Math 241 Midterm 1 Review Problems

These problems are intended to help you prepare for the test. Test problems will be similar to, but not the same as, the problems below. *This list of problems is not all inclusive; it does not represent every possible type of problem.* It is suggested that you review lectures and homework problems.

Previous Semester's Exam Problems

(1) Evaluate the following limits. Be as specific as possible (i.e write ∞ or $-\infty$ instead of DNE when applicable):

(a) $\lim_{x\to -1} \dfrac{x^2-5x-6}{x+1}$

Solution

$$\begin{aligned}\lim_{x\to -1} \frac{x^2-5x-6}{x+1} &= \lim_{x\to -1} \frac{(x-6)\cancel{(x+1)}}{\cancel{x+1}}\\ &= \lim_{x\to -1}(x-6)\\ &= -1-6\\ &= \boxed{-7}\end{aligned}$$

□

(b) $\lim_{x\to 3^-} \dfrac{4}{(x-3)^2}$

Solution

$$\begin{aligned}\lim_{x\to 3^-} \frac{4}{(x-3)^2} &\to \frac{4}{(0^-)^2}\\ &\to \frac{4}{0^+}\\ &\to \boxed{\infty}\end{aligned}$$

□

(c) $\lim_{x\to\infty} \dfrac{x^3+x^2+2}{7x^3+x+1}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+x^2+2}{7x^3+x+1} &= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{7x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}\\ &= \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{2}{x^3}}{7+\frac{1}{x^2}+\frac{1}{x^3}}\\ &= \frac{1+0+0}{7+0+0}\\ &= \boxed{\frac{1}{7}}\end{aligned}$$

□

(d) $\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$

Solution

$$\begin{aligned}
\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5} &= \lim_{x\to 25} \frac{x-25}{\sqrt{x}-5} \cdot \frac{\sqrt{x}+5}{\sqrt{x}+5} \\
&= \lim_{x\to 25} \frac{\cancel{(x-25)}(\sqrt{x}+5)}{\cancel{x-25}} \\
&= \lim_{x\to 25} (\sqrt{x}+5) \\
&= \sqrt{25}+5 \\
&= \boxed{10}
\end{aligned}$$

□

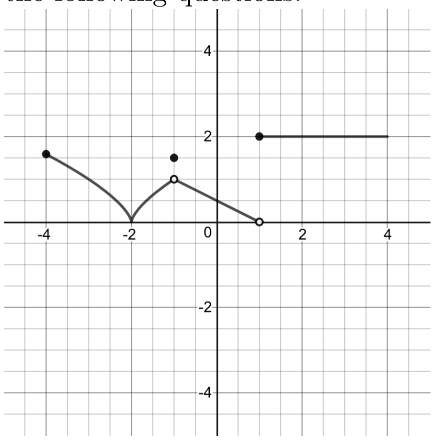
(e) $\lim_{x\to 0} \frac{\sin 4x}{5x}$

Solution

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{5x} &= \lim_{x\to 0} \frac{\sin 4x}{5x} \cdot \frac{4}{4} \\
&= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{4}{5} \\
&= 1 \cdot \lim_{x\to 0} \frac{4}{5} \\
&= \boxed{\frac{4}{5}}
\end{aligned}$$

□

(2) The function $f(x)$ is defined for $-4 \le x \le 4$ and is graphed below. Use the graph to answer the following questions:



(a) What is $\lim_{x \to -1} f(x)$?

Solution $\boxed{1}$

☐

(b) What is $\lim_{x \to 1} f(x)$?

Solution $\boxed{\text{DNE}}$

☐

(c) Give the intervals where $f(x)$ is continuous, be careful to include the endpoints if necessary.

Solution $\boxed{[-4, -1), (-1, 1), [1, 4]}$

☐

(d) Does the function appear to be differentiable at $x = -2$? Explain why or why not.

Solution $\boxed{\text{No, there is a corner}}$

☐

(3) Consider the function $f(x)$ given below. Find

(i) $\lim_{x \to k^-} f(x)$

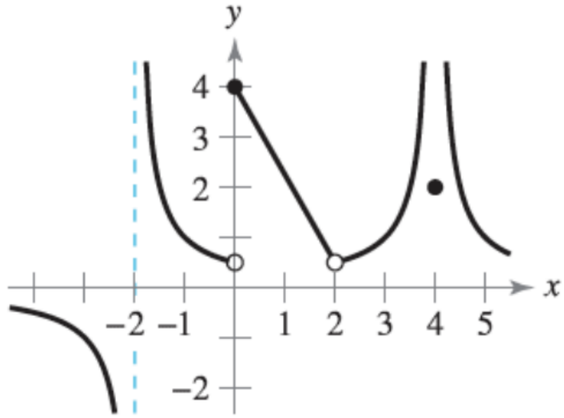
(ii) $\lim_{x \to k^+} f(x)$

(iii) $\lim_{x \to k} f(x)$

(iv) $f(k)$

(v) Is $f(x)$ continuous at k? (yes or no)

for each of the given values of k. If the given value does not exist, write "DNE", ∞, $-\infty$, or "undefined" as necessary:



