Final Exam Practice Questions - SOLUTIONS

Exponential and Logarithmic Functions

(1) The half-life of a radioactive substance is 9 years. Initially a sample has 20 grams. How many grams remain after 8 years?

Solution The half-life equation (which is on your reference sheet) is: $A(t) = C\left(\frac{1}{2}\right)^{t/k}$. We know from this problem that $C = 20$ and $k = 9$. We want to find A when $t = 8$.

$$\begin{aligned} A(8) &= 20\left(\frac{1}{2}\right)^{8/9} \leftarrow \text{exact answer} \\ &\approx 10.80 \leftarrow \text{rounded answer using a calculator} \end{aligned}$$

Note that the exact answer is the one that you should always use, unless the problem indicates that you may round it to a certain amount of decimal places.

□

(2) Initially, 200 bacteria are present in a colony. Eight hours later there are 500. What is the population two hours after the start?

Solution The formula we're using (from the reference sheet) is $f(x) = Ca^x$. We are given the initial population, so $C = 200$, and are told that when $x = 8$, $f(8) = 500$. Plugging in this information and solving for a, we have

$$\begin{aligned} 500 = 200a^8 &\Leftrightarrow \frac{5}{2} = a^8 \\ &\Leftrightarrow a = \left(\frac{5}{2}\right)^{1/8} \end{aligned}$$

Now, our general equation is

$$f(x) = 200\left(\frac{5}{2}\right)^{1/8 \cdot x} \quad \text{OR} \quad f(x) = 200\left(\frac{5}{2}\right)^{x/8}$$

Plugging in $x = 2$ for 2 years, we get

$$\begin{aligned} f(5) &= 200\left(\frac{5}{2}\right)^{2/8} \\ &= 200\left(\frac{5}{2}\right)^{1/4} \leftarrow \text{exact answer} \\ &\approx 251.49 \leftarrow \text{rounded answer using a calculator} \end{aligned}$$

Note that the exact answer is the one that you should always use, unless the problem indicates that you may round it to a certain amount of decimal places.

□

(3) Initially a bank account that is compounded continosly has \$4,000. 10 years later it has \$12,000. Find the amount after 5 years.

Solution Given:

$$N_0 = 4000 \text{ and } N(10) = 12,000$$

We want to find $N(5)$

$$\begin{aligned} N(10) = 12000 &\Leftrightarrow 12000 = 4000e^{10k} \\ &\Leftrightarrow 3 = e^{10k} \\ &\Leftrightarrow \ln 3 = \ln\left(e^{10k}\right) \\ &\Leftrightarrow \ln 3 = 10k \\ &\Leftrightarrow k = \frac{1}{10}\ln 3 \\ &\Rightarrow N(t) = 4000e^{\frac{t}{10}\ln 3} \\ &\Rightarrow N(5) = 4000e^{\frac{5}{10}\ln 3} \\ &\Rightarrow N(5) = \boxed{4000e^{\frac{1}{2}\ln 3} \text{ dollars}} \end{aligned}$$

□

(4) Graph $y = 1 - \ln(2 - x)$ labelling all asymptotes. Then find the domain and range.

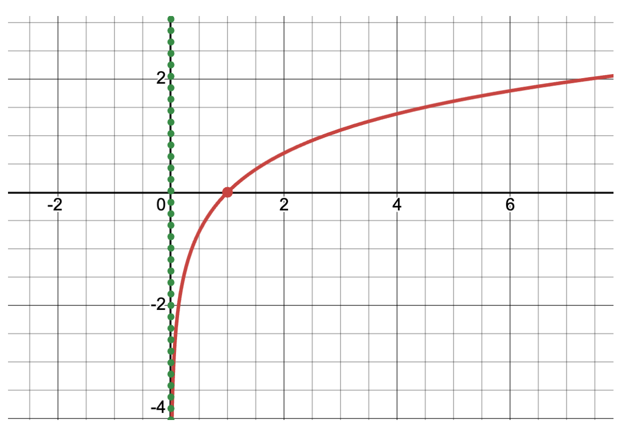
Solution Rewriting the equation to put it in the normal order, we have

$$y = -\ln(-x + 2) + 1 = -\ln(-(x - 2)) + 1$$

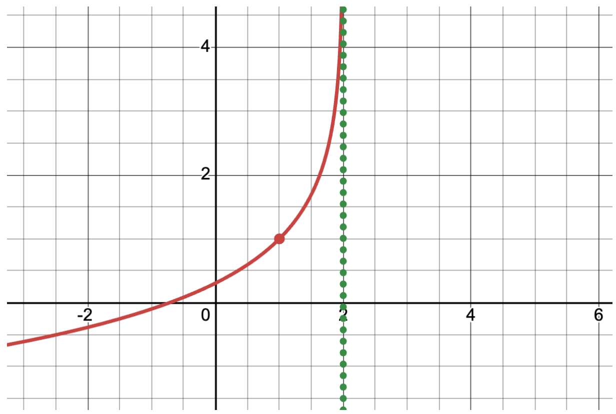
This means that we need to take the regular graph of $\ln x$ and:

- Reflect across the x-axis
- Reflect across the y-axis
- Shift right 2
- Shift up 1

The original graph of $y = \ln x$ looks like:



After applying the above steps, we get:



□

(5) Graph $y = \ln(-3 - x) - 2$ and answer the following questions:

a) What is the equation of the asymptote?

b) What is the domain?

c) What is the range?

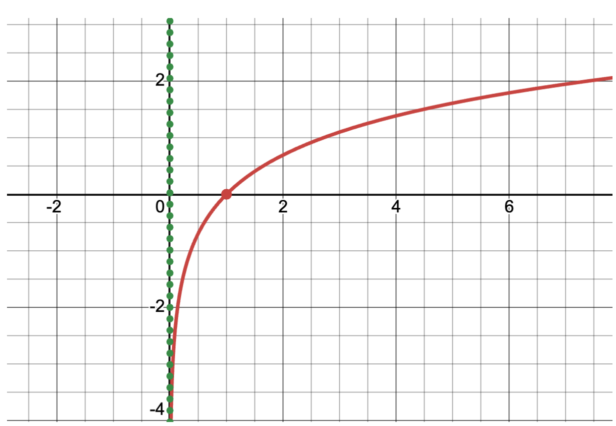
Solution

$$\begin{aligned} y = \ln(-3 - x) - 2 &\Leftrightarrow y = \ln(-x - 3) - 2 \\ &\Leftrightarrow y = \ln(-(x + 3)) - 2 \end{aligned}$$

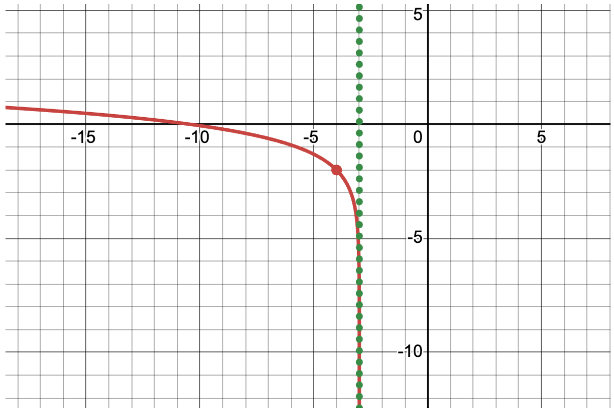
To graph we have to use the following steps:

- Reflect across the y-axis
- Shift left 3
- Shift down 2

The original graph of $y = \ln x$ looks like:



After applying the above steps, we get:



Our answers for the questions are therefore:

(a) $\boxed{x = -3}$ (since we moved left 3)

(b) $\boxed{(-\infty, -3)}$ (can use the graph or the fact that $-3 - x > 0$)

