Midterm 2 – Math 241

Monday, April 20, 2020

(1) Show that $x^3 + 6x + 3 = 0$ has exactly one solution in the interval $[-3, 0]$. Make sure you name what theorems you are using and why you can use them.

Solution Let $f(x) = x^3 + 6x + 3$. We want to show that $f(x) = 0$.

$$\begin{aligned} f(-3) &= (-3)^3 + 6(-3) + 3 \\ &= -27 - 18 + 3 \\ &= -42 \end{aligned}$$

$$\begin{aligned} f(0) &= 0^3 + 6(0) + 3 \\ &= 3 \end{aligned}$$

Since f is continuous on the interval $[-3, 0]$ with $f(-3) < 0$ and $f(0) > 0$, by the Intermediate Value Theorem, there is at least one solution to $f(x) = 0$.

f is differentiable on the interval, $(-3, 0)$, so by the Mean Value Theorem, if there were more than one solution to $f(x) = 0$, then $f'(x) = 0$ for some x in $(-3, 0)$.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 3x^2 + 6 = 0 \\ &\Leftrightarrow x^2 = -2 \end{aligned}$$

which is impossible. Thus there must be exactly one solution to $f(x) = 0$ in the interval $[-3, 0]$.

□

(2) Consider the function $f(x) = 9x - \frac{1}{3}x^3$

(a) Find the intervals where f is increasing and the intervals where f is decreasing.

(b) Find both coordinates of any local max and local min of f.

(c) Find the intervals where f is concave up, and the intervals where f is concave down.

(d) Find both coordinates of any inflection point(s) of f.

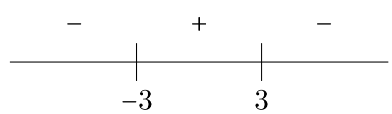
(e) Graph the function

Solution

(a)

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 9 - x^2 = 0 \\ &\Leftrightarrow x^2 = 9 \\ &\Leftrightarrow x = \pm 3 \end{aligned}$$

Thus our critical numbers are ± 3



From the number line we get that f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3) \cup (3, \infty)$

(b) The local max occurs at $x = 3$

$$f(3) = 27 - \frac{27}{3} = 18$$

Therefore the local max is (3,18)

The local min occurs at $x = -3$

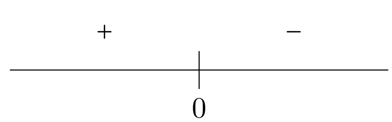
$$f(-3) = -27 + \frac{27}{3} = -18$$

Therefore the local min is $(-3, -18)$

(c)

$$\begin{aligned} f''(x) = 0 &\Leftrightarrow \frac{d}{dx}(f'(x)) = 0 \\ &\Leftrightarrow \frac{d}{dx}(9 - x^2) = 0 \\ &\Leftrightarrow 2x = 0 \\ &\Leftrightarrow x = 0 \end{aligned}$$

So our only important point is at $x = 0$



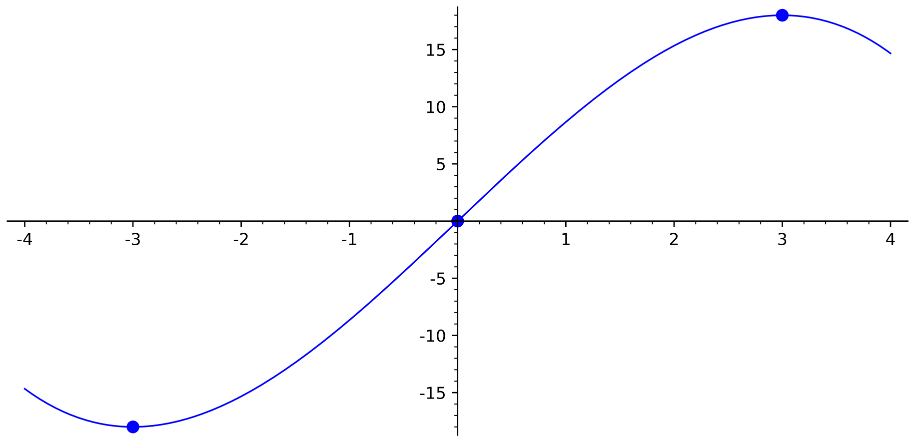
Thus we have f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$

