Math 241 Midterm 2 Review Problems

These problems are intended to help you prepare for the test. Test problems will be similar to, but not the same as, the problems below. *This list of problems is not all inclusive; it does not represent every possible type of problem.* It is suggested that you review lectures and homework problems.

Previous Semester's Exam Problems

(1) Show that $x^4 - 4x = 1$ has exactly one solution on $[-1, 0]$. Please state explicitly any theorems and how you are using them.

Solution Let $f(x) = x^4 - 4x - 1$. $(x^4 - 4x = 1 \Leftrightarrow f(x) = 0)$

$$\begin{aligned} f(-1) &= (-1)^4 - 4(-1) - 1 \\ &= 1 + 4 - 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^4 - 4(0) - 1 \\ &= -1 \end{aligned}$$

Since $f(x)$ is a continuous function on $[-1.0]$ with $f(-1) > 0$ and $f(0) < 0$, by the intermediate value theorem there is at least one solution to $f(x) = 0$. Assume there is more than one solution. Since $f(x)$ is differentiable on $(-1, 0)$, by the mean value theorem (or by Rolle's theorem), $f'(x) = 0$ in $(-1, 0)$.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 4x^3 - 4 = 0 \\ &\Leftrightarrow 4x^3 = 4 \\ &\Leftrightarrow x^3 = 1 \\ &\Leftrightarrow x = 1 (\text{not in } (-1, 0)) \end{aligned}$$

Thus there can't be more than one solution. I.e. there is exactly one.

□

(2) Show that $f(x) = 2x^3 + 3x^2 + 6x + 1$ has exactly one real root in $[-1, 0]$. Be sure to state and explain any theorems that you use.

Solution

$$\begin{aligned} f(-1) &= 2(-1)^3 + 3(-1)^2 + 6(-1) + 1 \\ &= 2(-1) + 3(1) - 6 + 1 \\ &= -2 + 3 - 6 + 1 \\ &= -4 \end{aligned}$$

$$\begin{aligned} f(0) &= 2(0)^3 + 3(0)^2 + 6(0) + 1 \\ &= 1 \end{aligned}$$

Since f is continuous, by IVT there is at least one solution. If there were more than one then since f is differentiable, by MVT $f'(x) = 0$ at some point in $[-1, 0]$

$$f'(x) = 6x^2 + 6x + 6 = 6(x^2 + x + 1)$$

$$\begin{aligned} x^2 + x + 1 = 0 \Rightarrow x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\ \Rightarrow x &= \frac{-1 \pm \sqrt{-3}}{2} \end{aligned}$$

Which is impossible since you cant square root a negative number. Thus there is exactly one solution.

□

(3) Let $f(x) = x^3 + 3x^2$

(a) Find the (open) intervals where f is increasing and where f is decreasing.

(b) Find all relative extrema (both x and y coordinates). Indicate whether it is a relative maximum or relative minimum.

(c) Find the (open) intervals where f is concave up and where f is concave down

(d) Find all inflection point(s) (both x and y coordinates)

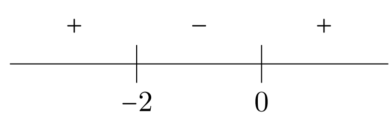
(e) Using the information from parts (a)-(d), graph the function. Label all relative extrema and inflection point(s).

Solution

(a)

$$f'(x) = 3x^2 + 6x = 3x(x + 2)$$

So the critical numbers are $x = 0, -2$. Plotting these and testing the intervals we have



Thus f is $\boxed{\text{increasing on } (-\infty, -2),\ (0, \infty) \text{ and decreasing on } (-2, 0)}$

(b) There is a relative max at $x = -2$ and a relative min at $x = 0$. Plugging these into the original function we have'

$$\begin{aligned} f(-2) &= (-2)^3 + 3(-2)^2 \\ &= -8 + 12 \\ &= 4 \end{aligned}$$

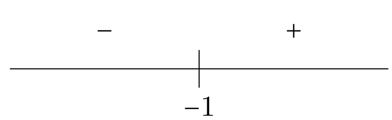
$$\begin{aligned} f(0) &= (0)^3 + 3(0) \\ &= 0 \end{aligned}$$

So the $\boxed{\text{relative max is } (-2, 4) \text{ and the relative min is } (0, 0)}$

(c)

$$f''(x) = 6x + 6 = 6(x + 1)$$

So the point we need to plot is $x = -1$. Testing the intervals we have



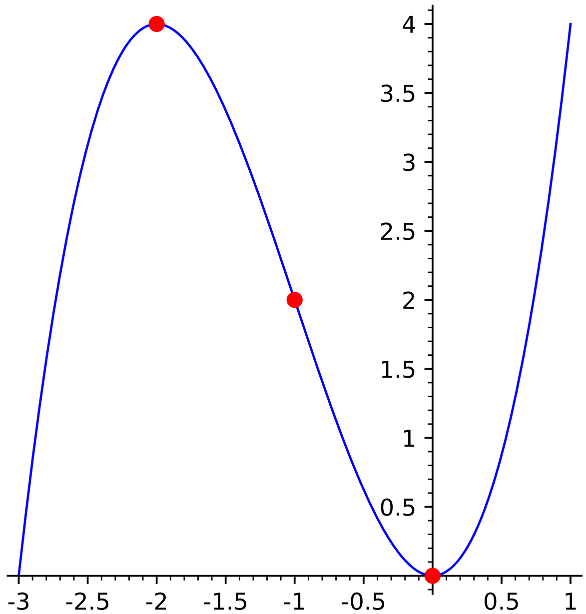
So f is $\boxed{\text{concave down on } (-\infty, -1) \text{ and concave up on } (-1, \infty)}$

(d) There is an inflection point at $x = -1$. Plugging this in gives

$$\begin{aligned} f(-1) &= (-1)^3 + 3(-1)^2 \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

So the inflection point is $\boxed{(-1, 2)}$

(e)



$\square$

(4) Consider the function $f(x) = x^3 - 6x^2 + 9x$

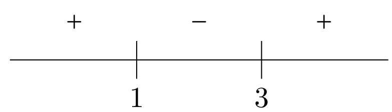
(a) Find the open intervals where f is increasing and the intervals where f is decreasing.
(b) Find both coordinates of any local extrema of the graph of f.
(c) Find the intervals where f is concave up, and the intervals where f is concave down.
(d) Find the both coordinates of any inflection point(s) of f.