(c) $\boxed{(-\infty, \infty)}$

□

(6) Combine into a single logarithm: $\ln(x + y) - \ln(xy) - \frac{1}{2}\ln(x - 1)$

Solution

$$\begin{aligned}\ln(x + y) - \ln(xy) - \frac{1}{2}\ln(x - 1) &= \ln(x + y) - \ln(xy) - \ln(x - 1)^{1/2}\\ &= \ln(x + y) - \left(\ln(xy) + \ln(x - 1)^{1/2}\right)\\ &= \ln(x + y) - \ln\left(xy \cdot (x - 1)^{1/2}\right)\\ &= \boxed{\ln \frac{x + y}{xy\sqrt{x - 1}}}\end{aligned}$$

□

(7) Combine into a single logarithm: $2\ln\left(\frac{1}{x}\right) + 3\ln(x+1)$

Solution

$$\begin{aligned}
2\ln\left(\frac{1}{x}\right) + 3\ln(x+1) &= \ln\left(\frac{1}{x}\right)^{1/2} + \ln(x+1)^3 \\
&= \ln\left(\sqrt{\frac{1}{x}}\cdot(x+1)^3\right) \\
&= \boxed{\ln\frac{(x+1)^3}{\sqrt{x}}}
\end{aligned}$$

□

(8) Combine into a single logarithm (with no coefficient): $\ln(x^2+y^2) + 2\ln(y^3)$

Solution

$$\begin{aligned}
\ln\big(x^2+y^2\big) + 2\ln\big(y^3\big) &= \ln\big(x^2+y^2\big) + \ln\big((y^3)^2\big) \\
&= \ln\big(x^2+y^2\big) + \ln\big(y^6\big) \\
&= \ln\big((x^2+y^2)y^6\big) \\
&= \boxed{\ln\big(x^2y^6+y^8\big)}
\end{aligned}$$

□

(9) Write as a sum/difference of multiples of the simplest possible logarithms: $\log_2\sqrt{\frac{xy}{(y-x)^3}}$

Solution

$$\begin{aligned}
\log_2\sqrt{\frac{xy}{(y-x)^3}} &= \log_2\left(\frac{xy}{(y-x)^3}\right)^{1/2} \\
&= \frac{1}{2}\log_2\frac{xy}{(y-x)^3} \\
&= \frac{1}{2}\big(\log_2 xy - \log_2(y-x)^3\big) \\
&= \frac{1}{2}\big(\log_2 x + \log_2 y - \log_2(y-x)^3\big) \\
&= \frac{1}{2}(\log_2 x + \log_2 y - 3\log_2(y-x)) \\
&= \boxed{\frac{1}{2}\log_2 x + \frac{1}{2}\log_2 y - \frac{3}{2}\log_2(y-x)}
\end{aligned}$$

□

(10) Write as a sum/difference of multiples of the simplest possible logarithms: $\log_b\left(\frac{\sqrt[3]{x}}{x+1}\right)$

Solution

$$\begin{aligned}\log_b\left(\frac{\sqrt[3]{x}}{x+1}\right) &= \log_b \frac{x^{1/3}}{x+1} \\ &= \log_b x^{1/3} - \log_b(x+1) \\ &= \boxed{\frac{1}{3}\log_b x - \log(x+1)}\end{aligned}$$

□

(11) Write as sum/difference of **multiples** of the simplest possible logarithms: $\log_b\left(\frac{\sqrt[4]{2y}}{y^3+a}\right)$

Solution

$$\begin{aligned}\log_b\left(\frac{\sqrt[4]{2y}}{y^3+a}\right) &= \log_b(\sqrt[4]{2y}) - \log_b(y^3+a) \\ &= \log_b((2y)^{1/4}) - \log_b(y^3+a) \\ &= \boxed{\frac{1}{4}\log_b(2y) - \log_b(y^3+a)} \\ &\text{OR} \\ &= \boxed{\frac{1}{4}\log_b(2) + \frac{1}{4}\log_b(y) - \log_b(y^3+a)}\end{aligned}$$

□

(12) Simplify $\log_9\left(\frac{1}{3}\right)$

Solution First, we want to write $\frac{1}{3}$ as a power of 9.

$$3^2 = 9 \Rightarrow 3 = 9^{1/2} \Rightarrow \frac{1}{3} = \frac{1}{9^{1/2}} = 9^{-1/2}$$

Now we have:

$$\begin{aligned}\log_9\left(\frac{1}{3}\right) &= \log_9(9^{-1/2}) \\ &= \boxed{-\frac{1}{2}}\end{aligned}$$

□

(13) Simplify $\log_2\left(\frac{1}{\sqrt{2}}\right)$

Solution

$$\begin{aligned}\log_2\left(\frac{1}{\sqrt{2}}\right) &= \log_2\left(\frac{1}{2^{1/2}}\right)\\ &= \log_2 2^{-1/2}\\ &= \boxed{-\frac{1}{2}}\end{aligned}$$

□

(14) Simplify $\ln(e\sqrt{e})$

Solution

$$\begin{aligned}\ln(e\sqrt{e}) &= \ln\left(e\cdot e^{1/2}\right)\\ &= \ln\left(e^{1+1/2}\right)\\ &= \ln e^{3/2}\\ &= \boxed{\frac{3}{2}}\end{aligned}$$

□

(15) Express using natural logarithms: $\log_{10}(14)$

Solution

$$\log_{10}(14) = \boxed{\frac{\ln 14}{\ln 10}}$$

□

(16) Solve $e^{3x} = 8$

Solution

$$\begin{aligned}e^{3x} = 8 &\Leftrightarrow \ln\left(e^{3x}\right) = \ln 8\\ &\Leftrightarrow 3x = \ln 8\\ &\Leftrightarrow x = \frac{1}{3}\ln 8\\ &\Leftrightarrow x = \ln\left(8^{1/3}\right)\\ &\Leftrightarrow \boxed{x = \ln 2}\end{aligned}$$

□

(17) Solve $e^{2x+4} = 9$

Solution

$$\begin{aligned} e^{2x+4} = 9 &\Rightarrow \ln\left(e^{2x+4}\right) = \ln 9 \\ &\Rightarrow 2x + 4 = \ln 9 \\ &\Rightarrow 2x = \ln 9 - 4 \\ &\Rightarrow \boxed{x = \frac{\ln 9 - 4}{2}} \end{aligned}$$

□

(18) Solve $\ln(x + 1) + \ln(x + 2) = \ln(6)$

Solution

$$\begin{aligned} \ln(x+1) + \ln(x+2) = \ln 6 &\Leftrightarrow \ln((x+1)(x+2)) = \ln 6 \\ &\Rightarrow e^{\ln((x+1)(x+2))} = e^{\ln 6} \\ &\Leftrightarrow (x+1)(x+2) = 6 \\ &\Leftrightarrow x^2 + 3x + 2 = 6 \\ &\Leftrightarrow x^2 + 3x - 4 = 0 \\ &\Leftrightarrow (x+4)(x-1) = 0 \\ &\Leftrightarrow x = -4, 1 \end{aligned}$$

Check: -4 is not valid, so the answer is $\boxed{x = 1}$

□

(19) Solve $\ln(x + 3) + \ln(x + 4) = \ln(2)$

Solution

$$\begin{aligned} \ln(x+3) + \ln(x+4) = \ln(2) &\Rightarrow \ln(x+3)(x+4) = \ln 2 \\ &\Rightarrow e^{\ln(x+3)(x+4)} = e^{\ln 2} \\ &\Rightarrow (x+3)(x+4) = 2 \\ &\Rightarrow x^2 + 7x + 12 = 2 \\ &\Rightarrow x^2 + 7x + 10 = 0 \\ &\Rightarrow (x+5)(x+2) = 0 \\ &\Rightarrow x = -4, -2 \end{aligned}$$