(d) The inflection point occurs at $x = 0$

$$f(0) = 0 - 0 = 0$$

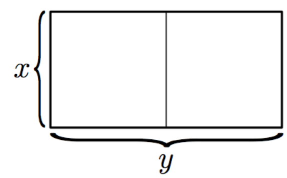
Thus the inflection point is (0,0)

(e) Putting the above information together, we have the following graph:



□

(3) A 24 m^2 rectangular farm is to be enclosed by a fence and divided into two equal parts by another fence (picture below). What dimensions will require the smallest total length of fence? How much fence will be needed?



Solution The picture is as follows We are given that $24 = xy \Leftrightarrow y = \frac{24}{x}$ and we want to minimize

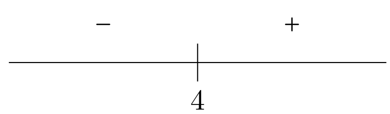
$$P = 3x + 2y = 3x + \frac{48}{x}$$

Taking the derivative we have

$$\frac{dP}{dx} = 3 - \frac{48}{x^2}$$

$$\frac{dP}{dx} = 0 \Leftrightarrow 3x^2 = 48 \Leftrightarrow x^2 = 16 \Rightarrow x = \pm 4$$

Since x must be positive we have one critical point of $x = 4$. Testing the intervals we have that this critical point is a minimum



Thus the dimensions needed are $x = \boxed{4\text{m}}$ and $y = \frac{24}{4} = \boxed{6\text{m}}$. The amount of fencing that will be used is

$$P = 3(4) + 2(6) = \boxed{24\text{m}}$$

□

(4) An object is thrown upward off a cliff that is 25m tall ($s(0) = 25$) with an initial velocity of 20 m/s ($v(0) = 20$). The acceleration of the object is given by

$$a(t) = -10 \text{ m/s}^2$$

(a) Find the equation for the position $s(t)$.
(b) When does the object reach its maximum height?
(c) When does the object hit the bottom ground?

Solution

(a)

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int -10\ dt \\ &= -10t + C \end{aligned}$$

$$\begin{aligned} s(t) &= \int v(t)\ dt \\ &= \int (-10t + C)\ dt \\ &= -5t^2 + Ct + D \end{aligned}$$

Using the given information we have

$$\begin{aligned} v(0) = 0 &\Leftrightarrow -10(0) + C = 20 \\ &\Leftrightarrow C = 20 \\ &\Leftrightarrow v(t) = -10t + 20 \\ &\Rightarrow s(t) = -5t^2 + 20t + D \end{aligned}$$

$$\begin{aligned} s(0) = 25 &\Leftrightarrow -5(0)^2 + 20(0) + D = 25 \\ &\Leftrightarrow D = 25 \\ &\Rightarrow \boxed{s(t) = -5t^2 + 20t + 25} \end{aligned}$$

(b) The object reaches its maximum height when $v(t) = 0$.

$$v(t) = 0 \Leftrightarrow -10t + 20 = 0 \Leftrightarrow \boxed{t = 2 \text{ sec}}$$

(c) The object hits the ground when $s(t) = 0$.

$$\begin{aligned} s(t) = 0 &\Leftrightarrow -5t^2 + 30t + 25 = 0 \\ &\Leftrightarrow -5(t^2 - 4t + 5) = 0 \\ &\Leftrightarrow -5(t - 5)(t + 1) = 0 \\ &\Rightarrow t = 5, -1 \end{aligned}$$

Since time has to be positive, the answer is $\boxed{t = 5 \text{ sec}}$

□

(5) Calculate the following integrals. Simplify your answers if possible.

