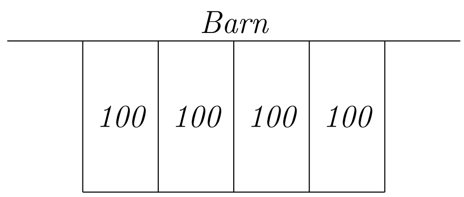
Math 241: Homework 9 Solutions

Graded Problems:
Section 4.5: 18, 20, 22
Section 4.6: 20, 62

Section 4.5

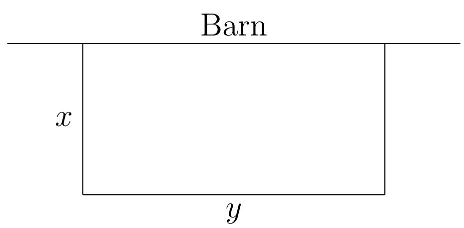
Problem 16.

(a) *A rectangular pen is built with one side against a barn. Two hundred meters of fencing are used for the other three sides of the pen. What dimensions maximize the area of the pen?*
(b) *A rancher plans to make four identical and adjacent rectangular pens against a barn, each with an area of 100 m^2 (see figure). What are the dimensions of each pen that minimize the amount of fence that must be used?*



Solution

(a) Here is our picture:



We are given that $x + x + y = 200$ and we want to maximize $A = xy$.

$$2x + y = 200 \Rightarrow y = 200 - 2x \Rightarrow A = x(200 - 2x) = 200x - 2x^2$$

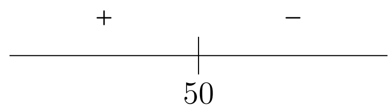
Taking the derivative we have

$$A'(x) = 200 - 4x$$

Setting this equal to 0 to get the critical number we have

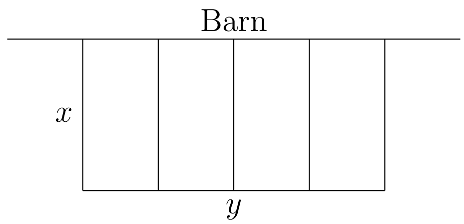
$$0 = 200 - 4x \Rightarrow 4x = 200 \Rightarrow x = 50$$

Testing the intervals we have for A'



Thus the maximum area happens with $x = 50$ and $y = 200-2(5) = 100$ i.e. $\boxed{50 \text{ m by } 100 \text{ m}}$

(b)



We are given

$$A = 100 + 100 + 100 + 100 = 400 = xy$$

We want to minimize the amount of fencing

$$F = x + x + x + x + x + y = 5x + y$$

Solving for one of the variables in the constraint equation and plugging it into the objective equation, we have:

$$400 = xy \Rightarrow y = \frac{400}{x} \Rightarrow F = 5x + \frac{400}{x}$$

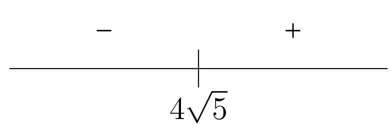
Taking the derivative we get:

$$F'(x) = 5 - \frac{400}{x^2}$$

Solving for the critical point we have:

$$\begin{aligned} 0 = 5 - \frac{400}{x^2} &\Rightarrow \frac{400}{x^2} = 5 \\ &\Rightarrow 5x^2 = 400 \\ &\Rightarrow x^2 = 80 \\ &\Rightarrow x = \sqrt{80} \\ &\Rightarrow x = 4\sqrt{5} \end{aligned}$$

Testing the intervals we have



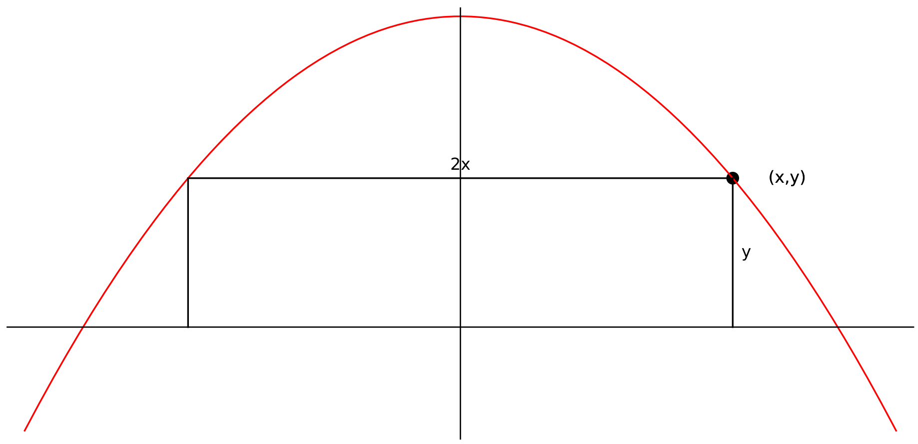
So the dimensions are

$$x = 4\sqrt{5} \text{ m and } y = \frac{400}{4\sqrt{5}} = \frac{100}{\sqrt{5}} \text{ m.}$$

$\square$

Problem 18. *A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?*

Solution Our picture looks like this:



Using this we have

$$A = 2xy = 2x(48 - x^2) = 96x - 2x^3$$

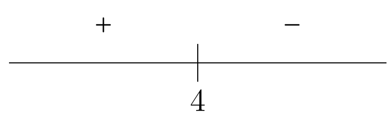
Taking the derivative we have

$$A'(x) = 96 - 6x^2$$

Solving for the critical numbers we have

$$0 = 96 - 6x^2 \Rightarrow x^2 = 16 \Rightarrow x = 4$$

Testing that this a max we have



Plugging this in to our equation for y we have that the dimensions are

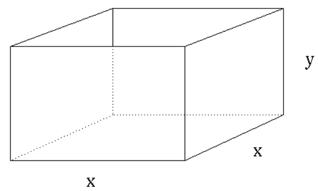
$$x = 4 \text{ (so total length of the rectangle is 8) and } y = 48 - 4^2 = 32$$

This gives an area of $A = 32(8) = 256$.

□

Problem 20. *Suppose an airline policy states that all baggage must be box-shaped with sum of length, width, and height not exceeding 108 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?*

Solution The picture looks like this



We are given

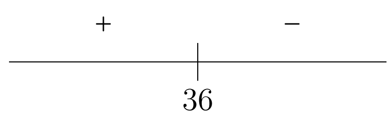
$$x + x + y = 2x + y = 108 \text{ and we want to maximize } V = x^2 y$$

$$2x + y = 108 \Rightarrow y = 108 - 2x \Rightarrow V = x^2(108 - 2x) = 108x^2 - 2x^3$$

Taking the derivative we have

$$V'(x) = 216x - 6x^2 = 6x(36 - x)$$

The only critical number in our interval (since x has to be positive) is $x = 36$. Testing we get



So the maximum volume happens when $x = 36$ in and $y = 108 - 2(36) = 36$ in. So this gives a volume of

$$V = 36(36)(36) = 46,656 \text{ in}^3$$

□

Problem 22. *What point on the line $y = 3x + 4$ is closest to the origin?*

Solution The distance between a point and the origin is given by

$$d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (3x + 4)^2}$$

Minimizing d is the same as minimizing $d^2 = D$ which is

$$D = x^2 + 9x^2 + 24x + 16 = 10x^2 + 24x + 16$$

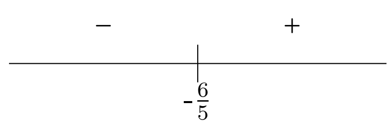
Taking the derivative we have

$$D'(x) = 20x + 24$$

Solving for the critical number we have

$$0 = 20x + 24 \Rightarrow x = -\frac{24}{20} = -\frac{6}{5}$$

Checking that this is a min we have



This gives a y of

$$y = 3\left(-\frac{6}{5}\right) + 4 = -\frac{18}{5} + \frac{20}{5} = \frac{2}{5}$$

So the answer is $\boxed{\left(-\frac{6}{5}, \frac{2}{5}\right)}$

$\square$

Section 4.6

Problem 20. *Find the linear approximation to the following function at the given point a:*

$$f(x) = x^3 - 5x + 3; \quad a = 2$$

Solution We have

$$f(2) = (2)^3 - 5(2) + 3 = 1$$

$$f'(x) = 3x^2 - 5 \Rightarrow f(2) = 3(2)^2 - 5 = 7$$

Using our equation for linear approximation we have

$$L(x) = f(2) + f'(2)(x - 2) = 1 + 7(x - 2)$$

So the answer is

$$\boxed{L(x) = 1 + 7(x - 2)} \text{ or } \boxed{L(x) = 7x - 13}$$

□

Problem 40. *Use linear approximations to estimate the following quantity. Choose a value of a to produce a small error:* $\sqrt[3]{65}$

Solution We are using the function $f(x) = \sqrt[3]{x}$ and $a = 64$.

$$f(64) = \sqrt[3]{64} = 4$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \Rightarrow f'(64) = \frac{1}{3(64)^{2/3}} = \frac{1}{48}$$

Using the linear approximation equation we have

$$\begin{aligned} \sqrt[3]{65} &\approx L(65) \\ &= f(64) + f'(64)(65 - 64) \\ &= 4 + \frac{1}{48} \\ &= \boxed{\frac{193}{48}} \end{aligned}$$

□

Problem 62. *Consider the following function and express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)dx$.*

$$f(x) = \sqrt{x^2+1}$$

Solution

$$f(x) = (x^2+1)^{1/2} \Rightarrow \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

This gives

$$\boxed{dy = \frac{x}{\sqrt{x^2+1}}\ dx}$$

□

Problem 64. *Consider the following function and express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)dx$.*

$$f(x) = \frac{x+4}{4-x}$$

Solution

$$\begin{aligned} f'(x) &= \frac{(4-x)(1)-(x+4)(-1)}{(4-x)^2} \\ &= \frac{4-x+x+4}{(4-x)^2} \\ &= \frac{8}{(4-x)^2} \end{aligned}$$

This gives

$$\boxed{dy = \frac{8}{(4-x)^2}\ dx}$$

□

Common Mistakes

- Make sure you test that something is a maximum or minimum in optimization problems