Check: -4 is not valid, so the answer is $\boxed{x = -2}$

□

(20) Solve $\ln x = 1 + \ln(x-1)$

Solution

$$\begin{aligned}
\ln x = 1 + \ln(x-1) &\Rightarrow \ln x - \ln(x-1) = 1 \\
&\Rightarrow \ln \frac{x}{x-1} = 1 \\
&\Rightarrow e^{\ln \frac{x}{x-1}} = e^1 \\
&\Rightarrow \frac{x}{x-1} = e \\
&\Rightarrow x = e(x-1) \\
&\Rightarrow x = ex - e \\
&\Rightarrow x - ex = -e \\
&\Rightarrow x(1-e) = -e \\
&\Rightarrow \boxed{x = \frac{-e}{1-e}}
\end{aligned}$$

□

(21) Solve $\ln(2x) = 1 + \ln(x-1)$

Solution

$$\begin{aligned}
\ln 2x = 1 + \ln(x-1) &\Rightarrow \ln 2x - \ln(x-1) = 1 \\
&\Rightarrow \ln \frac{2x}{x-1} = 1 \\
&\Rightarrow e^{\ln \frac{2x}{x-1}} = e^1 \\
&\Rightarrow \frac{2x}{x-1} = e \\
&\Rightarrow 2x = e(x-1) \\
&\Rightarrow 2x = ex - e \\
&\Rightarrow 2x - ex = -e \\
&\Rightarrow x(2-e) = -e \\
&\Rightarrow \boxed{x = \frac{-e}{2-e}}
\end{aligned}$$

□

(22) Solve $\ln(x) + \ln(x-1) = \ln 2$

Solution

$$\begin{aligned}
\ln(x) + \ln(x-1) = \ln 2 &\Rightarrow \ln(x(x-1)) = \ln 2 \\
&\Rightarrow e^{\ln(x(x-1))} = e^{\ln 2} \\
&\Rightarrow x(x-1) = 2 \\
&\Rightarrow x^2 - x = 2 \\
&\Rightarrow x^2 - x - 2 = 0 \\
&\Rightarrow (x-2)(x+1) = 0 \\
&\Rightarrow x = 2, -1
\end{aligned}$$

Check: -1 is not valid, so the answer is $\boxed{x = 2}$

□

(23) Solve $(3^{2x})^5 = 10$

Solution

$$\begin{aligned}
(3^{2x})^5 = 10 &\Leftrightarrow 3^{2x\cdot 5} = 10 \\
&\Leftrightarrow 3^{10x} = 10 \\
&\Leftrightarrow \cancel{\log_3}(\cancel{3}^{10x}) = \log_3 10 \\
&\Leftrightarrow 10x = \log_3 10 \\
&\Leftrightarrow \boxed{x = \frac{\log_3 10}{10}}
\end{aligned}$$

□

(24) Solve $3^x = 2^{x+1}$

Solution

$$\begin{aligned}
3^x = 2^{x+1} &\Leftrightarrow \ln 3^x = \ln 2^{x+1} \\
&\Leftrightarrow x \ln 3 = (x+1)\ln 2 \\
&\Leftrightarrow x \ln 3 = x \ln 2 + \ln 2 \\
&\Leftrightarrow x \ln 3 - x \ln 2 = \ln 2 \\
&\Leftrightarrow x(\ln 3 - \ln 2) = \ln 2 \\
&\Leftrightarrow x\left(\ln \frac{3}{2}\right) = \ln 2 \\
&\Leftrightarrow \boxed{x = \frac{\ln 2}{\ln 3/2}}
\end{aligned}$$

□

(25) Solve $3^{2x-3} = \sqrt{3}$

Solution

$$\begin{aligned}
3^{2x-3} = \sqrt{3} &\Rightarrow 3^{2x-3} = 3^{1/2} \\
&\Rightarrow 2x - 3 = \frac{1}{2} \\
&\Rightarrow 2x = \frac{1}{2} + 3 \\
&\Rightarrow 2x = \frac{1}{2} + \frac{6}{2} \\
&\Rightarrow 2x = \frac{7}{2} \\
&\Rightarrow x = \frac{7/2}{2} \\
&\Rightarrow x = \frac{7}{2} \cdot \frac{1}{2} \\
&\Rightarrow \boxed{x = \frac{7}{4}}
\end{aligned}$$

$\square$

Trigonometric Functions and Inverse Trigonometric Functions

(1) Convert 300° to radians.

Solution

$$\begin{aligned} 300^\circ &= 300^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} \\ &= \frac{300\pi}{180} \\ &= \boxed{\frac{5\pi}{3} \text{ radians}} \end{aligned}$$

□

(2) Find the radian measure of an angle which intercepts a 10 inch arc on a circle of radius 36 inches.

Solution

$$\begin{aligned} \theta = \frac{s}{r} &\Rightarrow \theta = \frac{10}{36} \\ &\Rightarrow \theta = \frac{5}{18} \text{ radians} \end{aligned}$$

Converting to degrees:

$$\begin{aligned} \frac{5}{18} &= \frac{5}{18} \cdot \frac{180^\circ}{\pi} \\ &= \frac{5}{1} \cdot \frac{10^\circ}{\pi} \\ &= \boxed{\frac{50^\circ}{\pi}} \end{aligned}$$

□

(3) Find $\cot\left(\frac{11\pi}{3}\right)$

Solution We want to find the corresponding angle on the unit circle, so we can subtract 2π.

$$\frac{11\pi}{3} - 2\pi = \frac{11\pi}{3} - \frac{6\pi}{3} = \frac{5\pi}{3}$$

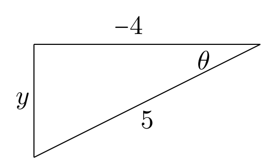
Looking at the unit circle, the coordinates of $\theta = \frac{5\pi}{3}$ are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Cotangent is $\frac{x}{y}$, so we have

$$\begin{aligned}\cot\left(\frac{11\pi}{3}\right) &= \cot\left(\frac{5\pi}{3}\right)\\ &= \frac{1/2}{-\sqrt{3}/2}\\ &= \frac{1}{2} \cdot -\frac{2}{\sqrt{3}}\\ &= \boxed{-\frac{1}{\sqrt{3}}}\end{aligned}$$

□

(4) If $\cos(\theta) = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\sin\theta$.

Solution If $\pi < \theta < \frac{3\pi}{2}$, then θ is in quadrant 3. Drawing the triangle, we have:



Using the pythagorean theorem to find y (keeping in mind that since we're in QIII, y is negative):

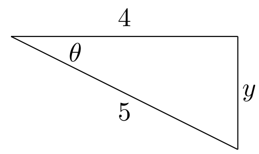
$$\begin{aligned}(-4)^2 + y^2 = 5^2 &\Rightarrow 16 + y^2 = 25\\ &\Rightarrow y^2 = 9\\ &\Rightarrow y = -3\end{aligned}$$

This give us that $\sin\theta = \frac{y}{5} = \boxed{-\frac{3}{5}}$

□

(5) If $\cos\theta = \dfrac{4}{5}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$

Solution If $\frac{3\pi}{2} < \theta < 2\pi$, then θ is in quadrant 4. Drawing the triangle, we have:



Using the pythagorean theorem to find y (keeping in mind that since we're in QIV, y is negative):

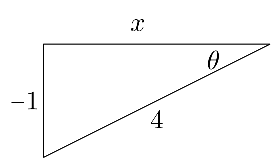
$$\begin{aligned}(4)^2 + y^2 = 5^2 &\Rightarrow 16 + y^2 = 25\\ &\Rightarrow y^2 = 9\\ &\Rightarrow y = -3\end{aligned}$$

This give us that $\sin\theta = \dfrac{y}{5} = \boxed{-\dfrac{3}{5}}$

$\square$

(6) Suppose $\sin(\theta) = -\dfrac{1}{4}$ and $\pi < \theta < \dfrac{3\pi}{2}$. Find $\cos(\theta)$

Solution If $\pi < \theta < \frac{3\pi}{2}$, then θ is in quadrant 3. Drawing the triangle, we have:



Using the pythagorean theorem to find x (keeping in mind that since we're in QIII, x is negative):

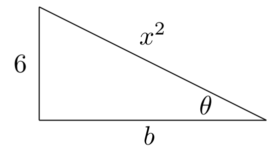
$$\begin{aligned}(-1)^2 + x^2 = 4^2 &\Rightarrow 1 + x^2 = 16\\ &\Rightarrow x^2 = 15\\ &\Rightarrow x = -\sqrt{15}\end{aligned}$$

This give us that $\cos\theta = \dfrac{x}{4} = \boxed{-\dfrac{\sqrt{15}}{4}}$

$\square$

(7) The side opposite angle θ of a right triangle is 6. The hypotenuse length is x^2, find $\tan(2\pi - \theta)$ and $\tan\left(\frac{\pi}{2} - \theta\right)$.