Solution

(a)

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$$

This gives critical points of $x = 1, 3$. Testing the intervals we have



Thus $\boxed{f \text{ is increasing on } (-\infty, 1),\ (3, \infty) \text{ and decreasing on } (1, 3)}$

(b) There is a relative max at $x = 1$ and a relative min at $x = 3$. Plugging these into the original function we have

$$\begin{aligned} f(1) &= 1 - 6 + 9 \\ &= 4 \end{aligned}$$

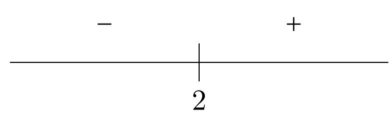
$$\begin{aligned} f(3) &= 27 - 6(9) + 27 \\ &= 27 - 54 + 27 \\ &= 0 \end{aligned}$$

Thus the $\boxed{\text{relative max is } (1, 4) \text{ and the relative min is } (3, 0)}$

(c)

$$f''(x) = 6x - 12 = 6(x-2)$$

Plotting $x = 2$ and testing we have



Thus $\boxed{f \text{ is concave down on } (-\infty, 2) \text{ and concave up on } (2, \infty)}$

(d) The inflection point occurs at $x = 2$. Plugging it into the original we have

$$\begin{aligned} f(2) &= 8 - 6(4) + 18 \\ &= 8 - 24 + 18 \\ &= 2 \end{aligned}$$

Thus the inflection point is $\boxed{(2,2)}$

□

(5) An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with two additional fences across its width, as shown in the sketch. Find the **dimensions** of the pen that has the maximum area she can enclose with 3600 m of fencing.

Solution Labeling the width as x and the length as y the equations we have are

$$3600 = 4x + 2y \text{ and } A = xy$$

Solving the first equation for y gives

$$2y = 3600 - 4x \Rightarrow y = 1800 - 2x$$

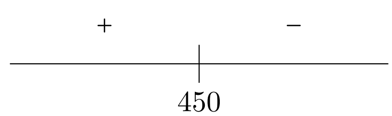
Plugging this into the equation for area gives

$$A(x) = 1800x - 2x^2$$

Taking the derivative we have

$$A'(x) = 1800 - 4x = 4(450 - x)$$

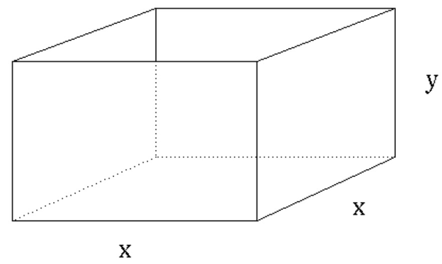
Thus the critical number is $x = 450$. Testing that this is a max we have



Thus $\boxed{\text{the max occurs when } x = 450m \text{ and } y = 1800 - 2(450) = 900m}$

□

(6) A box with a square base must have a volume of 8 in^3. What are the dimensions of the box that will minimize the amount of material needed to build it (i.e. minimize surface area).



Solution

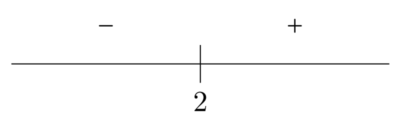
$$x^2 y = 8 \Rightarrow y = \frac{8}{x^2}$$

$$\begin{aligned} A &= 2x^2 + 4xy \\ &= 2x^2 + 4x\left(\frac{8}{x^2}\right) \\ &= 2x^2 + \frac{32}{x} \end{aligned}$$

Taking the derivative and solving for the critical point we have

$$\begin{aligned} A'(x) = 0 &\Leftrightarrow 4x - \frac{32}{x^2} = 0 \\ &\Leftrightarrow \frac{32}{x^2} = 4x \\ &\Leftrightarrow 32 = 4x^3 \\ &\Leftrightarrow x^3 = 8 \\ &\Leftrightarrow x = 2 \end{aligned}$$

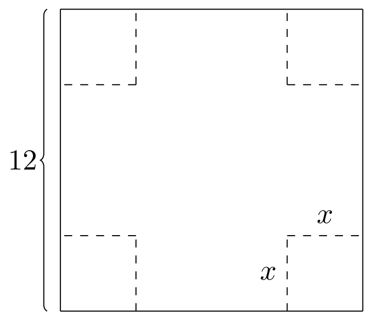
Checking that this is a minimum we have

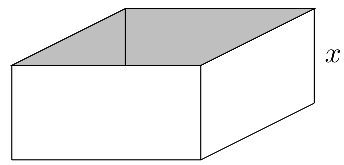


Thus the min occurs when $x = 2$ and $y = \frac{8}{2^2} = 2$. The dimensions are $\boxed{\text{2 in by 2 in by 2in}}$

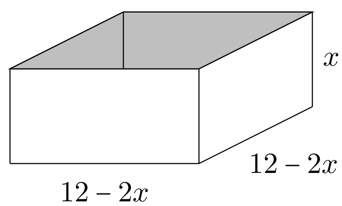
□

(7) A box with no top is constructed by cutting equal-sized squares from the corners of a 12 cm by 12 cm sheet of metal and bending up the sides. What is the largest possible volume of such a box? See the pictures below. (Note: The domain of x is $(0, 6)$.)





Solution Using the information provided, the picture can be drawn as:



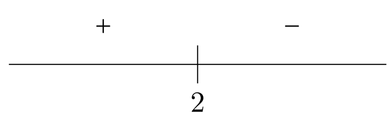
This says that the volume is given by

$$V = (12 - 2x)(12 - 2x)(x) = 4x^3 - 48x^2 + 144x$$

Taking the derivative we have

$$V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x - 6)(x - 2)$$

This gives critical numbers of $x = 6$ and $x = 2$ but since our domain is $(0, 6)$, the only critical number we care about is $x = 2$. Testing the intervals we have



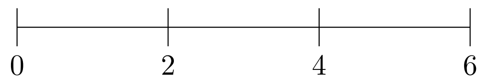
Thus there is a max at $x = 2$ which gives a volume of

$$V = (12 - 2(2))(12 - 2(2))(2) = 8 \cdot 8 \cdot 2 = \boxed{128 \text{ cm}^3}$$

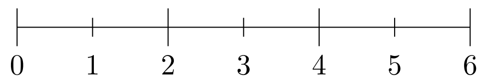
□

(8) Use mid-points to approximate the area above the x-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

Solution Splitting the interval into 3 subintervals will give us this



So each rectangle has width 2. Finding the midpoint of each interval will give us



Thus the area is approximately

$$\begin{aligned} M_3 &= 2f(1) + 2f(3) + 2f(5) \\ &= 2(1^2 + 6) + 2(3^3 + 6) + 2(5^2 + 6) \\ &= 2(7) + 2(15) + 2(31) \\ &= 14 + 30 + 62 \\ &= \boxed{106} \end{aligned}$$

□

(9) A particle's acceleration is given by $a(t) = 6t + 2$. Its velocity at 1 sec is -1 m/s. Its initial position is given by $s(0) = 5$. Find the position function $s(t)$.