(a) $k = -1$

Solution

(i) $\boxed{1}$
(ii) $\boxed{1}$
(iii) $\boxed{1}$
(iv) $\boxed{1}$
(v) $\boxed{\text{yes}}$

□

(b) $k = 0$

Solution

(i) $\boxed{\frac{1}{2}}$
(ii) $\boxed{4}$
(iii) $\boxed{\text{DNE}}$
(iv) $\boxed{4}$
(v) $\boxed{\text{no}}$

□

(c) $k = 2$

Solution

(i) $\boxed{\frac{1}{2}}$

(ii) $\boxed{\frac{1}{2}}$

(iii) $\boxed{\frac{1}{2}}$

(iv) $\boxed{\text{undefined}}$

(v) $\boxed{\text{no}}$

$\square$

(d) $k = 4$

Solution

(i) $\boxed{\infty}$

(ii) $\boxed{\infty}$

(iii) $\boxed{\infty}$

(iv) $\boxed{2}$

(v) $\boxed{\text{no}}$

$\square$

(4) On which interval(s) is the following function continuous. Justify your answers using limits:

$$f(x) = \begin{cases} 1 - x^2 & x < -1 \\ 1 + x & -1 \le x \le 1 \\ -3 & x > 1 \end{cases}$$

Solution Each piece is continuous so we just have to check $x = -1$ and $x = 1$.

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (1 - x^2) \\ &= 1 - (-1)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (1 + x) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-1) &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (1 + x) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (-3) \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(1) &= 1 + 1 \\ &= 2 \end{aligned}$$

So the function is continuous at $x = -1$ and not at $x = 1$ (continuous from the left). Accepted answers are either $\boxed{(-\infty, 1) \cup (1, \infty)}$ or $\boxed{(-\infty, 1], (1, \infty)}$

□

(5) Evaluate $\lim_{x \to 0} (x^2 \sin(4x) + 1)$

Solution

$$
\begin{aligned}
-1 \leq \sin(4x) \leq 1 &\Rightarrow -x^2 \leq x^2 \sin(4x) \leq x^2 \\
&\Rightarrow -x^2 + 1 \leq x^2 \sin(4x) + 1 \leq x^2 + 1
\end{aligned}
$$

$$\lim_{x \to 0} (-x^2 + 1) = 1 \text{ and } \lim_{x \to 0} (x^2 + 1) = 1$$

Thus by the squeeze theorem $\lim_{x \to 0} (x^2 \sin(4x) + 1) = \boxed{1}$

Note: You could also just plug in $x = 0$

□

(6) Show that the equation $x^3 - x^2 + 2x - 7 = 0$ has a solution in the interval $[1, 2]$. State any theorems you use to support your answer.

Solution Let $f(x) = x^3 - x^2 + 2x - 7$

$$
\begin{aligned}
f(1) &= 1 - 1 + 2 - 7 \\
&= -5
\end{aligned}
$$

$$
\begin{aligned}
f(2) &= 8 - 4 + 4 - 7 \\
&= 1
\end{aligned}
$$

f is continuous with $f(1) < 0$ and $f(2) > 0$ so by the Intermediate Value Theorem, there exists at least one solution in $[1, 2]$

□

(7) Let $f(x) = 5 + x - x^4$. Use the intermediate value theorem to show that there is at least one point where $f(x) = 0$.

Solution

$$
\begin{aligned}
f(0) &= 5 + 0 - 0 \\
&= 5
\end{aligned}
$$

$$
\begin{aligned}
f(2) &= 5 + 2 - 16 \\
&= -9
\end{aligned}
$$

f is continuous with $f(0) > 0$ and $f(2) < 0$ so by the Intermediate Value Theorem there is at least one point where $f(x) = 0$.

□

(8) (a) Using the limit definition of a derivative, differentiate the following:

$$f(x) = x^2 - 3x - 1$$

(b) Find the equation of the line tangent to $f(x)$ at $x = 1$

Solution

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2 - 3(x+h) - 1] - (x^2 - 3x - 1)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{1} - \cancel{x^2} + \cancel{3x} + \cancel{1}}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} \\ &= \lim_{h\to 0} (2x + h - 3) \\ &= \boxed{2x - 3} \end{aligned}$$

(b)

$$f'(1) = -1 \text{ and } f(1) = -3$$

Using point-slope form we have

$$\boxed{y + 3 = -(x - 1)}$$

□

(9) Differentiate the following functions. You do not need to simplify your answers.