Solution Using the information we have about the triangle, we have:



Using the pythagorean theorem to find the missing side, we have:

$$\begin{aligned} 6^2 + b^2 = (x^2)^2 &\Rightarrow 36 + b^2 = x^4 \\ &\Rightarrow b^2 = x^4 - 36 \\ &\Rightarrow b = \sqrt{x^4 - 36} \end{aligned}$$

Using trigonometric identities, we know that

$$\tan(2\pi - \theta) = \tan(-\theta) = -\tan\theta$$

and

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

So, we have

$$\boxed{\tan(2\pi - \theta) = -\frac{6}{\sqrt{x^4 - 36}} \text{ and } \tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sqrt{x^4 - 36}}{6}}$$

□

(8) Graph $y = -2\sin\left(\frac{x}{4} - \frac{\pi}{2}\right)$ over one period. Label the x-values of the four parts.

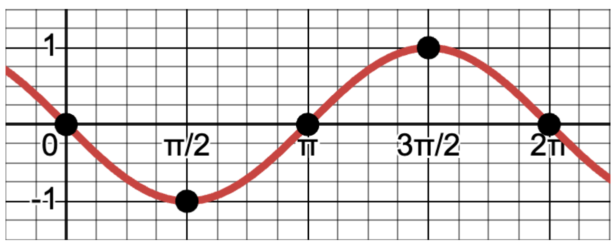
Solution When graphing $y = -A\sin(Bx - C)$, we should first factor out B. In this problem, we would get:

$$y = -2\sin\left(\frac{1}{4}\left(x - \frac{\pi/2}{1/4}\right)\right) \Rightarrow y = -2\sin\left(\frac{1}{4}(x - 2\pi)\right)$$

A gives us the amplitude, $\frac{2\pi}{B}$ gives us the period, and C gives us the phase shift.

- General shape: $y = -\sin x$
- Amplitude = 2
- Period = $P = \frac{2\pi}{1/4} = 8\pi$
- Phase shift = 2π to the right

The general graph of $-\sin x$ looks like:



We need to adjust the graph to have a period of 8π, amplitude of 2, and shifted 2π to the right.

We can take the point at the origin $(0,0)$ and shift it to the right to $(2\pi, 0)$. To ensure that we graph at least one period, we can go out 8π to the right.

$$2\pi + 8\pi = 10\pi$$

This means that our graph will start at 2π and go out to 10π on the x axis. We then need to divide that segment into 4 pieces to label our x axis accurately. You can do this either my finding midpoints, or by repeatedly adding $\frac{1}{4}$ of the period, P.

Method 1: Using Midpoints

The point between 2π and 10π is:

$$\frac{2\pi + 10\pi}{2} = \frac{12\pi}{2} = 6\pi$$

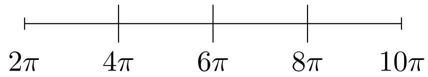
The point between 2π and 6π is:

$$\frac{2\pi + 6\pi}{2} = \frac{8\pi}{2} = 4\pi$$

The point between 6π and 10π is:

$$\frac{6\pi + 10\pi}{2} = \frac{16\pi}{2} = 8\pi$$

So now we know that our x axis is going to be labeled:



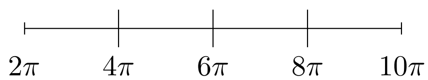
<u>Method 2: Adding $\frac{P}{4}$</u>

For our problem $P/4 = 8\pi/4 = 2\pi$, so we're going to start with our starting point at 2π (from the shift) and add 2π ($P/4$) each time to get the next tick mark.

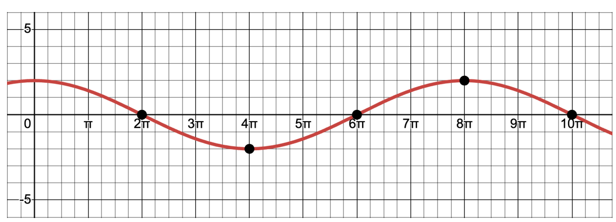
$$2\pi + 2\pi = 4\pi \qquad 4\pi + 2\pi = 6\pi$$

$$6\pi + 2\pi = 8\pi \qquad 8\pi + 2\pi = 10\pi$$

So now we know that our x axis is going to be labeled:



The graph shape of $y = -\sin x$ starts at the middle, goes down to $-A$, goes back to the middle, goes up to A, and then goes back to the middle. So, our final graph (over one period) looks like:



$\square$

(9) Graph $y = \cos(\pi x - \pi)$ **over one period**.

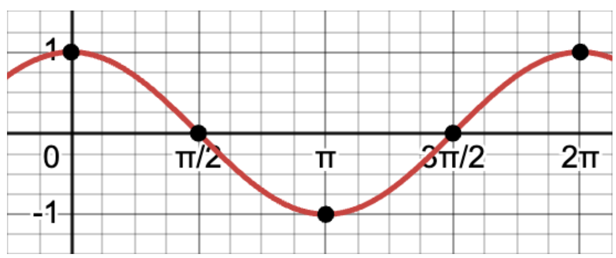
Solution The steps for this problem are very similar to the steps from Problem 8, so I'm not going to go into as much detail.

$$y = \cos(\pi x - \pi) \Rightarrow y = \cos(\pi(x - 1))$$

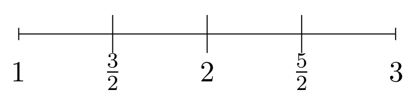
From the equation, we know:

- General shape: $y = \cos x$
- Amplitude = 1
- Period = $\frac{2\pi}{\pi} = 2$
- Phase shift = 1 to the right
- Vertical shift: none

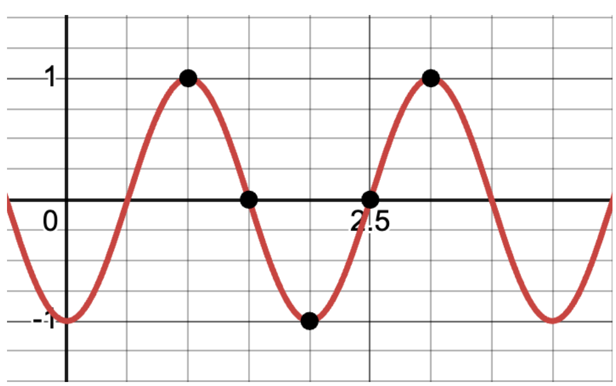
The general graph of $\cos x$ looks like:



Applying the phase shift and period, our start point is at $x = 1$, and our end point will be at $1 + 2 = 3$. Using either of the methods listed above, our x axis should look like the following:



Keeping in mind the general shape of a cosine graph, we get:



□

(10) Graph $y = \tan\left(-\frac{x}{2} + \pi\right)$ **over one period**

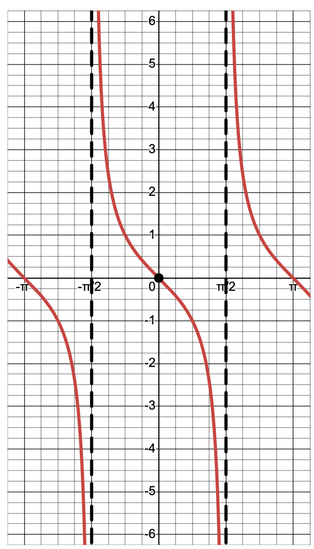
Solution The steps for this problem are very similar to the steps from Problem 8 (though slightly altered, since it's a tangent graph).