Solution

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int (6t+2)\ dt \\ &= 6\left(\frac{t^2}{2}\right) + 2t + C \\ &= 3t^2 + 2t + C \end{aligned}$$

We are given that $v(1) = -1$

$$\begin{aligned} v(1) = -1 &\Leftrightarrow 3 + 2 + C = -1 \\ &\Leftrightarrow 5 + C = -1 \\ &\Leftrightarrow C = -6 \\ &\Rightarrow v(t) = 3t^2 + 2t - 6 \end{aligned}$$

$$\begin{aligned} s(t) &= \int v(t)\ dt \\ &= \int (3t^2 + 2t - 6)\ dt \\ &= 3\left(\frac{t^3}{3}\right) + 2\left(\frac{t^2}{2}\right) - 6t + D \\ &= t^3 + t^2 - 6t + D \end{aligned}$$

We are given that $s(0) = 5$

$$\begin{aligned} s(0) = 5 &\Leftrightarrow 0 + 0 - 0 + D = 5 \\ &\Leftrightarrow D = 5 \\ &\Rightarrow \boxed{s(t) = t^3 + t^2 - 6t + 5} \end{aligned}$$

□

(10) Solve the initial value problem $\dfrac{dy}{dx} = 9x^2 - 4x + 5,\ y(-1) = 0$

Solution

$$\begin{aligned} y &= \int (9x^2 - 4x + 5)\ dx \\ &= 9\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5x + C \\ &= 3x^3 - 2x^2 + 5x + C \end{aligned}$$

Using the initial value we were given we have

$$\begin{aligned} y(-1) = 0 &\Leftrightarrow 3(-1)^3 - 2(-1)^2 + 5(-1) + C = 0 \\ &\Leftrightarrow -3 - 2 - 5 + C = 0 \\ &\Leftrightarrow -10 + C = 0 \\ &\Leftrightarrow C = 10 \\ &\Leftrightarrow y = \boxed{3x^3 - 2x^2 + 5x + 10} \end{aligned}$$

□

(11) A ball is thrown from a cliff that is 6 feet from the ground ($s(0) = 6$) with initial velocity 100ft/sec ($v(0) = 100$). If the acceleration due to gravity is -32 ft/sec^2 ($a(t) = -32$), find the equation $s(t)$ for the position of the ball at time t.

Solution This is an initial value problem where we have

$$a(t) = -32, \quad v(0) = 100, \quad s(0) = 6$$

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int -32\ dt \\ &= -32t + C \end{aligned}$$

$$\begin{aligned} s(t) &- \int v(t)\ dt \\ &= \int (-32t + C)\ dt \\ &= -32\left(\frac{t^2}{2}\right) + Ct + D \\ &= -16t^2 + Ct + D \end{aligned}$$

Since $v(0) = 100$ we have

$$v(0) = C = 100 \Rightarrow v(t) = -32t + 100 \text{ and } s(t) = -16t^2 + 100t + D$$

And since $s(0) = 6$ we have

$$s(0) = D = 6 \Rightarrow \boxed{s(t) = -16t^2 + 100t + 6}$$

□

(12) Find the following integrals:

(a) $\int_0^4 2(\sqrt{t}-t)\ dt$

(b) $\int \dfrac{1+2t^3}{t^3}\ dt$

(c) $\int \tan^4 x \sec^2 x dx$

(d) $\int_0^{\pi} 2\sin x \cos^2 x$

(e) $\int \dfrac{x}{(x^2+2)^3}$

Solution

(a)

$$\begin{aligned}
\int_0^4 2(\sqrt{t}-t)\ dt &= \int_0^4 (2\sqrt{t}-2t)\ dt \\
&= \int_0^4 (2t^{1/2}-2t)\ dt \\
&= \left[2\left(\frac{t^{3/2}}{3/2}\right)-2\left(\frac{t^2}{2}\right)\right]_0^4 \\
&= \left[2\cdot\frac{2}{3}t^{3/2}-t^2\right]_0^4 \\
&= \left(\frac{4}{3}t^{3/2}-t^2\right)\Big|_9^4 \\
&= \left(\frac{4}{3}(4)^{3/2}-(4)^2\right)-\left(\frac{4}{3}(0)-(0)^2\right) \\
&= \frac{4}{3}(2)^3-16 \\
&= \frac{4}{3}(8)-16 \\
&= \frac{32}{3}-\frac{48}{3} \\
&= \boxed{-\frac{16}{3}}
\end{aligned}$$

(b)

$$\int \frac{1+2t^3}{t^3}\ dt = \int \left(\frac{1}{t^3} + \frac{2t^3}{t^3}\right)\ dt$$
$$= \int (t^{-3}+2)\ dt$$
$$= \boxed{-\frac{t^{-2}}{2} + 2t + C}$$

(c)

$$u = \tan x \Rightarrow du = \sec^2 x\ dx \Rightarrow dx = \frac{du}{\sec^2 x}$$

$$\int \tan^4 x \sec^2 x dx = \int u^4 \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}}$$
$$= \int u^4\ du$$
$$= \frac{u^5}{5} + C$$
$$= \boxed{\frac{\tan^5 x}{5} + C}$$

(d)

$$u = \cos x \Rightarrow du = -\sin x\ dx \Rightarrow dx = \frac{-\sin x}{du}$$

Since this is a definite integral we have to change the bounds.

