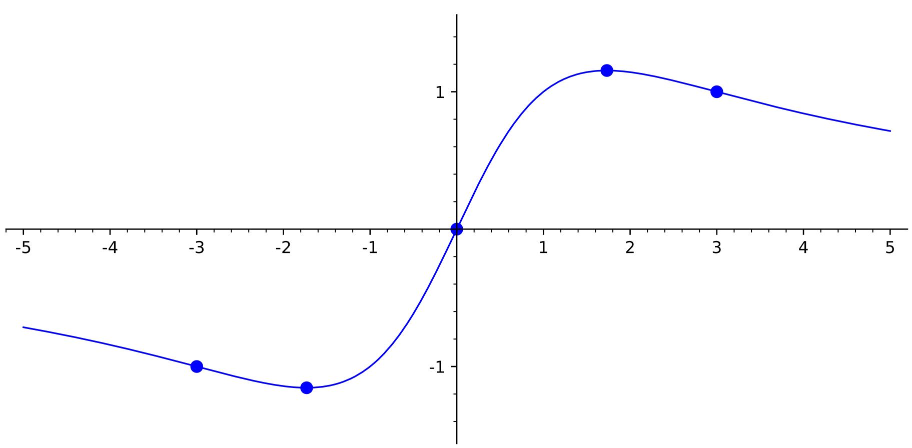
$$\begin{aligned} f(-3) &= \frac{4(-3)}{(-3)^3 + 3} \\ &= \frac{-12}{12} \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(3) &= \frac{4(3)}{(3)^3 + 3} \\ &= \frac{12}{12} \\ &= 1 \end{aligned}$$

$$f(0) = 0$$

So our inflection points are $(-3, -1), (0, 0), (3, 1)$.

<u>Graph</u>
Putting everything together we have



□