(a) $y = 3x \sin x$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{d}{dx}(3x)\right) \sin x + 3x \frac{d}{dx}(\sin x) \\ &= \boxed{3 \sin x + 3x \cos x} \end{aligned}$$

□

(b) $y = \tan\left(x^2 + 1\right)$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(x^2+1) \cdot \frac{d}{dx}(x^2+1) \\ &= \boxed{\sec^2(x^2+1)(2x)}\end{aligned}$$

□

(c) $y = \dfrac{x+1}{x^2+2}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+2)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^2+2)}{(x^2+2)^2} \\ &= \boxed{\frac{(x^2+2)(1) - (x+1)(2x)}{(x^2+2)^2}}\end{aligned}$$

□

(d) $f(x) = 12x^2 - \dfrac{5}{\sqrt{x}} + 78$

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}(12x^2 - 5x^{-1/2} + 78) \\ &= 12(2x) - 5\left(-\frac{1}{2}x^{-3/2}\right) \\ &= \boxed{24x + \frac{5}{2}x^{-3/2}}\end{aligned}$$

□

(e) $y = \dfrac{\sin^2 x}{2x^3 + 4x^2 + 7}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x^3+4x^2+7)\frac{d}{dx}(\sin^2 x) - \sin^2 x \frac{d}{dx}(2x^3+4x^2+7)}{(2x^3+4x^2+7)^2} \\ &= \frac{(2x^3+4x^2+7)\left(2\sin x \frac{d}{dx}(\sin x)\right) - \sin^2 x(2(3x^2) + 4(2x)}{(2x^3+4x^2+7)^2} \\ &= \boxed{\frac{(2x^3+4x^2+7)(2\sin x\cos x) - \sin^2 x(6x^2+8x)}{(2x^3+4x^2+7)^2}}\end{aligned}$$

□

(10) Calculate the following derivatives:

(a)

$$y = \frac{x^3 + 1}{2 - x}, \qquad \frac{dy}{dx} =$$

Solution

$$\frac{dy}{dx} = \frac{(2-x)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}(2-x)}{(2-x)^2}$$

$$= \boxed{\frac{(2-x)(3x^2) - (x^3+1)(-1)}{(2-x)^2}}$$

□

(b)

$$h(t) = \cos^2(\pi t) + 3, \qquad h'(t) =$$

Solution

$$h'(t) = 2\cos(\pi t)\frac{d}{dt}(\cos(\pi t)) + 0$$

$$= 2\cos(\pi t) \cdot -\sin(\pi t)\frac{d}{dt}(\pi t)$$

$$= \boxed{2\cos(\pi t) \cdot -\sin(\pi t) \cdot (\pi)}$$

□

(c)

$$f(x) = \sin(x)(x^2+3) + x^{4/3}, \qquad f'(x) =$$

Solution

$$f'(x) = \sin(x)\frac{d}{dx}(x^2+3) + (x^2+3)\frac{d}{dx}(\sin(x)) + \frac{4}{3}x^{4/3-1}$$

$$= \boxed{\sin(x)(2x) + (x^2+3)\cos(x) + \frac{4}{3}x^{1/3}}$$

□

(d)

$$y^4 - \sqrt{x} + 2y = 2, \qquad y' \text{ at } (1,1)$$

Solution

$$\frac{d}{dx}(y^4 - \sqrt{x} + 2y) = \frac{d}{dx}(2)$$

$$4y^3y' - \frac{d}{dx}(x^{1/2}) + 2y' = 0$$

$$4y^3y' - \frac{1}{2}x^{-1/2} + 2y' = 0$$

Plugging in $x = 1$ and $y = 1$, we get

$$4 \cdot 1^3 y' - \frac{1}{2} \cdot 1^{-1/2} + 2y' = 0$$

$$6y' - \frac{1}{2} = 0$$

$$6y' = \frac{1}{2}$$

$$y' = \boxed{\frac{1}{12}}$$

□

(11) Find an equation of the tangent line to $x^2 + xy + y^2 = 3$ at $(1,1)$

Solution Taking the derivative of both sides with respect to x we have

$$2x + \left[\left(\frac{d}{dx}(x)\right)y + x\frac{d}{dx}(y)\right] + 2y\frac{dy}{dx} = 0 \Leftrightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Plugging in $x = 1$ and $y = 1$ we have

$$2(1) + (1) + (1)\frac{dy}{dx} + 2(1)\frac{dy}{dx} = 0 \Leftrightarrow 3 + 3\frac{dy}{dx} = 0$$

$$\Leftrightarrow 3\frac{dy}{dx} = -3$$

$$\Leftrightarrow \frac{dy}{dx} = -1$$

Using point-slope form we have

$$\boxed{y - 1 = -(x - 1)}$$

□

(12) Let a particle's motion be given by $s(t) = \sqrt{t - \cos t}$ for $t > 1$.

(a) Find the particle's **velocity** and **acceleration** as functions of t.

(b) What is the particle's speed at $t = 3\pi/2$?

