Supplementary Midterm 2 Practice Problems

Note: There are practice problems on MathXL. These problems are meant to supplement the ones that are already online.

Graphing Problems: Graph the following functions

(1) $y = 2\sin\left(2x - \frac{\pi}{2}\right)$

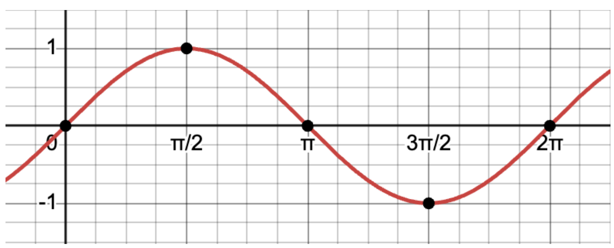
Solution When graphing $y = A\sin(Bx - C)$, we should first factor out B. In this problem, we would get:

$$y = 2\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$$

A gives us the amplitude, $\frac{2\pi}{B}$ gives us the period, and C gives us the phase shift.

- General shape: $y = \sin x$
- Amplitude = 2
- Period = $P = \frac{\cancel{2}\pi}{\cancel{2}} = \pi$
- Phase shift = $\frac{\pi}{4}$ to the right

The general graph of $\sin x$ looks like:



We need to adjust the graph to have a period of π, amplitude of 2, and shifted $\frac{\pi}{4}$ to the right.

We can take the point at the origin $(0,0)$ and shift it to the right to $\left(\frac{\pi}{4}, 0\right)$. To ensure that we graph at least one period, we can go out π to the right.

$$\frac{\pi}{4} + \pi = \frac{\pi}{4} + \frac{4\pi}{4} = \frac{5\pi}{4}$$

This means that our graph will start at $\frac{\pi}{4}$ and go out to $\frac{5\pi}{4}$ on the x axis. We then need to divide that segment into 4 pieces to label our x axis accurately. You can do this either my finding midpoints, or by repeatedly adding $\frac{1}{4}$ of the period, P.

<u>Method 1: Using Midpoints</u>

The point between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ is:

$$\frac{\frac{\pi}{4} + \frac{5\pi}{4}}{2} = \frac{\frac{6\pi}{4}}{2} = \frac{6\pi}{4} \cdot \frac{1}{2} = \frac{6\pi}{8} = \frac{3\pi}{4}$$

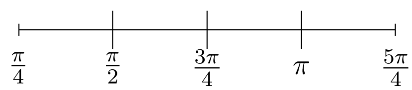
The point between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is:

$$\frac{\frac{\pi}{4}+\frac{3\pi}{4}}{2}=\frac{\frac{4\pi}{4}}{2}=\frac{4\pi}{4}\cdot\frac{1}{2}=\frac{4\pi}{8}=\frac{\pi}{2}$$

The point between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ is:

$$\frac{\frac{3\pi}{4}+\frac{5\pi}{4}}{2}=\frac{\frac{8\pi}{4}}{2}=\frac{8\pi}{4}\cdot\frac{1}{2}=\frac{8\pi}{8}=\pi$$

So now we know that our x axis is going to be labeled:



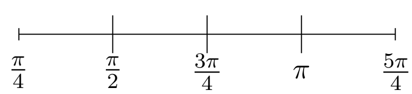
Method 2: Adding $\frac{P}{4}$

For our problem $P/4 = \pi/4$, so we're going to start with our starting point at $\pi/4$ (from the shift) and add $\pi/4$ ($P/4$) each time to get the next tick mark.