$$x = 0 \Rightarrow u = \cos(0) = 1 \text{ and } x = \pi \Rightarrow u = \cos(\pi) = -1$$

$$\int_0^\pi 2 \sin x \cos^2 x = \int_1^{-1} 2\cancel{\sin x}u^2 \cdot \frac{du}{-\cancel{\sin x}}$$
$$= -2 \int_1^{-1} u^2\ du$$
$$= -2\left(\frac{u^3}{3}\right)\Bigg|_1^{-1}$$
$$= -2\left(\frac{(-1)^3}{3} - \frac{(1)^3}{3}\right)$$
$$= -2\left(-\frac{1}{3} - \frac{1}{3}\right)$$
$$= -2\left(\frac{-2}{3}\right)$$
$$= \boxed{\frac{4}{3}}$$

(e)

$$u = x^2 + 2 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned}\int \frac{x}{(x^2+2)^3} &= \int \frac{\cancel{x}}{u^3} \cdot \frac{du}{2\cancel{x}} \\ &= \frac{1}{2} \int u^{-3}\ du \\ &= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C \\ &= -\frac{1}{4} u^{-2} + C \\ &= \boxed{-\frac{1}{4}(x^2+2)^{-2} + C}\end{aligned}$$

□

(13) Calculate the following integrals.

(a) $\displaystyle\int \left(\frac{x^2+7x^5+5}{x^2}\right)\,dx$

(b) $\displaystyle\int \tan^4 x\sec^2 x\,dx$

(c) $\displaystyle\int_1^2 (x^2+3x-1)\,dx$

(d) $\displaystyle\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

Solution

(a)

$$\begin{aligned}\int \left(\frac{x^2+7x^5+5}{x^2}\right)\,dx &= \int \left(\frac{x^2}{x^2}+\frac{7x^5}{x^2}+\frac{5}{x^2}\right)\,dx \\ &= \int (1+7x^3+5x^{-2})\,dx \\ &= x+7\left(\frac{x^4}{4}\right)+5\left(\frac{x^{-1}}{-1}\right)+C \\ &= \boxed{x+\frac{7x^4}{4}-\frac{5}{x}+C}\end{aligned}$$

(b)

$$u=\tan x \Rightarrow du=\sec^2 x\;dx \Rightarrow dx=\frac{du}{\sec^2 x}$$

$$\begin{aligned}\int \tan^4 x\sec^2 x\;dx &= \int u^4\cancel{\sec^2 x}\cdot\frac{du}{\cancel{\sec^2 x}} \\ &= \int u^4\;du \\ &= \frac{u^5}{5}+C \\ &= \boxed{\frac{\tan^5 x}{5}+C}\end{aligned}$$

(c)

$$\int_1^2 (x^2+3x-1)\ dx = \left(\frac{x^3}{3}+3\left(\frac{x^2}{2}\right)-x\right)\Bigg|_1^2$$
$$= \left(\frac{x^3}{3}+\frac{3x^2}{2}-x\right)\Bigg|_1^2$$
$$= \left(\frac{8}{3}+\frac{12}{2}-2\right)-\left(\frac{1}{3}+\frac{3}{2}-1\right)$$
$$= \frac{8}{3}+6-2-\frac{1}{3}-\frac{3}{2}+1$$
$$= \frac{7}{3}+5-\frac{3}{2}$$
$$= \frac{14}{6}+\frac{30}{6}-\frac{9}{6}$$
$$= \boxed{\frac{35}{6}}$$

(d)

$$u = x^2+9 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

Changing the bounds we have

$$x = 0 \Rightarrow u = 0^2+9 = 9 \text{ and } u = 4 \Rightarrow u = 4^2+9 = 25$$

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\ dx = \int_9^{25} \frac{\cancel{x}}{\sqrt{u}}\cdot\frac{du}{2\cancel{x}}$$
$$= \frac{1}{2}\int_9^{25} u^{-1/2}\ du$$
$$\frac{\cancel{1}}{\cancel{2}}\left(\frac{u^{1/2}}{\cancel{1/2}}\right)\Bigg|_9^{25}$$
$$= \sqrt{u}\big|_9^{25}$$
$$= \sqrt{25}-\sqrt{9}$$
$$= 5-3$$
$$= \boxed{2}$$

□

Extra Practice Problems

(1) Find the absolute maximum and minimum values of the following functions of the given intervals.

(a) $f(x) = x^2 - 1, \ -1 \le x \le 2$

(b) $f(x) = \sqrt[3]{x}, \ -1 \le x \le 8$

Solution

(a)

$$f'(x) = 2x \text{ so } f'(x) = 0 \Leftrightarrow x = 0$$

$$\begin{aligned} f(-1) &= (-1)^2 - 1 \\ &= 0 \\ f(2) &= 2^2 - 1 \\ &= \boxed{3 \leftarrow \text{absolute max}} \\ f(0) &= 0^2 - 1 \\ &= \boxed{-1 \leftarrow \text{absolute min}} \end{aligned}$$

(b)

$$h'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

This there is a critical point at $x = 0$

$$\begin{aligned} h(-1) &= \boxed{-1 \leftarrow \text{abs min}} \\ h(0) &= 0 \\ h(8) &= \sqrt[3]{8} \\ &= \boxed{2 \leftarrow \text{abs max}} \end{aligned}$$

□

(2) Explain why $g(t) = \sqrt{t} + \sqrt{1+t} - 4$ has exactly one solution in the interval $(0, \infty)$. State any theorems used.

Solution

$$g(1) = 1 + \sqrt{2} - 4 < 0 \text{ and } g(5) = \sqrt{5} + \sqrt{6} - 4 > 0$$

Thus by the Intermediate Value Theorem we have that there is at least on zero on $(0, \infty)$. Assume there are two zeroes. Then by Rolle's Theorem there is a c in $(0, \infty)$ such that $g'(c) = 0$

$$\begin{aligned} g'(t) &= \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{1+t}} \\ &= \frac{1}{2}\left(\frac{\sqrt{1+t} + \sqrt{t}}{\sqrt{t+t^2}} \cdot \frac{\sqrt{1+t} - \sqrt{t}}{\sqrt{1+t} - \sqrt{t}}\right) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{t+t^2}(\sqrt{1+t} - \sqrt{t})}\right) \end{aligned}$$

$$f'(c) = 0 \Leftrightarrow \frac{1}{\sqrt{c+c^2}(\sqrt{1+c} - \sqrt{c})} = 0 \Leftrightarrow 1 = 0$$

which is impossible thus there cannot be two solutions.