Solution

(a)

$$\begin{aligned} v(t) &= s'(t) \\ &= \frac{1}{2}(t - \cos t)^{-1/2} \cdot \frac{d}{dt}(t - \cos t) \\ &= \boxed{\frac{1}{2}(t - \cos t)^{-1/2}(1 + \sin t)} \end{aligned}$$

$$\begin{aligned} a(t) &= v'(t) \\ &= \left(\frac{d}{dt}\left(\frac{1}{2}(t - \cos t)^{-1/2}\right)\right)(1 + \sin t) + \left(\frac{1}{2}(t - \cos t)^{-1/2}\right)\frac{d}{dt}(1 + \sin t) \\ &= \left(-\frac{1}{4}(t - \cos t)^{-3/2}\frac{d}{dt}(t - \cos t)\right)(1 + \sin t) + \left(\frac{1}{2}(t - \cos t)^{-1/2}\right)(\cos t) \\ &= \boxed{-\frac{1}{4}(t - \cos t)^{-3/2}(1 + \sin t)^2 + \frac{1}{2}(t - \cos t)^{-1/2}(\cos t)} \end{aligned}$$

(b)

$$\begin{aligned} \text{Speed at } \frac{3\pi}{2} &= \left|v\left(\frac{3\pi}{2}\right)\right| \\ &= \left|\frac{1}{2}\left(\frac{3\pi}{2} - \cos\frac{3\pi}{2}\right)^{-1/2}\left(1 + \sin\frac{3\pi}{2}\right)\right| \\ &= \left|\frac{1}{2}\left(\frac{3\pi}{2} - 0\right)^{-1/2}(1 - 1)\right| \\ &= |0| \\ &= \boxed{0} \end{aligned}$$

□

(13) A spherical snowball is placed in the sun. The sun melts the snowball so that its radius **decreases** 1/4 in. per hour. Find the rate of change of the volume with respect to time at the instant the radius is 4 in. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Solution

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

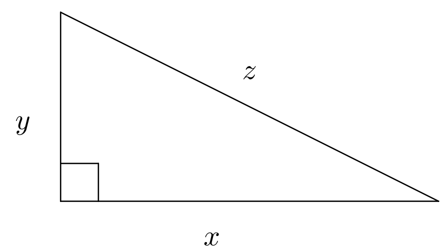
We are given that $\frac{dr}{dt} = \frac{-1}{4}$ and we want to find $\frac{dV}{dt}$ when $r = 4$. Plugging these in we have

$$\frac{dV}{dt} = 4\pi(4)^2\left(-\frac{1}{4}\right) = \boxed{-16\pi \text{ in}^3/\text{hr}}$$

□

(14) A person leaves a given point and travels north at 3 mph. Another person leaves the same point at the same time and travels east at 4 mph. At what rate is the distance between the two people changing at the instant when they have traveled 2 hours?

Solution The following is what the problem sketch will look like:



We are given

$$\frac{dy}{dt} = 3 \text{ and } \frac{dx}{dt} = 4$$

We want to find $\frac{dz}{dt}$ when $x = 8$ and $y = 6$ (as that is how far the cars with have traveled in 2 hours)

When $x = 8$ and $y = 6$ we have

$$z^2 = (8)^2 + (6)^2 = 100 \Rightarrow z = 10$$

Taking the derivative with respect to time and plugging in our given information we have

$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow 2(8)(4) + 2(6)(3) = 2(10)\frac{dz}{dt}$$

$$\Leftrightarrow 64 + 36 = 20\frac{dz}{dt}$$

$$\Leftrightarrow 100 = 20\frac{dz}{dt}$$

$$\Leftrightarrow \frac{dz}{dt} = \boxed{5 \text{ mph}}$$

□

Extra Practice Problems

(1) Calculate the following limits.

(a) $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$

Solution

$$\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} \to \frac{0}{0}$$

$$\begin{aligned} \lim_{t\to 1} \frac{t^2+t-2}{t^2-1} &= \lim_{t\to 1} \frac{(t+2)\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} \\ &= \lim_{t\to 1} \frac{(t+2)}{(t+1)} \\ &= \frac{1+2}{1+1} \\ &= \boxed{\frac{3}{2}} \end{aligned}$$

□

(b) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

Solution

$$\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} \to \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} &= \lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} \cdot \frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4} \\
&= \lim_{x\to 2} \frac{(x^2+12)-4^2}{(x-2)(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+12}+4} \\
&= \frac{2+2}{\sqrt{2^2+12}+4} \\
&= \frac{4}{\sqrt{16}+4} \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

□

(c) $\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x}$

Solution

$$\begin{aligned}
\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} &= \lim_{x\to\infty} \frac{\frac{7x^3}{x^3}}{\frac{x^3}{x^3}-\frac{3x^2}{x^3}+\frac{6x}{x^3}} \\
&= \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \\
&= \frac{7}{1-0+0} \\
&= \boxed{7}
\end{aligned}$$

□

(d) $\lim_{x \to 0} \frac{1}{x^3 - 1} + 1$

Solution

$$\begin{aligned}
\lim_{x \to 0} \frac{1}{x^3 - 1} + 1 &= \frac{1}{0 - 1} + 1 \\
&= -1 + 1 \\
&= \boxed{0}
\end{aligned}$$