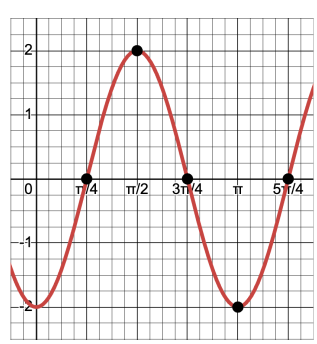
$$\frac{\pi}{4}+\frac{\pi}{4}=\frac{2\pi}{4}=\frac{\pi}{2} \qquad \frac{\pi}{2}+\frac{\pi}{4}=\frac{2\pi}{4}+\frac{\pi}{4}=\frac{3\pi}{4}$$

$$\frac{3\pi}{4}+\frac{\pi}{4}=\frac{4\pi}{4}=\pi \qquad \pi+\frac{\pi}{4}=\frac{4\pi}{4}+\frac{\pi}{4}=\frac{5\pi}{4}$$

So now we know that our x axis is going to be labeled:



The graph shape of $y = \sin x$ starts at the middle, goes up to A, goes back to the middle, goes down to $-A$, and then goes back to the middle. So, our final graph (over one period) looks like:



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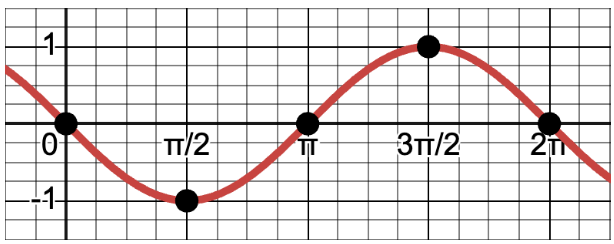
(2) $y = -3\sin\left(\frac{1}{2}(x - \pi)\right) + 1$

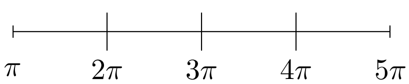
Solution The steps for this problem are very similar to the steps from Problem 1, so I'm not going to go into as much detail. B is already factored out, so we don't need to do that step. From the equation, we know:

- General shape: $y = -\sin x$
- Amplitude = 3
- Period = $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot 2 = 4\pi$
- Phase shift = π to the right
- Vertical shift = 1 up

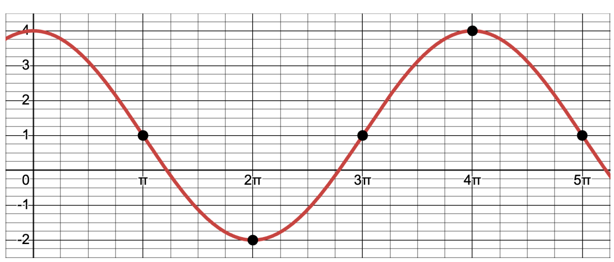
The general graph of $-\sin x$ looks like:



Applying the phase shift and period, our start point is at $x = \pi$, and our end point will be at $\pi + 4\pi = 5\pi$. Using either of the methods listed above, our x axis should look like the following:



Keeping in mind that the graph needs to be shifted up 1, that the amplitude is 3, and the general shape of a negative sine graph, we get:



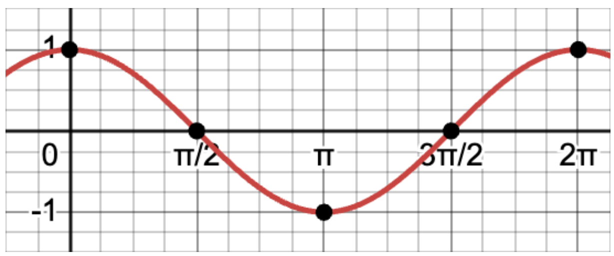
$\square$

(3) $y = \cos\left(\frac{\pi}{2}(x-1)\right) - 2$

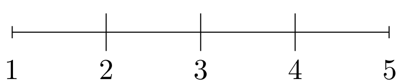
Solution The steps for this problem are very similar to the steps from Problem 1, so I'm not going to go into as much detail. B is already factored out, so we don't need to do that step. From the equation, we know:

- General shape: $y = \cos x$
- Amplitude = 1
- Period = $\frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$
- Phase shift = 1 to the right
- Vertical shift = 2 down

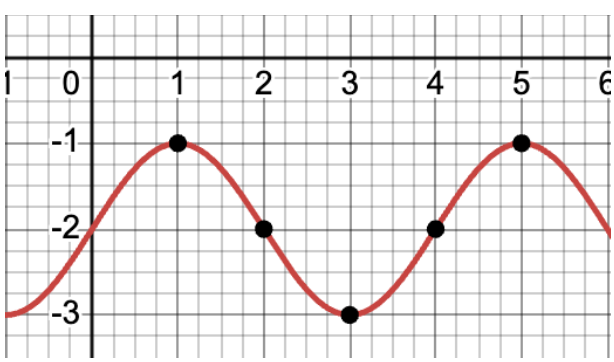
The general graph of $\cos x$ looks like:



Applying the phase shift and period, our start point is at $x = 1$, and our end point will be at $1 + 4 = 5$. Using either of the methods listed above, our x axis should look like the following:



Keeping in mind that the graph needs to be shifted down 2 and the general shape of a negative sine graph, we get:



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(4) $f(x) = -2\tan(2x + \pi)$

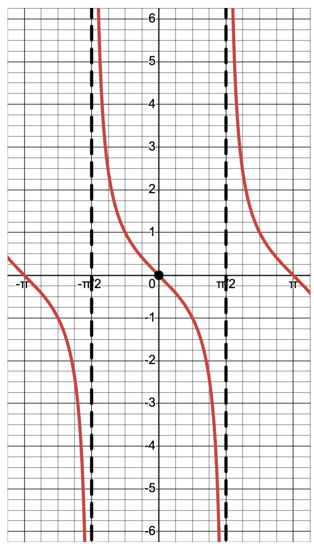
Solution The steps for this problem are very similar to the steps from Problem 1 (though slightly altered, since it's a tangent graph).

First, we need to factor out B:

$$-2\tan(2x+\pi) = -2\tan\left(2\left(x+\frac{\pi}{2}\right)\right)$$

- General shape: $y = -\tan x$
- Amplitude: none
- Period = $\frac{\pi}{2}$
- Phase shift = $\frac{\pi}{2}$ to the left

The general graph of $-\tan x$ looks like:

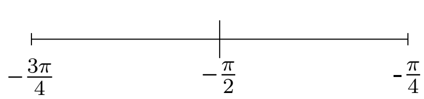


For tangent graphs, we just need to figure out the center point and the asymptotes. Applying the phase shift, our center point shifts from $(0,0)$ to $\left(-\frac{\pi}{2}, 0\right)$. Since the distance between the asymptotes is P, we can find the new asymptotes by adding $\frac{P}{2}$ and subtracting $\frac{P}{2}$ from the center point.

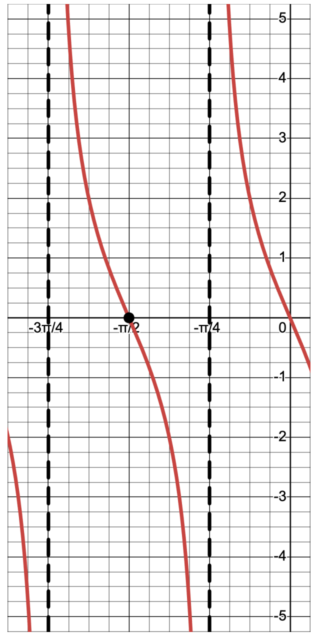
$$\frac{P}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$$

$$\frac{-\pi}{2} + \frac{\pi}{4} = -\frac{2\pi}{4} + \frac{\pi}{4} = -\frac{\pi}{4} \qquad -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{2\pi}{4} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

So our x axis would look like



Keeping in mind that the graph has the general shape of a negative tangent graph, we get:



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Proofs: Verify the following

Note: These solutions only give one possible way to prove the problem. There are many other ways you could do these.

(1) $1 + \sec^2 x \sin^2 x = \sec^2 x$

Solution

$1 + \sec^2 x \sin^2 x$	$\sec^2 x$
$1 + \frac{1}{\cos^2 x} \cdot \sin^2 x$	
$1 + \frac{\sin^2 x}{\cos^2 x}$	
$1 + \tan^2 x$	(Using the quotient identity $\tan x = \frac{\sin x}{\cos x}$)
$\sec^2 x$ ✓	(Using the pythagorean identity $1 + \tan^2 x = \sec^2 x$)