First, we need to factor out B and use the property that $\tan(-\theta) = -\tan\theta$:

$$\tan\left(-\frac{x}{2} + \pi\right) = \tan\left(-\frac{1}{2}(x - 2\pi)\right) = -\tan\left(\frac{1}{2}(x - 2\pi)\right)$$

- General shape: $y = -\tan x$
- Amplitude: none
- Period = $\frac{\pi}{1/2} = 2\pi$
- Phase shift = 2π to the right

The general graph of $-\tan x$ looks like:

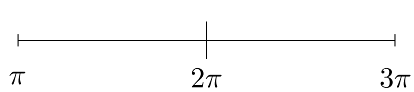


For tangent graphs, we just need to figure out the center point and the asymptotes. Applying the phase shift, our center point shifts from $(0, 0)$ to $(2\pi, 0)$. Since the distance between the asymptotes is P, we can find the new asymptotes by adding $\frac{P}{2}$ and subtracting $\frac{P}{2}$ from the center point.

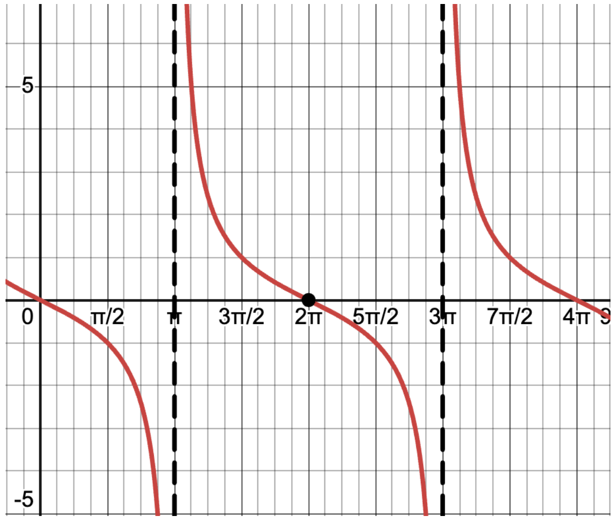
$$\frac{P}{2} = \frac{2\pi}{2} = \pi$$

$$2\pi + \pi = 3\pi \qquad -2\pi - \pi = \pi$$

So our x axis would look like



Keeping in mind that the graph has the general shape of a negative tangent graph, we get:

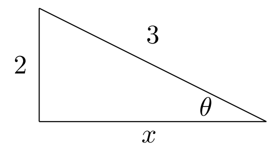


□

(11) Find $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

Solution Since $\frac{2}{3}$ isn't a value on the unit circle, we have to draw the corresponding triangle.

$$\sin^{-1}\left(\frac{2}{3}\right) = \theta \Rightarrow \sin\theta = \frac{2}{3} = \frac{\text{opposite}}{\text{hyoptenuse}}$$



Solving for the missing side using the pythagorean theorem:

$$\begin{aligned} 2^2 + x^2 = 3^2 &\Rightarrow 4 + x^2 = 9 \\ &\Rightarrow x^2 = 5 \\ &\Rightarrow x = \sqrt{5} \end{aligned}$$

Finally, we have

$$\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \boxed{\frac{2}{\sqrt{5}}}$$

□

(12) Find $\sin^{-1}\left(\sin\left(\frac{5\pi}{7}\right)\right)$

Solution Arcsine and sine only cancel each other out if the angle is within the restricted area of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since $\frac{5\pi}{7}$ is in the second quadrant, it is not in the restricted area. However, it has the same y coordinate (i.e. the same sine) as $\frac{2\pi}{7}$ in Quadrant I. This gives us:

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{7}\right)\right) = \cancel{\sin^{-1}}\left(\cancel{\sin}\left(\frac{2\pi}{7}\right)\right) = \boxed{\frac{2\pi}{7}}$$

□

Trigonometric Identities and Equations

(1) Simplify $\dfrac{\sin(\theta)}{1+\cos(\theta)} + \dfrac{1+\cos(\theta)}{\sin(\theta)}$

Solution

$$\begin{aligned}
\frac{\sin(\theta)}{1+\cos(\theta)} + \frac{1+\cos(\theta)}{\sin(\theta)} &= \frac{\sin(\theta)}{1+\cos(\theta)} \cdot \frac{\sin\theta}{\sin\theta} + \frac{1+\cos(\theta)}{\sin(\theta)} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
&= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} + \frac{(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} \\
&= \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta)(1+\cos\theta)} \\
&= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} \\
&= \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} \\
&= \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} \\
&= \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} \\
&= \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}} \\
&= \frac{2}{\sin\theta} \\
&= \boxed{2\csc\theta}
\end{aligned}$$

□

(2) Simplify $\tan(t)(\cos t + \cos(-t))$

Solution

$$\begin{aligned}
\tan(t)(\cos t + \cos(-t)) &= \tan(t)(\cos t + \cos t) \\
&= \tan t(2\cos t) \\
&= \frac{\sin t}{\cancel{\cos t}} \cdot 2\cancel{\cos t} \\
&= \boxed{2\sin t}
\end{aligned}$$

□

(3) Simplify $\dfrac{\cot^2 t(\sec^2 t - 1)}{\sec^2 t - \tan^2 t + 1}$

Solution

$$\begin{aligned} \frac{\cot^2 t(\sec^2 t - 1)}{\sec^2 t - \tan^2 t + 1} &= \frac{\cot^2 t(\tan^2 t)}{1 + \tan^2 t - \tan^2 t + 1} \\ &= \frac{\frac{1}{\cancel{\tan^2 t}} \cdot \cancel{\tan^2 t}}{2} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

□

(4) Prove $\csc t = \sin t + \cot t \cos t$

Solution

$\csc t$	$\sin t + \cot t \cos t$
	$\sin t + \frac{\cos t}{\sin t} \cdot \cos t$
	$\sin t + \frac{\cos^2 t}{\sin t}$
	$\sin t \cdot \frac{\sin t}{\sin t} + \frac{\cos^2 t}{\sin t}$
	$\frac{\sin^2 t}{\sin t} + \frac{\cos^2 t}{\sin t}$
	$\frac{\sin^2 t + \cos^2 t}{\sin t}$
	$\frac{1}{\sin t}$
	$\csc t$ ✓

□

(5) Prove $\dfrac{\sec^2 \theta - \tan^2 \theta}{1 + \cot^2 \theta} = \sin^2 \theta$

Solution

$\dfrac{\sec^2 \theta - \tan^2 \theta}{1 + \cot^2 \theta}$	$\sin^2 \theta$
$\frac{1 \cancel{+ \tan^2 \theta - \tan^2 \theta}}{\csc^2 \theta}$	
$\frac{1}{\csc^2 \theta}$	
$\sin^2 \theta$ ✓	

□

(6) Simplify $\sin(x+y)\cos x - \cos(x+y)\sin x$

Solution We are going to use the fact that

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\begin{aligned} \sin(x+y)\cos x - \cos(x+y)\sin x &= \sin((x+y)-x) \\ &= \sin(\cancel{x} + y - \cancel{x}) \\ &= \boxed{\sin y} \end{aligned}$$