□

(e) $\lim_{x \to 1} \frac{3x - 4}{x^2 + x + 1}$

Solution

$$\begin{aligned}
\lim_{x \to 1} \frac{3x - 4}{x^2 + x + 1} &= \frac{3(1) - 4}{(1)^2 + 1 + 1} \\
&= \boxed{-\frac{1}{3}}
\end{aligned}$$

□

(f) $\lim_{x \to -1} \frac{x^2 + x - 2}{x^3 + 1}$

Solution

$$\begin{aligned}
\lim_{x \to -1} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{(-1)^2 - 1 - 2}{(-1)^3 + 1} \\
&\to \frac{1 - 1 - 2}{-1 + 1} \\
&\to -\frac{2}{0}
\end{aligned}$$

So we have to check the one-sided limits.

$$\begin{aligned}
\lim_{x \to -1^-} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{-2}{0^-} \\
&= \to \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to -1^+} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{-2}{0^+} \\
&\to -\infty
\end{aligned}$$

Thus the limit $\boxed{\text{DNE}}$

□

(g) $\lim\limits_{x\to -1^-} \dfrac{|x+1|}{x+1}$

Solution

$$\begin{aligned}
\lim_{x\to -1^-} \frac{|x+1|}{x+1} &= \lim_{x\to -1^-} \frac{-\cancel{(x+1)}}{\cancel{x+1}} \\
&= \lim_{x\to -1^-} -1 \\
&= \boxed{-1}
\end{aligned}$$

□

(h) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} \\
&= \lim_{x\to 1}(x+1) \\
&= 1+1 \\
&= \boxed{2}
\end{aligned}$$

□

(i) $\lim\limits_{x\to \infty} \dfrac{2x^3-4x^2+5x}{17x^2+1}$

Solution

$$\begin{aligned}
\lim_{x\to \infty} \frac{2x^3-4x^2+5x}{17x^2+1} &= \lim_{x\to \infty} \frac{\frac{2x^3}{x^2} - \frac{4x^2}{x^2} + \frac{5x}{x^2}}{\frac{17x^2}{x^2} + \frac{1}{x^2}} \\
&= \lim_{x\to \infty} \frac{2x-4+\frac{5}{x}}{17+\frac{1}{x^2}} \\
&\to \frac{2(\infty)-4+0}{17+0} \\
&\to \boxed{\infty}
\end{aligned}$$

□

(j) $\lim_{x\to-\infty} \frac{3x^2+4}{x+7}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2+4}{x+7} = \lim_{x\to-\infty} \frac{\frac{3x^2}{x}+\frac{4}{x}}{\frac{x}{x}+\frac{7}{x}}$$

$$= \lim_{x\to-\infty} \frac{3x+\frac{4}{x}}{1+\frac{7}{x}}$$

$$\to \frac{3(-\infty)+0}{1+0}$$

$$\to \boxed{-\infty}$$

□

(k) $\lim_{t\to2^-} \frac{t+2}{t-2}$

Solution

$$\lim_{t\to2^-} \frac{t+2}{t-2} \to \frac{2+2}{0^-}$$

$$\to \frac{4}{0^-}$$

$$\to \boxed{-\infty}$$

□

(l) $\lim_{x\to0} \frac{\sin x}{3x}$

Solution

$$\lim_{x\to0} \frac{\sin x}{3x} = \lim_{x\to0} \frac{\sin x}{x} \cdot \frac{1}{3}$$

$$= 1 \cdot \frac{1}{3}$$

$$= \boxed{\frac{1}{3}}$$

□

(m) $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$

Solution

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x \leq x\cos\left(\frac{1}{x}\right) \leq x$$

$$\lim_{x \to 0}(-x) = 0 \text{ and } \lim_{x \to 0}(x) = 0$$

Thus by squeeze theorem $\lim_{x \to 0} x\cos\left(\frac{1}{x}\right) = \boxed{0}$

□

(2) Calculate the following limits. State clearly any theorems that you use.

(a)

$$\lim_{x \to 3^-} \frac{|x-3|}{(6-2x)}$$

Solution

$$\lim_{x \to 3^-} \frac{|x-3|}{(6-2x)} = \lim_{x \to 3^-} \frac{-\cancel{(x-3)}}{-2\cancel{(x-3)}}$$

$$= \lim_{x \to 3^-} \frac{-1}{-2}$$

$$= \boxed{\frac{1}{2}}$$

□

(b)

$$\lim_{t \to 0} \frac{\sin(3t)}{t}$$

Solution

$$\lim_{t \to 0} \frac{\sin(3t)}{t} = \lim_{t \to 0} \frac{\sin(3t)}{t} \cdot \frac{3}{3}$$

$$= \lim_{t \to 0} \frac{\sin(3t)}{3t} \cdot 3$$

$$= 1 \cdot 3$$

$$= \boxed{3}$$

□

(c)