(a) $\int_0^1 (x^2 + \sqrt{x})\ dx$

(b) $\int \left(\frac{1}{x^3} - 1 + 2\sec^2 x\right)\ dx$

(c) $\int \cos^5(2x)\sin(2x)\ dx$

(d) $\int_0^1 t^3(1+t^4)^3\ dt$

Solution

(a)

$$\begin{aligned}\int_0^1 (x^2 + \sqrt{x})\ dx &= \int_0^1 (x^2 + x^{1/2})\ dx \\ &= \left(\frac{x^3}{3} + \frac{x^{3/2}}{3/2}\right)\Bigg|_0^1 \\ &= \left(\frac{x^3}{3} + \frac{2}{3}x^{3/2}\right)\Bigg|_0^1 \\ &= \left(\frac{1}{3} + \frac{2}{3}\right) - (0+0) \\ &= \boxed{1}\end{aligned}$$

(b)

$$\begin{aligned}\int \left(\frac{1}{x^3} - 1 + 2\sec^2 x\right)\ dx &= \int (x^{-3} - 1 + 2\sec^2 x)\ dx \\ &= \boxed{\frac{x^{-2}}{-2} - x + 2\tan x + C}\end{aligned}$$

(c)

$$u = \cos(2x) \Rightarrow du = -2\sin(2x)\ dx \Rightarrow dx = \frac{du}{-2\sin(2x)}$$

$$\begin{aligned}\int \cos^5(2x)\sin(2x)\ dx &= \int u^5 \cancel{\sin(2x)} \cdot \frac{du}{-2\cancel{\sin(2x)}} \\ &= -\frac{1}{2}\int u^5\ du \\ &= -\frac{1}{2}\cdot\frac{u^6}{6} + C \\ &= \boxed{-\frac{\cos^6(2x)}{12} + C}\end{aligned}$$

(d)

$$u = 1 + t^4 \Rightarrow du = 4r^2 \; dt \Rightarrow dt = \frac{du}{4t^3}$$

New bounds:

$$t = 0 \Rightarrow u = 1 \text{ and } t = 1 \Rightarrow u = 2$$

$$\begin{aligned}
\int_0^1 t^3(1+t^4)^3 \; dt &= \int_1^2 \cancel{t^3} u^3 \cdot \frac{du}{4\cancel{t^3}} \\
&= \frac{1}{4} \int_1^2 u^3 \; du \\
&= \frac{1}{4} \cdot \frac{u^4}{4} \bigg|_1^2 \\
&- \frac{u^4}{16} \bigg|_1^2 \\
&= -\frac{1}{16} + \frac{16}{16} \\
&= \boxed{\frac{15}{16}}
\end{aligned}$$

□

(6) Find the following derivative:

$$\frac{d}{dx}\int_0^{x^3} \sin t \ dt$$

Optional Extra Credit: Solve Problem 6 (correctly lol) using two different methods.

Solution
Method 1: Let $u = x^3$

$$\begin{aligned}\frac{d}{dx}\int_0^{x^3} \sin t \ dt &= \left(\frac{d}{du}\int_0^{u} \sin t \ dt\right) \cdot \frac{du}{dx} \\ &= \sin u \cdot 3x^2 \\ &= \boxed{3x^2 \sin x^3}\end{aligned}$$

Method 2:

$$\frac{d}{dx}\int_0^{x^3} \sin t \ dt\& = \frac{d}{dx}(-\cos t|_0^{x^3})$$

$$\begin{aligned}&= \frac{d}{dx}(-\cos\left(x^3\right) + \cos 0) \\ &= \sin\left(x^3\right) \cdot 3x^2 \\ &= \boxed{3x^2 \sin x^3}\end{aligned}$$

□