□

(7) Simplify $\dfrac{\tan\left({}^{\pi}\!/_{5}\right) - \tan\left({}^{\pi}\!/_{30}\right)}{1 + \tan\left({}^{\pi}\!/_{5}\right)\tan\left({}^{\pi}\!/_{30}\right)}$

Solution We're going to use the fact that

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\begin{aligned} \frac{\tan\left({}^{\pi}\!/_{5}\right) - \tan\left({}^{\pi}\!/_{30}\right)}{1 + \tan\left({}^{\pi}\!/_{5}\right)\tan\left({}^{\pi}\!/_{30}\right)} &= \tan\left(\frac{\pi}{5} - \frac{\pi}{30}\right) \\ &= \tan\left(\frac{6\pi}{30} - \frac{\pi}{30}\right) \\ &= \tan\frac{5\pi}{30} \\ &= \tan\frac{\pi}{6} \\ &= \frac{{}^{1}\!/_{2}}{{}^{\sqrt{3}}\!/_{2}} \\ &= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} \\ &= \boxed{\frac{1}{\sqrt{3}}} \end{aligned}$$

□

(8) If $\tan(s) = 2$ and $\tan(t) = 3$, find $\tan(s+t)$.

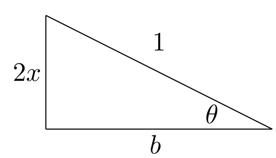
Solution

$$\begin{aligned}\tan(s+t) &= \frac{\tan s + \tan t}{1 - \tan s \tan t} \\ &= \frac{2+3}{1-(2)(3)} \\ &= \frac{5}{1-6} \\ &= \frac{5}{-5} \\ &= \boxed{-1}\end{aligned}$$

□

(9) If $\sin\theta = 2x$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin(2\theta)$

Solution If $\frac{\pi}{2} < \theta < \pi$, then θ is in Quadrant II. We can draw the following triangle:



Using the pythagorean theorem to find the missing side (keeping in mind b must be negative in QII):

$$\begin{aligned}(2x)^2 + b^2 = 1^2 &\Rightarrow 4x^2 + b^2 = 1 \\ &\Rightarrow b^2 = 1 - 4x^2 \\ &\Rightarrow b = -\sqrt{1-4x^2}\end{aligned}$$

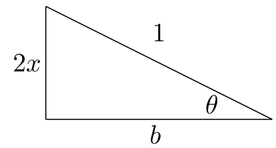
Now we have:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{2x}{1}\right)\left(\frac{-\sqrt{1-4x^2}}{1}\right) \\ &= \boxed{-4x\sqrt{1-4x^2}}\end{aligned}$$

□

(10) If $\sin\theta = 2x$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin\left(\frac{\theta}{2}\right)$

Solution If $\frac{\pi}{2} < \theta < \pi$, then θ is in Quadrant II. We can draw the following triangle:



Using the pythagorean theorem to find the missing side (keeping in mind b must be negative in QII):

$$\begin{aligned}(2x)^2 + b^2 = 1^2 &\Rightarrow 4x^2 + b^2 = 1\\ &\Rightarrow b^2 = 1 - 4x^2\\ &\Rightarrow b = -\sqrt{1-4x^2}\end{aligned}$$

Since we're looking at $\frac{\theta}{2}$, we also have to note:

$$\frac{\pi}{2} < \theta < \pi \Rightarrow \frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$$

This tells us that $\frac{\pi}{2}$ is in Quadrant I, so $\sin\left(\frac{\theta}{2}\right)$ is positive

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right) &= +\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\frac{-\sqrt{1-4x^2}}{1}}{2}}\\ &= \boxed{\sqrt{\frac{1+\sqrt{1-4x^2}}{2}}}\end{aligned}$$

□

(11) Find all solutions to $2\cos^2 x - \sin x - 1 = 0$

Solution

$$\begin{aligned}
2\cos^2 x - \sin x - 1 = 0 &\Rightarrow 2(1-\sin^2 x) - \sin x - 1 = 0\\
&\Rightarrow 2 - 2\sin^2 x - \sin x - 1 = 0\\
&\Rightarrow -2\sin^2 x - \sin x + 1 = 0\\
&\Rightarrow 2\sin^2 x + \sin x - 1 = 0\\
&\Rightarrow (2\sin x - 1)(\sin x + 1) = 0\\
&\Rightarrow 2\sin x - 1 = 0 \text{ or } \sin x + 1 = 0\\
&\Rightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -1\\
&\Rightarrow \boxed{x = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n, \frac{3\pi}{2} + 2\pi n}
\end{aligned}$$

□

(12) Find all solutions to $2\cos^2 x - 5\cos x = -2$

Solution

$$\begin{aligned}
2\cos^2 x - 5\cos x = -2 &\Rightarrow 2\cos^2 x - 5\cos x + 2 = 0\\
&\Rightarrow (2\cos x - 1)(\cos x - 2) = 0\\
&\Rightarrow 2\cos x - 1 = 0 \text{ or } \cos x - 2 = 0\\
&\Rightarrow \cos x = \frac{1}{2} \text{ or } \cos x = 2\\
&\Rightarrow \boxed{x = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n}
\end{aligned}$$

□

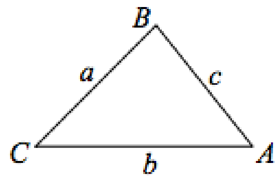
(13) Find all solutions to $\cos^2 x + \sin x + 1 = 0$

Solution

$$\begin{aligned}
\cos^2 x + \sin x + 1 = 0 &\Rightarrow 1 - \sin^2 x + \sin x + 1 = 0\\
&\Rightarrow -\sin^2 x + \sin x + 2 = 0\\
&\Rightarrow \sin^2 x - \sin x - 2 = 0\\
&\Rightarrow (\sin x - 2)(\sin x + 1) = 0\\
&\Rightarrow \sin x = 2 \text{ or } \sin x = -1\\
&\Rightarrow \boxed{x = \frac{3\pi}{2} + 2\pi n}
\end{aligned}$$

□

(14) Consider the following triangle



(a) If $a = 3$, $b = 1$, and $c = 3$, find $\angle B$.

Solution

$$\begin{aligned} b^2 = a^2 + c^2 - 2ac\cos B &\Rightarrow 1^2 = 3^2 + 3^2 - 2(3)(3)\cos B \\ &\Rightarrow 1 = 9 + 9 - 18\cos B \\ &\Rightarrow 1 = 18 - 18\cos B \\ &\Rightarrow -17 = -18\cos B \\ &\Rightarrow \boxed{\cos B = \frac{17}{18}} \end{aligned}$$

□

(b) If $a = 3$, $\angle C = 60^\circ$, and $\angle A = 50^\circ$, find b.

Solution

$$60^\circ + 50^\circ + \angle B = 180^\circ \Rightarrow \angle B = 70^\circ$$

$$\begin{aligned} \frac{b}{\sin B} = \frac{a}{\sin A} &\Rightarrow \frac{b}{\sin 70^\circ} = \frac{3}{\sin 50^\circ} \\ &\Rightarrow \boxed{b = \frac{3\sin 70^\circ}{\sin 50^\circ}} \end{aligned}$$

□

(c) If $a = 3$, $b = 2$, and $\angle B = 30^\circ$, solve for $\angle C$.

Solution

$$\begin{aligned} \frac{\sin A}{a} = \frac{\sin B}{b} &\Rightarrow \frac{\sin A}{3} = \frac{\sin 30^\circ}{2} \\ &\Rightarrow \sin A = \frac{3\sin 30^\circ}{2} \\ &\Rightarrow \sin A = \frac{3 \cdot \frac{1}{2}}{2} \\ &\Rightarrow \sin A = \frac{3}{4} \\ &\Rightarrow A = \sin^{-1}\left(\frac{3}{4}\right) \end{aligned}$$

We know:

$$\angle A + \angle B + \angle C = 180^\circ \Rightarrow \angle C = 180^\circ - 30^\circ - \sin^{-1}\left(\frac{3}{4}\right) = \boxed{150^\circ - \sin^{-1}\left(\frac{3}{4}\right)}$$

□

(d) If $a = 3$, $c = 2$, and $\angle C = 60^\circ$, find b.