$$\lim_{h\to 1} \frac{\sqrt{h+8}-3}{1-h}$$

Solution

$$\begin{aligned}
\lim_{h\to 1} \frac{\sqrt{h+8}-3}{1-h} &= \lim_{h\to 1} \frac{\sqrt{h+8}-3}{1-h} \cdot \frac{\sqrt{h+8}+3}{\sqrt{h+8}+3} \\
&= \lim_{h\to 1} \frac{(h+8)-9}{(1-h)\sqrt{h+8}+3} \\
&= \lim_{h\to 1} \frac{\cancel{h-1}}{-\cancel{(h-1)}(\sqrt{h+8}+3)} \\
&= \lim_{h\to 1} \frac{1}{-(\sqrt{h+8}+3)} \\
&= -\frac{1}{\sqrt{9}+3} \\
&= \boxed{-\frac{1}{6}}
\end{aligned}$$

□

(d)

$$\lim_{x\to 0} x^2 \cos\left(\frac{3}{x}\right) - 11$$

Solution

$$\begin{aligned}
-1 \le \cos\left(\frac{3}{x}\right) \le 1 &\Rightarrow -x^2 \le x^2 \cos\left(\frac{3}{x}\right) \le x^2 \\
&\Rightarrow -x^2 - 11 \le x^2 \cos\left(\frac{3}{x}\right) - 11 \le x^2 - 11
\end{aligned}$$

$$\lim_{x\to 0}(-x^2 - 11) = -11 \text{ and } \lim_{x\to 0}(x^2 - 11) = -11$$

Thus by the squeeze theorem $\lim_{x\to 0} x^2 \cos\left(\frac{3}{x}\right) - 11 = \boxed{-11}$

□

(3) Describe on which intervals the following functions are continuous (show your work):

(a) $y = \dfrac{\sin x}{x - 2}$

Solution

$$x - 2 \neq 0 \Rightarrow x \neq 2 \Rightarrow \boxed{(-\infty, 2), (2, \infty)}$$

$\square$

(b) $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & 2 \leq x < 4 \\ 3, & x \geq 4 \end{cases}$

Solution Each piece is continuous. Check the breaks.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (3 - x) \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} \left(\frac{x}{2} + 1 \right) \\ &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(2) &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 4^-} f(x) &= \lim_{x \to 4^-} \left(\frac{x}{2} + 1 \right) \\ &= \frac{4}{2} + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 4^+} f(x) &= \lim_{x \to 4^+} 3 \\ &= 3 \end{aligned}$$

$$f(4) = 3$$

Thus f is not continuous at $x = 2$ (but is continuous from the right) and is continuous at $x = 4$. Accepted answers are either $\boxed{(-\infty, 2) \cup (2, \infty)}$ or $\boxed{(-\infty, 2), [2, \infty)}$

$\square$

(c) $f(x) = \begin{cases} 1 - x^2 & x < -1 \\ 1 + x & -1 \le x \le 1 \\ -3 & x > 1 \end{cases}$

Solution Each piece is continuous so we just have to check $x = -1$ and $x = 1$.

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (1 - x^2) \\ &= 1 - (-1)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (1 + x) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(=1) &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (1 + x) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (-3) \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(1) &= 1 + 1 \\ &= 2 \end{aligned}$$

So the function is continuous at $x = -1$ and not at $x = 1$ (continuous from the left). Accepted answers are either $\boxed{(-\infty, 1) \cup (1, \infty)}$ or $\boxed{(-\infty, 1], (1, \infty)}$

□

(4) Show that the equation $x^3 - 15x + 1 = 0$ has at least three solutions in the interval $[-4, 4]$

Solution Let $f(x) = x^3 - 15x + 1$

$$\begin{aligned} f(-4) &= (-4)^3 - 15(-4) + 1 \\ &= -64 + 60 + 1 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(0) &= 0 - 0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(1) &= 1 - 15 + 1 \\ &= -15 \end{aligned}$$

$$\begin{aligned} f(4) &= (4)^3 - 15(4) + 1 \\ &= 64 - 60 + 1 \\ &= 5 \end{aligned}$$

Since f is continuous and changes sign at least three times in $[-4, 4]$ there is at least three solutions by the Intermediate Value Theorem.

□

(5) (a) **Use the definition of derivative** to show that the derivative of $f(x) = x^2 - x$ at $x = -2$ is -5, i.e. $f'(-2) = -5$.

(b) Find an equation for the tangent line to $f(x) = x^2 - x$ at $x = -2$.