□

(3) For the following functions, **a)** find the critical points, **b)** classify them as local maxima, local minima, or neither, **c)** find where the function is increasing, **d)** find where the function is concave up, and **e)** sketch the graph.

(a) $y = x^4 - 2x^2$

(b) $y = x^5 - 5x^4$

Solution

(a) Let $y = f(x)$

(a)

$$f'(x) = 4x^3 - 4x$$

$$f'(x) = 0 \Leftrightarrow 4x(x^2 - 1) = 0 \Leftrightarrow x = 0, \pm 1$$

So the critical points are $\boxed{x = -1,\ x = 0, \text{ and } x = 1}$

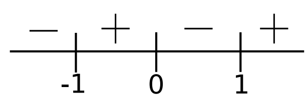
(b)

$$f''(-1) = 8 > 0,\ f''(0) = -4 < 0, \text{ and } f''(1) = 8 > 0$$

$$f(-1) = -1,\ f(0) = 0, \text{ and } f(1) = -1$$

Thus there are $\boxed{\text{relative mins of -1 at } x = \pm 1 \text{ and a relative max of 0 at } x = 0.}$

(c) The intervals for f' are given by



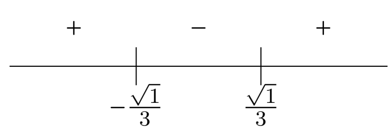
Thus f is $\boxed{\text{increasing on } (-1, 0),\ (1, \infty) \text{ and decreasing on } (-\infty, -1),\ (0, 1)}$

(d)

$$f''(x) = 12x^2 - 4$$

$$f''(x) = 0 \Leftrightarrow 4(3x^2 - 1) = 0 \Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$$

Plotting and testing the intervals we have



Thus f is $\boxed{\text{concave up on } (-\infty, -\sqrt{1/3}, (\sqrt{1/3}, \infty) \text{ and concave down on } (-\sqrt{1/3}, \sqrt{1/3}).}$

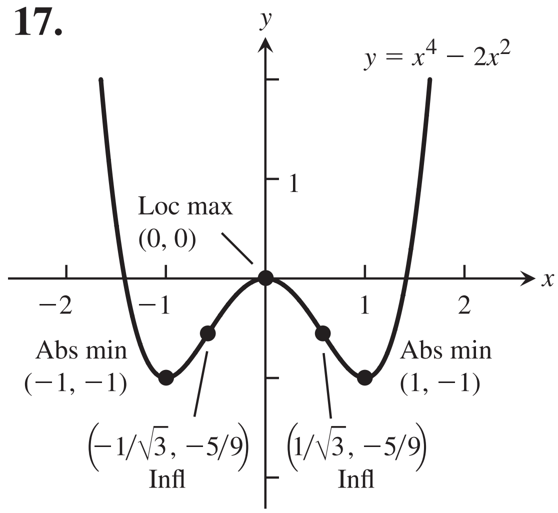
$$f\left(-\sqrt{\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1}{3} - 2\right)$$
$$= \frac{1}{3} \cdot -\frac{5}{3}$$
$$= -\frac{5}{9}$$

$$f\left(\sqrt{\frac{1}{3}}\right) = \frac{1}{3} \cdot -\frac{5}{3}$$
$$= -\frac{5}{9}$$

Therefore there are two inflection points of $\boxed{\left(-\sqrt{\frac{1}{3}}, -\frac{5}{9}\right) \text{ and } \left(\sqrt{\frac{1}{3}}, -\frac{5}{9}\right)}$

(e) Using the above information we have:

17.



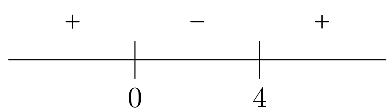
(b) Let $y = f(x) = x^4(x-5)$

(a)

$$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$$

So the critical points are $\boxed{x = 0 \text{ and } x = 4.}$

(b) Testing the intervals we have



$$f(0) = 0 \text{ and } f(4) = -256$$

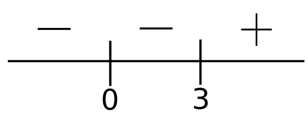
So there is a $\boxed{\text{relative max of 0 at } x = 0 \text{ and a relative min of } -256 \text{ at } x = 4}$

(c) From the above number line we have that f is $\boxed{\text{increasing on } (-\infty, 0),\ (4, \infty) \text{ and decreasing on } (0, 4}$

(d)

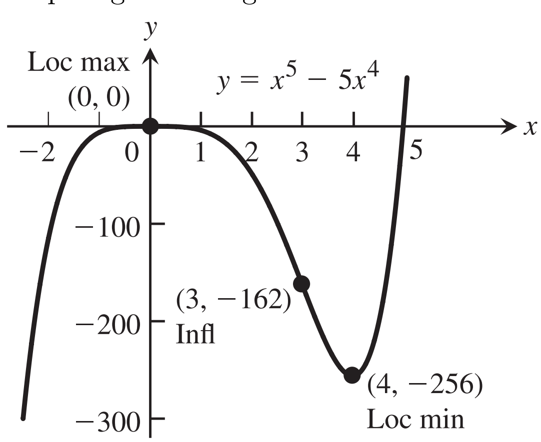
$$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$$

This gives important points of $x = 0$ and $x = 3$. Testing the intervals we have



So f is $\boxed{\text{concave down on } (-\infty, 0),\ (0, 3) \text{ and concave up on } (3, \infty).}$

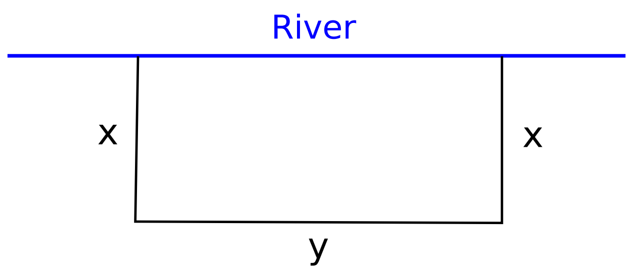
(e) Putting the previous steps together we get



□

(4) A rectangular plot of land will be bounded on one side by a river and on the other three sides by some sort of fence. With 800 m of fencing at your disposal, what is the largest area you can enclose, and what are its dimensions?