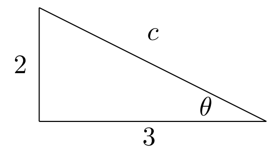
Solution

$$\begin{aligned} c^2 = a^2 + b^2 - 2ab\cos C &\Rightarrow 2^2 = 3^2 + b^2 - 2(3)(b)\cos 60^\circ \\ &\Rightarrow 4 = 9 + b^2 - 6b\left(\frac{1}{2}\right) \\ &\Rightarrow 4 = 9 + b^2 - 3b \\ &\Rightarrow b^2 - 3b + 5 = 0 \\ &\Rightarrow b = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(5)}}{2(1)} \\ &\Rightarrow b = \frac{3 \pm \sqrt{-11}}{2} \\ &\Rightarrow \boxed{\text{No (real) solutions}} \end{aligned}$$

□

(15) The two legs of a right triangle are 2 and 3. What is the cosine of the smallest angle? (Hint: the smallest angle is opposite the smallest side)

Solution We have a right triangle and we want to look at the angle across from 2.



Using the pythagorean theorem:

$$\begin{aligned} 2^2 + 3^2 = c^2 &\Rightarrow 4 + 9 = c^2 \\ &\Rightarrow 13 = c^2 \\ &\Rightarrow c = \sqrt{13} \\ &\Rightarrow \boxed{\cos\theta = \frac{3}{\sqrt{3}}} \end{aligned}$$

□

Polar Equations, Trigonometric Forms of Complex Numbers, and Systems of Equations

(1) Convert the polar equation to a rectangular equation: $2\sin\theta - 3\cos\theta = r$

Solution

$$x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r} \quad \text{and} \quad y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$$

$$\begin{aligned} 2\sin\theta - 3\cos\theta = r &\Rightarrow 2\cdot\frac{y}{r} - 3\cdot\frac{x}{r} = r \\ &\Rightarrow 2y - 3x = r^2 \\ &\Rightarrow \boxed{2y - 3x = x^2 + y^2} \end{aligned}$$

□

(2) Convert the rectangular equation to a polar equation: $y = x^2$

Solution

$$y = x^2 \Rightarrow \boxed{r\sin\theta = (r\cos\theta)^2}$$

□

(3) Convert $\left(2, \frac{\pi}{4}\right)$ to rectangular coordinates. Convert (π, π) to polar coordinates.

Solution

$$\begin{aligned} r = 2, \theta = \frac{\pi}{4} &\Rightarrow x = 2\cos\frac{\pi}{4} \text{ and } y = 2\sin\frac{\pi}{4} \\ &\Rightarrow x = 2\cdot\frac{\sqrt{2}}{2} \text{ and } y = 2\cdot\frac{\sqrt{2}}{2} \\ &\Rightarrow x = \sqrt{2} \text{ and } y = \sqrt{2} \end{aligned}$$

So, $\left(2, \frac{\pi}{4}\right)$ is $\boxed{(\sqrt{2}, \sqrt{2})}$ in rectangular coordinates.

To convert $(\pi.\pi)$ to polar coordinates, we have to note that the point lies in QI.

$$\begin{aligned} x = \pi, y = \pi &\Rightarrow r^2 = \pi^2 + \pi^2 \text{ and } \tan\theta = \frac{\pi}{\pi} \\ &\Rightarrow r^2 = 2\pi^2 \text{ and } \tan\theta = 1 \\ &\Rightarrow r = \sqrt{2}\pi \text{ and } \theta = \frac{\pi}{4} \end{aligned}$$

So, (π, π) is $\boxed{\left(\pi\sqrt{2}, \frac{\pi}{4}\right)}$ in polar coordinates.

□

(4) Convert the polar equation to a rectangular equation: $r^2 = \cos(2\theta)$

Solution

$$\begin{aligned} r^2 = \cos(2\theta) &\Rightarrow r^2 = \cos^2\theta - \sin^2\theta \\ &\Rightarrow x^2 + y^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\ &\Rightarrow x^2 + y^2 = \frac{x^2}{r^2} - \frac{y^2}{r^2} \\ &\Rightarrow x^2 + y^2 = \frac{x^2 - y^2}{r^2} \\ &\Rightarrow \boxed{x^2 + y^2 = \frac{x^2 - y^2}{x^2 + y^2}} \end{aligned}$$

□

(5) Convert the rectangular equation to a polar equation: $x^2 - y^2 = 1$

Solution

$$x^2 - y^2 = 1 \Rightarrow \boxed{(r\cos\theta)^2 - (r\sin\theta)^2 = 1}$$

□

(6) Write $\cos 150° + i\sin 150°$ in standard form

Solution Note: 150° is on the unit circle.

$$\cos 150° + i\sin 150° = -\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right) = \boxed{-\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

□

(7) Write $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ in standard form

Solution

$$\begin{aligned} 5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) &\Rightarrow 5\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \\ &\Rightarrow \boxed{-\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i} \end{aligned}$$

□

(8) Write $-\sqrt{2} - i\sqrt{2}$ in trigonometric form

Solution To convert to trigonometric form, we have to note that the point $-2 - i\sqrt{2}$ is in QIII.

$$a = -\sqrt{2}, b = -\sqrt{2} \Rightarrow r = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} \text{ and } \tan\theta = \frac{-\sqrt{2}}{-\sqrt{2}}$$

$$\Rightarrow r = \sqrt{2+2} \text{ and } \tan\theta = 1$$

$$\Rightarrow r = 2 \text{ and } \theta = \frac{5\pi}{4}$$

So, we have $-2 - i\sqrt{2} = \boxed{2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)}$

□

(9) Write $\sqrt{3} + i$ in trigonometric form

Solution To convert to trigonometric form, we have to note that the point $\sqrt{3} + i$ is in QI.

$$a = \sqrt{3}, b = 1 \Rightarrow r = \sqrt{(\sqrt{3})^2 + 1^2} \text{ and } \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow r = 2 \text{ and } \theta = \frac{\pi}{6}$$

So, we have $\sqrt{3} + i = \boxed{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$

□

(10) Write $-1 + i\sqrt{3}$ in trigonometric form

Solution To convert to trigonometric form, we have to note that the point $-1 + i\sqrt{3}$ is in QII.

$$a = -1, b = \sqrt{3} \Rightarrow r = \sqrt{(-1)^2 + (\sqrt{3})^2} \text{ and } \tan\theta = \frac{\sqrt{3}}{-1}$$

$$\Rightarrow r = \sqrt{4} \text{ and } \tan\theta = -\sqrt{3}$$

$$\Rightarrow r = 2 \text{ and } \theta = \frac{2\pi}{3}$$

So, we have $-1 + i\sqrt{3} = \boxed{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$

□

(11) Compute the operation and leave your answer in trigonometric form:
$6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

Solution

$$\begin{aligned}
6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) &= \frac{6}{1}\left(\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right) \\
&= 6\left(\sin\frac{2\pi}{4}+i\sin\frac{2\pi}{4}\right) \\
&= \boxed{6\left(\sin\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}
\end{aligned}$$

□

(12) Compute the operation, then convert your answer to standard form:
$5(\cos 90^\circ + i\sin 90^\circ)\div 2(\cos 30^\circ + i\sin 30^\circ)$