Solution

(a)

$$\begin{aligned} f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h\to 0} \frac{[(-2+h)^2 - (-2+h)] - [(-2)^2 - (-2)]}{h} \\ &= \lim_{h\to 0} \frac{[(4 - 4h + h^2) + 2 - h] - [4+2]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{4} - 4h + h^2 + \cancel{2} - h - \cancel{4} - \cancel{2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-4 + h - 1)}{\cancel{h}} \\ &= \lim_{h\to 0} (-4 + h - 1) \\ &= -4 - 1 \\ &= -5 \checkmark \end{aligned}$$

(b)

$$f(-2) = (-2)^2 - (-2) = 6$$

Using the slope $m = -5$ and the point $(-2, 6)$ we have

$$\boxed{y - 6 = -5(x+2)}$$

□

(6) Use the definition of the derivative (limit definition) to find the derivatives of the following:

(a) $f(x) = \sqrt{x}$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \boxed{\frac{1}{2\sqrt{x}}}
\end{aligned}
$$

□

(b) $f(x) = x^2 - x$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - (x+h)] - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x + h - 1) \\
&= \boxed{2x - 1}
\end{aligned}
$$

□

(c) $f(x) = \frac{1}{x}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \boxed{-\frac{1}{x^2}}
\end{aligned}$$

□

(7) Consider the function $f(x) = 5 - x^2$.

(a) Find the equation for the secant line to the graph of $f(x)$ that passes through the points $(1, 4)$ and $(2, 1)$.

Solution

$$m = \frac{4 - 1}{1 - 2} = -3$$

Using point-slope form we have

$$\boxed{y - 4 = -3(x - 1)}$$

□

(b) Find $f'(x)$ using the definition of a derivative.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[5 - (x+h)^2] - (5 - x^2)}{h} \\
&= \lim_{h\to 0} \frac{5 - (x^2 + 2xh + h^2) - 5 + x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5} - \cancel{x^2} - 2xh - h^2 - \cancel{5} + \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h}} \\
&= \lim_{h\to 0} (-2x - h) \\
&= \boxed{-2x}
\end{aligned}$$

□

(c) Find the equation for the tangent line to the graph of $f(x)$ at the point $(1, 4)$.

Solution

$$m = f'(1) = -2$$

Using point-slope form we have

$$\boxed{y - 4 = -2(x - 1)}$$

□

(d) Find the equation for the tangent line to the graph of $f(x)$ at the point $(2,1)$.

Solution

$$m = f'(2) = -4$$

Using point-slope form we have

$$\boxed{y - 1 = -4(x - 2)}$$

□

(8) A particle is moving along the x-axis. Its position at time t is given by the function $s(t) = -2t^2 + 5t - 2$.

(a) Find the particle's average velocity v_{av} between $t = 1$ and $t = 4$.

Solution

$$\begin{aligned} \text{Average Velocity} &= \frac{s(4) - s(1)}{4 - 1} \\ &= \frac{[-2(4)^2 + 5(4) - 2] - [-2(1)^2 + 5(1) - 2]}{3} \\ &= \frac{-2(16) + 20 - 2 + 2 - 5 + 2}{3} \\ &= \frac{-15}{3} \\ &= \boxed{-5} \end{aligned}$$

□

(b) Find the particle's instantaneous velocity at $t = 1$.

Solution

$$v(t) = s'(t) = -4t + 5$$

$$v'(1) = -4 + 5 = \boxed{1}$$

□

(9) If a particle's motion is given by the equation $s(t) = 4t^3 - 10t^2 + 5$, find its velocity and acceleration as functions of t. What is its speed at $t = 1$

Solution

$$v(t) = s'(t) = 12t^2 - 20t$$

$$a(t) = v'(t) = 24t - 20$$

$$\text{speed}|_{t=1} = |v(1)| = |12 - 20| = |-8| = 8$$

□

(10) Find the first derivatives of the following:

(a) $y = 6x^2 - 10x - 5x^{-2}$

Solution

$$\begin{aligned} \frac{dy}{dx} &= 6(2)x^{2-1} - 10(1)x^{1-1} - 5(-2)x^{-2-1} \\ &= \boxed{12x - 10 + 10x^{-3}} \end{aligned}$$

$\square$

(b) $y = x^2 \sin x + 2x \cos x - 2 \sin x$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{d}{dx}(x^2)\sin x\right) + x^2\left(\frac{d}{dx}(\sin x)\right) + \left(\frac{d}{dx}(2x)\right)\cos x + 2x\left(\frac{d}{dx}(\cos x)\right) - 2\cos x \\ &= \cancel{2x\sin x} + x^2\cos x + \cancel{2\cos x} - \cancel{2x\sin x} - \cancel{2\cos x} \\ &= \boxed{x^2\cos x} \end{aligned}$$

$\square$

(c) $h(x) = x\tan\left(2\sqrt{x}\right) + 7$

Solution

$$\begin{aligned} h'(x) &= \left(\frac{d}{dx}(x)\right)\tan\left(2\sqrt{x}\right) + x\left(\frac{d}{dx}\left(\tan\left(2\sqrt{x}\right)\right)\right) \\ &= \tan\left(2\sqrt{x}\right) + x\sec^2(2\sqrt{x})\frac{d}{dx}(2\sqrt{x}) \\ &= \tan\left(2\sqrt{x}\right) + x\sec^2(2\sqrt{x})(x^{-1/2}) \\ &= \boxed{\tan\left(2\sqrt{x}\right) + \sqrt{x}\sec^2(2\sqrt{x})} \end{aligned}$$