Solution The picture is



We are given

$$2x + y = 800 \Rightarrow y = 800 - 2x$$

Thus

$$A = xy = x(800 - 2x) = 800x - 2x^2$$

Differentiating with respect to x we have

$$\frac{dA}{dx} = 800 - 4x$$

This gives one critical point of $x = 200$. Testing the intervals we have that there is a relative and thus absolute max at $x = 200 \Rightarrow y = 800 - 400 = 400$. So the dimensions are $\boxed{200 \text{ m by } 400 \text{ m}}$

$$A = 200(400) = \boxed{80000\text{m}^2}$$

□

(5) Suppose you want to build a steel box with an open top and square base. Find the dimensions for a box of volume 500 ft^3 that will weigh as little as possible.

Solution We have

$$V = 500 = x^2 y \Rightarrow y = \frac{500}{x^2}$$

We want to minimize the material so we want to minimize surface area which is given by

$$S = x^2 + 4xy = x^2 + \frac{2000}{x}$$

Differentiating we have

$$\frac{dS}{dx} = 2x - \frac{2000}{x^2} = \frac{2x^3 - 2000}{x^2}$$

Since x must be positive this gives one critical point of $x = 10$. Testing we have that there is a relative and thus absolute minimum at $x = 10$.

(a)

$$x = 10 \Rightarrow y = \frac{500}{100} = 5$$

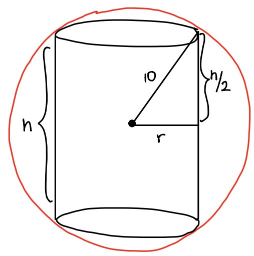
Thus the dimensions are $\boxed{10 \text{ ft by } 10 \text{ ft by } 5 \text{ ft}}$

(b) By minimizing the amount of material used, we minimize the weight used since the weight depends on the material.

□

(6) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

Solution The picture looks like



Using the picture we have that

$$\left(\frac{h}{2}\right)^2 + r^2 = 10^2 \Rightarrow r^2 = 100 - \frac{h^2}{4}$$

Plugging this in we have

$$V = \pi\left(100 - \frac{h^2}{4}\right)h = 100\pi h - \frac{\pi h^3}{4}$$

Differentiating with respect to h gives

$$\frac{dV}{dh} = 100\pi - \frac{3\pi h^2}{4} = \frac{400\pi - 3\pi h^2}{4}$$

Solving for the critical point we have

$$400\pi - 3\pi h^2 = 0 \Leftrightarrow 3\pi h^2 = 400\pi \Leftrightarrow h^2 = \frac{400}{3} \Leftrightarrow h = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}}$$

Since the domain for h is $[0, 20]$ we have

$$\begin{aligned} V(0) &= 0 \\ V(20) &= 0 \\ V\left(\frac{20}{\sqrt{3}}\right) &= 100\pi\left(\frac{20}{\sqrt{3}}\right) - \frac{\pi(20/\sqrt{3})^3}{4} \\ &= \frac{2000\pi}{\sqrt{3}} - \frac{20^3\pi}{4\cdot(\sqrt{3})^3} \\ &= \frac{2000\pi}{\sqrt{3}} - \frac{8000\pi}{4\cdot 3\sqrt{3}} \\ &= \frac{2000\pi}{\sqrt{3}} - \frac{2000\pi}{3\sqrt{3}} \\ &= \frac{6000\pi - 2000\pi}{3\sqrt{3}} \\ &= \frac{4000\pi}{3\sqrt{3}} \end{aligned}$$

Thus the maximum volume is $\boxed{\frac{4000\pi}{3\sqrt{3}}\text{cm}^3}$ □

(7) Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

Solution

$$f(x) = x^4 - 2 \Rightarrow f'(x) = 4x^3$$

$$\begin{aligned} x_1 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-1}{4} \\ &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{5}{4} - \frac{f(5/4)}{f'(5/4)} \\ &= \frac{5}{4} - \frac{(625/256) - 2}{(500/64)} \\ &= \frac{5}{4} - \frac{625 - 512}{2000} \\ &= \frac{5}{4} - \frac{113}{2000} \\ &= \boxed{\frac{2387}{2000}} \end{aligned}$$

□

(8) Find the most general antiderivative for the following. Check your answer by differentiation.

(a) $f(x) = \frac{1}{x^2} - x^2 - \frac{1}{3}$

(b) $f(x) = 2x(1 - x^{-3})$

Solution

(a)

$$f(x) = x^{-2} - x^2 - \frac{1}{3}$$

$$\begin{aligned} F(x) &= \frac{x^{-2+1}}{-2+1} - \frac{x^{2+1}}{2+1} - \frac{1}{3} \cdot \frac{x^{0+1}}{0+1} + C \\ &= \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{x}{3} + C \\ &= \boxed{-\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C} \end{aligned}$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C\right) = \frac{1}{x^2} - x^2 - \frac{1}{3}$$

(b)

$$f(x) = 2x - 2x^{-2}$$

$$\begin{aligned} F(x) &= 2 \cdot \frac{x^{1+1}}{1+1} - 2 \cdot \frac{x^{-2+1}}{-2+1} + C \\ &= 2 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^{-1}}{-1} + C \\ &= \boxed{x^2 + \frac{2}{x} + C} \end{aligned}$$

Check:

$$\frac{d}{dx}\left(x^2 + \frac{2}{x} + C\right) = 2x - \frac{2}{x^2} = 2x(1 - x^{-3})$$

□

(9) Solve the following initial value problems.

(a) $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \; r(0) = 0$

(b) $\dfrac{d^3y}{dx^3} = 6; \; y''(0) = -8, \; y'(0) = 0, \; y(0) = 5$

Solution

(a) Using $u = \pi\theta \Rightarrow du = \pi \; d\theta$ we have

$$r = \int (-\pi \sin(\pi\theta)) \; d\theta = \cos(\pi\theta) + C$$

$$0 = \cos(0) + C \Leftrightarrow C = -1 \Rightarrow \boxed{r = \cos(\pi\theta) - 1}$$

(b)

$$y''(x) = 6x + C$$

$$-8 = 0 + C \Leftrightarrow C = -8 \Rightarrow y''(x) = 6x - 8$$

$$y'(x) = 6 \cdot \frac{x^2}{2} - 8 \cdot \frac{x^{0+1}}{0+1} = 3x^2 - 8x + C$$

$$0 = 0 - 0 + C \Leftrightarrow C = 0 \Rightarrow y'(x) = 3x^2 - 8x$$

$$y = 3 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + C = x^3 - 4x^2 + C$$

$$5 = 0 - 0 + C \Leftrightarrow C = 5 \Rightarrow \boxed{y = x^3 - 4x^2 + 5}$$

□

(10) The acceleration of an object is given by $\frac{3t}{8}$ find the position given that $v(4) = 3$ and $s(4) = 4$.