Solution

$$\begin{aligned}
5(\cos 90^\circ + i\sin 90^\circ)\div 2(\cos 30^\circ + i\sin 30^\circ) &= \frac{5}{2}\left(\cos(90^\circ-30^\circ)+i\sin(90^\circ-30^\circ)\right) \\
&= \frac{5}{2}(\cos 60^\circ + i\sin 60^\circ) \\
&= \frac{5}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\
&= \boxed{\frac{5}{4}+\frac{5\sqrt{3}}{4}i}
\end{aligned}$$

□

(13) Compute the operation and leave your answer in trigonometric form:
$9(\cos 45^\circ + i\sin 45^\circ)\cdot 3(\cos 15^\circ + i\sin 15^\circ)$

Solution

$$\begin{aligned}
9(\cos 45^\circ + i\sin 45^\circ)\cdot 3(\cos 15^\circ + i\sin 15^\circ) &= 27(\cos(45^\circ+15^\circ)+i\sin(45^\circ+15^\circ)) \\
&= \boxed{27(\cos(60^\circ)+i\sin 60^\circ)}
\end{aligned}$$

□

(14) Compute the operation, then convert your answer to standard form:
$\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \cdot \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Solution

$$\begin{aligned}
\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \cdot \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) &= \cos\left(\frac{\pi}{12} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{12} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{12} - \frac{3\pi}{12}\right) + i\sin\left(\frac{\pi}{12} - \frac{3\pi}{12}\right) \\
&= \cos\left(-\frac{2\pi}{12}\right) + i\sin\left(-\frac{2\pi}{12}\right) \\
&= \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) \\
&= \frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right) \\
&= \boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}i}
\end{aligned}$$

□

(15) Compute the operation, then convert your answer to standard form:
$\left(3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^4$

Solution

$$\begin{aligned}
\left(3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^4 &= 3^4\left(\cos\frac{\pi}{4}\cdot 4 + i\sin\frac{\pi}{4}\cdot 4\right) \\
&= 81(\cos\pi + i\sin\pi) \\
&= 81(-1 + i(0)) \\
&= \boxed{-81}
\end{aligned}$$

□

(16) Compute the operation, then convert your answer to standard form: $(2(\cos 90^\circ + i\sin 90^\circ))^5$

Solution

$$\begin{aligned}(2(\cos 90^\circ + i\sin 90^\circ))^5 &= 2^5(\cos(5\cdot 90)^\circ + i\sin(5\cdot 90)^\circ)\\ &= 32(\cos 450^\circ + i\sin 450^\circ)\\ &= 32(\cos(450-360)^\circ + i\sin(450-360)^\circ)\\ &= 32(\cos 90^\circ + i\sin 90^\circ)\\ &= 32(0+i(1))\\ &= \boxed{32i}\end{aligned}$$

□

(17) Solve: $\begin{array}{l}4x = 3y+17\\ x = -2y-4\end{array}$

Solution Note: I'm doing this via elimination, but you can do it via substitution.

$4x = 3y+17$
$x = -2y-4$

Multiply the bottom by −4:

$4x = 3y+17$
$-4x = 8y+16$

Add together and solve for y:

$$0 = 11y+33 \Rightarrow 11y = -33 \Rightarrow y = -3$$

Solve for x:

$$x = -2(-3) - 4 = 6 - 4 = 2$$

So the solution is $\boxed{(2,-3)}$

□

(18) Solve: $\begin{aligned} 6x + 3y &= 30 \\ 2x + 3y &= 18 \end{aligned}$

Solution Note: I'm doing this via elimination, but you can do it via substitution.

$$\begin{aligned} 6x + 3y &= 30 \\ 2x + 3y &= 18 \end{aligned}$$

Subtract them and solve for x:

$$4x = 12 \Rightarrow x = 3$$

Plug in and solve for y:

$$2(3) + 3y = 18 \Rightarrow 6 + 3y = 18 \Rightarrow 3y = 12 \Rightarrow y = 4$$

So the solution is $\boxed{(3, 4)}$

□

(19) Solve: $\begin{aligned} x - 3y &= 4 \\ -2x + 6y &= 1 \end{aligned}$

Solution Note: I'm doing this via elimination, but you can do it via substitution.

$$\begin{aligned} x - 3y &= 4 \\ -2x + 6y &= 1 \end{aligned}$$

Multiply the top by 2:

$$\begin{aligned} 2x - 6y &= 8 \\ -2x + 6y &= 1 \end{aligned}$$

Add together:

$$0 = 9$$

which is false, so the answer is $\boxed{\text{Inconsistent or No Solutions}}$

□

(20) Solve: $\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned}$

Solution Note: There are many ways to solve this. I'm showing one way.

$2x + y - 2z = -1$
$3x - 3y - z = 5$
$x - 2y + 3z = 6$

Multiply the last equation by -2 and use it with the first equation:

$2x + y - 2z = -1$
$-2x + 4y - 6z = -12$

Add together:

$5y - 8z = -13$

Multiply the original last equation by -3 and use it with the second equation:

$3x - 3y - z = 5$
$-3x + 6y - 9z = -18$

Add together:

$3y - 10z = -13$

Now we have a new system of equations with y and z:

$5y - 8z = -13$
$3y - 10z = -13$

Multiply the top by -3 and the bottom by 5:

$-15y + 24z = 39$
$15y - 50z = -65$

Add together and solve for z:

$$-26z = -26 \Rightarrow z = 1$$

Plug this in and solve for y:

$$15y - 50(1) = -65 \Rightarrow 15y = -15 \Rightarrow y = -1$$

Plug both these in and solve for x:

$$x - 2(-1) + 3(1) = 6 \Rightarrow x + 2 + 3 = 6 \Rightarrow x = 1$$

So our solution is $\boxed{(1, -1, 1)}$

$\square$

(21) Solve: $\begin{array}{l} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{array}$

Solution Note: There are many ways to solve this. I'm showing one way.

$x + y + z = 2$
$y - 3z = 1$
$2x + y + 5z = 0$

Multiply the top by −2 and use it with the bottom equation:

$-2x - 2y - 2z = -4$
$2x + y + 5z = 0$

Add together:

$-y + 3z = -4$

Now we have a new system of equations with y and z:

$y - 3z = 1$
$-y + 3z = -4$

Add together:

$$0 = -3$$

which is false, so the answer is $\boxed{\text{Inconsistent or No Solutions}}$

□

(22) Solve:
$$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

Solution Note: There are many ways to solve this. I'm showing one way.

$2x + y - 3z = 0$
$4x + 2y - 6z = 0$
$x - y + z = 0$

Multiply the bottom by -2 and use it with the top equation:

$2x + y - 3z = 0$
$-2x + 2y - 2z = 0$

Add together:

$3y - 5z = 0$

Multiply the bottom by -4 and use it with the middle equation:

$4x + 2y - 6z = 0$
$-4x + 4y - 4z = 0$

Add together:

$6y - 10z = 0$

Now we have a new system of equations with y and z:

$3y - 5z = 0$
$6y - 10z = 0$

Multiply the top by -2:

$-6y + 10z = 0$
$6y - 10z = 0$

Add together:

$$0 = 0$$

Which is true, so we know the system is dependent. To find the form that each solution takes, we solve for y to get:

$$6y - 10z = 0 \Rightarrow 6y = 10z \Rightarrow y = \frac{10z}{6} = \frac{5z}{3} = \frac{5}{3}z$$

Plug this in and solve for x

$$\begin{aligned} x - y + z = 0 &\Rightarrow x - \frac{5}{3}z + z = 0 \\ &\Rightarrow x - \frac{5}{3}z + \frac{3}{3}z = 0 \\ &\Rightarrow x - \frac{2}{3}z = 0 \\ &\Rightarrow x = \frac{2}{3}z \end{aligned}$$

So our solutions look like: $\boxed{\left(\frac{2}{3}z, \frac{5}{3}z, z\right)}$

□