□

(2) $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

Solution

$\tan^2 x$	$\sec^2 x - \sin^2 x - \cos^2 x$
	$\sec^2 x - (\sin^2 x + \cos^2 x)$
(Using the pythagorean identity $\sin^2 x + \cos^2 x = 1$)	$= \sec^2 x - 1$
(Using the pythagorean identity $1 + \tan^2 x = \sec^2 x$)	$= \tan^2 x$ ✓

□

(3) $\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} = 2\csc x$

Solution

$\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}$	$2\csc x$
$\frac{\sin x}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} + \frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}$	
$\frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)} + \frac{\sin x(1-\cos x)}{(1-\cos x)(1+\cos x)}$	
$\frac{\sin x(1+\cos x)+\sin x(1-\cos x)}{(1-\cos x)(1+\cos x)}$	
$\frac{\sin x+\cancel{\sin x\cos x}+\sin x-\cancel{\sin x\cos x}}{1-\cancel{\cos x}+\cancel{\cos x}-\cos^2 x}$	
$\frac{2\sin x}{1-\cos^2 x}$	
$\frac{2\sin x}{\sin^2 x}$	(Using the pythagorean identity $\sin^2 x + \cos^2 x = 1$)
$\frac{2}{\sin x}$	
$2\csc x$ ✓	(Using the reciprocal identity $\frac{1}{\sin x} = \csc x$)

□

(4) $\frac{\cot^2 x}{\csc x-1} = \frac{1+\sin x}{\sin x}$

Solution

$\frac{\cot^2 x}{\csc x-1}$	$\frac{1+\sin x}{\sin x}$
$\frac{\csc^2 x-1}{\csc x-1}$	(Using the pythagorean identity $\cot^2 x + 1 = \csc^2 x$)
$\frac{\cancel{(\csc x-1)}(\csc x+1)}{\cancel{\csc x-1}}$	(Using $a^2 - b^2 = (a-b)(a+b)$)
$\csc x + 1$	
$\frac{1}{\sin x} + 1$	(Using the reciprocal $\csc x = \frac{1}{\sin x}$)
$\frac{1}{\sin x} + \frac{\sin x}{\sin x}$	
$\frac{1+\sin x}{\sin x}$ ✓	

□

(5) $\sin(a+b)+\sin(a-b)=2\sin a\cos b$

Solution

$\sin(a+b)+\sin(a-b)$	$2\sin a\cos b$
$\sin a\cos b+\cancel{\sin b\cos a}+\sin a\cos b-\cancel{\sin b\cos a}$	(Using sum/difference identity for sine)
$2\sin a\cos b$ ✓	

□

(6) $\frac{1-\tan\theta}{1+\tan\theta}=\frac{1-\sin 2\theta}{\cos 2\theta}$

Solution

$\frac{1-\tan\theta}{1+\tan\theta}$	$\frac{1-\sin 2\theta}{\cos 2\theta}$
$\frac{1-\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin\theta}{\cos\theta}}$	(Using quotient identity $\tan x=\frac{\sin x}{\cos x}$)
$\frac{1-\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin\theta}{\cos\theta}}\cdot\frac{\cos\theta}{\cos\theta}$	
$\frac{\cos\theta(1-\frac{\sin\theta}{\cos\theta})}{\cos\theta(1+\frac{\sin\theta}{\cos\theta})}$	
$\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$	
$\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\cdot\frac{\cos\theta-\sin\theta}{\cos\theta-\sin\theta}$	
$\frac{\cos^2\theta-\sin\theta\cos\theta-\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta-\cancel{\sin\theta\cos\theta}+\cancel{\sin\theta\cos\theta}-\sin^2\theta}$	
$\frac{\cos^2\theta-2\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$	
$\frac{\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta}{\cos 2\theta}$	(Using double-angle identity $\cos 2x=\cos^2 x-\sin^2 x$)
$\frac{1-2\sin\theta\cos\theta}{\cos 2\theta}$	(Using pythagorean identity $\sin^2 x+\cos^2 x=1$)
$\frac{1-\sin 2\theta}{\cos 2\theta}$ ✓	(Using double-angle identity $\sin 2x=2\sin x\cos x$)

□