$\square$

(d) $y = \dfrac{\cot x}{1 + \cot(x^2 + x)}$

Solution

$$\frac{dy}{dx} = \frac{(1 + \cot(x^2 + x))\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot(x^2 + x))}{(1 + \cot(x^2 + x))^2}$$

$$= \frac{(1 + \cot(x^2 + x))(-\csc^2 x) - (\cot x)(-\csc^2(x^2 + x)\frac{d}{dx}(x^2 + x))}{(1 + \cot(x^2 + x))^2}$$

$$= \boxed{\frac{(1 + \cot(x^2 + x))(-\csc^2 x) - (\cot x)(-\csc^2(x^2 + x)(2x + 1))}{(1 + \cot(x^2 + x))^2}}$$

□

(e) $y = \left(1 - \dfrac{x}{7}\right)^{-7}$

Solution

$$\frac{dy}{dx} = -7\left(1 - \frac{x}{7}\right)^{-8} \cdot \frac{d}{dx}\left(1 - \frac{x}{7}\right)$$

$$= -7\left(1 - \frac{x}{7}\right)^{-8}\left(-\frac{1}{7}\right)$$

$$= \boxed{\left(1 - \frac{x}{7}\right)^{-8}}$$

□

(11) Find equations for the tangent and normal lines to $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1, 0)$.

Solution

$$6x^2 + 3xy + 2y^2 + 17y - 6 = 0 \Rightarrow 12x + 3x\frac{dy}{dx} + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

Plugging in $x = -1$ and $y = 0$ gives:

$$-12 - 3\frac{dy}{dx} + 17\frac{dy}{dx} = 0 \Rightarrow 14\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{7}$$

Thus the slope is $6/7$ for the tangent line and the point is $(-1, 0)$. Using point-slope form we get:

$$\boxed{y = \frac{6}{7}(x+1) \Leftrightarrow y = \frac{6}{7}x + \frac{6}{7}}$$

The slope for the normal line is $-7/6$ so using point-slope form we get:

$$\boxed{y = -\frac{7}{6}(x+1)} \Leftrightarrow \boxed{y = -\frac{7}{6}x - \frac{7}{6}}$$

$\square$

(12) The length of a rectangle is decreasing at the rate of 2 cm/sec while the width is increasing at the rate of 2 cm/sec. When the length is 12cm and the width is 5cm, find the rates of change of **a)** the area, **b)** the perimeter, and **c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

Solution

(a)

$$A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt}$$

Plugging in $l = 12$, $w = 5$, $\frac{dl}{dt} = -2$, and $\frac{dw}{dt} = 2$ we have

$$\frac{dA}{dt} = -2(5) + (12)(2) = \boxed{14 \text{ cm}^2/\text{sec, which is increasing.}}$$

(b)

$$P = 2l + 2w \Rightarrow \frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

Plugging in $\frac{dl}{dt} = -2$ and $\frac{dw}{dt} = 2$ we have

$$\frac{dP}{dt} = 2(-2) + 2(2) = \boxed{0 \text{ cm/sec, which is neither decreasing nor increasing}}$$

(c) The diagonal is related to the sides by the pythagorean theorem:

$$D^2 = l^2 + w^2 \Rightarrow 2d\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt}$$

When $l = 12$ and $w = 5$ we have

$$D^2 = 144 + 25 = 169 \Rightarrow D = 13$$

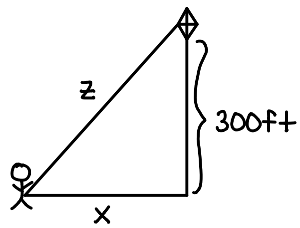
Plugging in $l = 12$, $w = 5$, $D = 13$, $\frac{dl}{dt} = -2$, and $\frac{dw}{dt} = 2$ we have

$$26\frac{dD}{dt} = 24(-2) + 10(2) \Leftrightarrow \frac{dD}{dt} = \boxed{-\frac{14}{13} \text{ cm/sec, which is decreasing}}$$

$\square$

(13) A child flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from them at a rate of 25 ft/sec. How fast must they let out the string when the kite is 500 ft away from them?

Solution The picture is the following:



Using the pythagorean theorem we have

$$x^2 + 300^2 = z^2 \Rightarrow 2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

When $z = 500$ we have $x^2 + 300^2 = 500^2 \Rightarrow x = 400$. We are also given $\frac{dx}{dt} = 25$ and we want to find $\frac{dz}{dt}$. Plugging in our known information we have

$$2(400)(25) = 2(500)\frac{dz}{dt} \Leftrightarrow \frac{dz}{dt} = \boxed{20 \text{ ft/sec}}$$

□