Solution

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int \frac{3}{8} t\ dt \\ &= \frac{3}{8}\left(\frac{t^2}{2}\right) + C \\ &= \frac{3t^2}{16} + C \end{aligned}$$

Since $v(4) = 3$ we have

$$\begin{aligned} v(4) = 3 &\Leftrightarrow \frac{3(16)}{16} + C = 3 \\ &\Leftrightarrow 3 + C = 3 \\ &\Leftrightarrow C = 0 \\ &\Leftrightarrow v(t) = \frac{3t^2}{16} \end{aligned}$$

We can now find $s(t)$

$$\begin{aligned} s(t) &= \int v(t)\ dt \\ &= \int \frac{3}{16} t^2\ dt \\ &= \frac{3}{16} \cdot \frac{t^3}{3} + D \\ &= \frac{t^3}{16} + D \end{aligned}$$

Using the fact that $s(4) = 4$ we have

$$\begin{aligned} s(4) = 4 &\Leftrightarrow \frac{64}{16} + D = 4 \\ &\Leftrightarrow 4 + D = 4 \\ &\Leftrightarrow D = 0 \\ &\rightarrow s(t) = \boxed{\frac{t^3}{16}} \end{aligned}$$

□

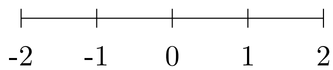
(11) Using 4 rectangles of equal length and the following rules find Riemann sums estimates for $f(x) = -x^2 + 16$ from $x = -2$ to $x = 2$.

(a) Left-hand endpoints

(b) Right-hand endpoints

(c) Midpoints

Solution Splitting the interval into 4 subintervals we have



So each rectangle has width 1.

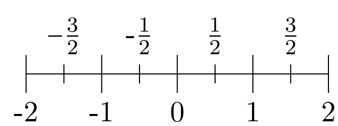
(a)

$$\begin{aligned} L_4 &= 1f(-2) + 1f(-1) + 1f(0) + 1f(1) \\ &= (-4+16) + (-1+16) + (16) + (-1+16) \\ &= 12 + 15 + 16 + 15 \\ &= \boxed{58} \end{aligned}$$

(b)

$$\begin{aligned} R_4 &= 1f(-1) + 1f(0) + 1f(1) + 1f(2) \\ &= (-1+16) + (16) + (-1+16) + (-4+16) \\ &= 15 + 16 + 15 + 12 \\ &= \boxed{58} \end{aligned}$$

(c) Finding the midpoint of each subinterval we have



$$\begin{aligned} M_4 &= 1f\left(-\frac{3}{2}\right) + 1f\left(-\frac{1}{2}\right) + 1f\left(\frac{1}{2}\right) + 1f\left(\frac{3}{2}\right) \\ &= \left(-\frac{9}{4} + 16\right) + \left(-\frac{1}{4} + 16\right) + \left(-\frac{1}{4} + 16\right) + \left(-\frac{9}{4} + 16\right) \\ &= -\frac{20}{4} + 64 \\ &= -5 + 64 \\ &= \boxed{59} \end{aligned}$$

□

(12) Find $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t \; dt$

(a) by evaluating the integral and differentiating the result.

(b) by differentiating the integral directly

Solution

(a)

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t \; dt &= \frac{d}{dx}\left(\sin t\big|_0^{\sqrt{x}}\right) \\ &= \frac{d}{dx}\left(\sin\sqrt{x} - \sin 0\right) \\ &= \frac{d}{dx}\left(\sin\sqrt{x}\right) \\ &= \cos\sqrt{x}\cdot\frac{d}{dx}\left(\sqrt{x}\right) \\ &= \boxed{(\cos\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right)}\end{aligned}$$

(b) Let $u = \sqrt{x}$

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t \; dt &= \left(\frac{d}{du}\int_0^{u} \cos t \; dt\right)\left(\frac{du}{dx}\right) \\ &= (\cos u)\left(\frac{1}{2}x^{-1/2}\right) \\ &= \boxed{(\cos\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right)}\end{aligned}$$

□

(13) Evaluate the following integrals

(a) $\int \tan x \sec^2 x \, dx$

(b) $\int \frac{x}{\sqrt{4x^2+9}} \, dx$

(c) $\int \sec^2(5x) \, dx$

(d) $\int x(2x+1)^5 \, dx$

(e) $\int_0^2 x\sqrt{x^2+1} \, dx$

(f) $\int_0^{\sqrt{\pi}/2} x \sin x^2 \, dx$

(g) $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx$

(h) $\int x^{-3}(x+1) \, dx$

(i) $\int_0^{\pi/3} 2\sec^2 x \, dx$

(j) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx$

(k) $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} \, dt$

(l) $\int_{-1}^{1} t^3(1+t^4)^3 \, dt$

(m) $\int_0^{\pi/6} (1-\cos 3t) \sin 3t \, dt$

Solution

(a)

$$u = \tan x \Rightarrow du = \sec^2 x \, dx \Rightarrow dx = \frac{du}{\sec 2x}$$

$$\begin{aligned} \int \tan x \sec^2 x \, dx &= \int u \cancel{sec^2 x} \cdot \frac{du}{\cancel{\sec 2x}} \\ &= \int u \, du \\ &= \frac{u^2}{2} + C \\ &= \boxed{\frac{\tan^2 x}{2} + C} \end{aligned}$$

(b)

$$u = 4x^2 + 9 \Rightarrow du = 8x \, dx \Rightarrow dx = \frac{du}{8x}$$

$$\begin{aligned} \int \frac{x}{\sqrt{4x^2+9}} \, dx &= \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{8\cancel{x}} \\ &= \frac{1}{8} \int u^{-1/2} \, du \\ &= \frac{1}{8} \cdot \frac{u^{1/2}}{1/2} + C \\ &= \frac{1}{4} u^{1/2} + C \\ &= \boxed{\frac{1}{4}(4x^2+9)^{1/2} + C} \end{aligned}$$

(c)

$$u = 5x \Rightarrow du = 5\ dx \Rightarrow dx = \frac{du}{5}$$

$$\begin{aligned}\int \sec^2(5x)\ dx &= \int \sec^2(u) \cdot \frac{du}{5} \\ &= \frac{1}{5}\int \sec^2 u\ du \\ &= \frac{1}{5}\tan u + C \\ &= \boxed{\frac{1}{5}\tan(5x) + C}\end{aligned}$$

(d)

$$u = 2x + 1 \Rightarrow du = 2\ dx \Rightarrow du = \frac{du}{2}$$

$$\begin{aligned}\int x(2x+1)^5\ dx &= \int x \cdot u^5 \cdot \frac{du}{2} \\ &= \frac{1}{2}\int xu^5\ du\end{aligned}$$

Using $u = 2x + 1$ we have $2x = u - 1 \Rightarrow x = \frac{u-1}{2}$

$$\begin{aligned}&= \frac{1}{2}\int \left(\frac{u-1}{2}\right)u^5\ du \\ &= \frac{1}{2}\int \left(\frac{u}{2} - \frac{1}{2}\right)u^5\ du \\ &= \frac{1}{2}\int \left(\frac{u^6}{2} - \frac{u^5}{2}\right)\ du \\ &= \frac{1}{2}\left(\frac{u^7}{14} - \frac{u^6}{12}\right) + C \\ &= \frac{u^7}{28} - \frac{u^6}{24} + C \\ &= \boxed{\frac{(2x+1)^7}{28} - \frac{(2x+1)^6}{24} + C}\end{aligned}$$

(e)

$$u = x^2 + 1 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

Changing the bounds we have

$$x = 2 \Rightarrow u = 2^2 + 1 = 5 \text{ and } x = 0 \Rightarrow u = 0^2 + 1 = 1$$

$$\begin{aligned}
\int_0^2 x\sqrt{x^1 + 1}\ dx &= \int_1^5 \cancel{x}\sqrt{u} \cdot \frac{du}{2\cancel{x}} \\
&= \frac{1}{2}\int_1^5 \sqrt{u}\ du \\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_1^5 \\
&= \frac{1}{3}u^{3/2}\Bigg|_1^5 \\
&= \boxed{\frac{1}{3}(5)^{3/2} - \frac{1}{3}}
\end{aligned}$$

(f)

$$u = x^2 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

Changing the bounds we have

$$x = 0 \Rightarrow u = 0^2 = 0 \text{ and } x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$$

$$\begin{aligned}
\int_0^{\sqrt{\pi}/2} x\sin x^2\ dx &= \int_0^{\pi/4} \cancel{x}\sin u \cdot \frac{du}{2\cancel{x}} \\
&= \frac{1}{2}\int_0^{\pi/4} \sin u\ du \\
&= \frac{1}{2}(-\cos u)\Bigg|_0^{\pi/4} \\
&= -\frac{1}{2}\cos\left(\frac{\pi}{4}\right) - \left(-\frac{1}{2}\cos 0\right) \\
&= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2}(1) \\
&= \boxed{-\frac{\sqrt{2}}{4} + \frac{1}{2}}
\end{aligned}$$

(g)

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) \, dx = \int \left(\frac{1}{2} x^{1/2} + 2x^{-1/2} \right) \, dx$$

$$= \frac{1}{2} \left(\frac{x^{3/2}}{3/2} \right) + 2 \left(\frac{x^{1/2}}{1/2} \right) + C$$

$$= \boxed{\frac{1}{3} x^{3/2} + 4x^{1/2} + C}$$

(h)

$$\int x^{-3}(x+1) \, dx = \int (x^{-2} + x^{-3}) \, dx$$

$$= \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C$$

$$= \boxed{-x^{-1} - \frac{1}{2} x^{-2} + C}$$

(i)

$$\int_0^{\pi/3} 2 \sec^2 x \, dx = 2 \tan x \Big|_0^{\pi/3}$$

$$= 2 \tan \left(\frac{\pi}{3} \right) - 2 \tan 0$$

$$= \boxed{2\sqrt{3} - 2}$$

(j)

$$u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2} x^{-1/2} \, dx \Rightarrow dx = 2\sqrt{x} \, du$$

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx = \int \frac{1}{\cancel{\sqrt{x}} u^2} \cdot 2 \cancel{\sqrt{x}} \, du$$

$$2 \int \frac{1}{u^2} \, du$$

$$= 2 \int u^{-2} \, du$$

$$= 2 \left(\frac{u^{-1}}{-1} \right) + C$$

$$= -2u^{-1} + C$$

$$= \boxed{-2(1+\sqrt{x})^{-1} + C}$$

(k)

$$u = \cos(2t+1) \Rightarrow du = -\sin(2t+1)(2)\ dt \Rightarrow dt = -\frac{du}{2\sin(2t+1)}$$

$$\begin{aligned}
\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\ dt &= \int \frac{\cancel{\sin(2t+1)}}{u^2} \cdot \frac{du}{-2\cancel{\sin(2t+1)}} \\
&= -\frac{1}{2}\int \frac{1}{u^{-2}}\ du \\
&= -\frac{1}{2}\int u^{-2}\ du \\
&= -\frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C \\
&= \frac{1}{2}u^{-1} + C \\
&= \boxed{\frac{1}{2}(\cos(2t+1))^{-1} + C}
\end{aligned}$$

(l) Since the integrand $f(t) = t^3(1+t^4)^3$ is odd, the integral equals $\boxed{0}$.

(m)

$$u = 1 - \cos(3t) \Rightarrow du = -(-\sin(3t)(3))\ dt = 3\sin(3t)\ dt \Rightarrow dt = \frac{du}{3\sin(3t)}$$

Changing the bounds we have

$$t = 0 \Rightarrow u = 1 - \cos 0 = 0 \text{ and } t = \frac{\pi}{6} \Rightarrow u = 1 - \cos\frac{\pi}{2} = 1$$

$$\begin{aligned}
int_0^{\pi/6}(1-\cos 3t)\sin 3t\ dt &= \int_0^1 u\cancel{\sin(3t)} \cdot \frac{du}{3\cancel{\sin(3t)}} \\
&= \frac{1}{3}\int_0^1 u\ du \\
&= \frac{1}{3}\left(\frac{u^2}{2}\right)\Bigg|_0^1 \\
&= \frac{u^2}{6}\Bigg|_0^1 \\
&= \boxed{\frac{1}{6}}
\end{aligned}$$

$\square$