Math 215 Class Lectures

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Contents

Lecture 1: Review of Algebra and Linear Functions

Algebra Review

This class requires you to be comfortable with algebraic manipulations. The following is a review of some types of problems you should be able to solve.

1. Simplify the following expression: $\frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{4}-\frac{1}{3}\right)+\frac{1}{6}\right]$

   Solution Remember the order of operations (PEMDAS) and do each operation in the correct order:

$$\begin{aligned}\frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{4}-\frac{1}{3}\right)+\frac{1}{6}\right] &= \frac{1}{3}\left[\frac{1}{2}\left(\frac{3}{12}-\frac{4}{12}\right)+\frac{1}{6}\right] \\ &= \frac{1}{3}\left[\frac{1}{2}\left(\frac{-1}{12}\right)+\frac{1}{6}\right] \\ &= \frac{1}{3}\left[\frac{-1}{24}+\frac{1}{6}\right] \\ &= \frac{1}{3}\left[\frac{-1}{24}+\frac{4}{24}\right] \\ &= \frac{1}{\cancel{3}}\left[\frac{\cancel{3}}{24}\right] \\ &= \frac{1}{24}\end{aligned}$$

□

2. Factor $7a^3 + 14a^2$

   Solution There is a 7 and an a^2 in common in both so we can factor them out to get

$$7a^3 + 14a^2 = 7a^2(a+2)$$

□

3. Expand $(9k+q)(2k-q)$

   Solution

$$\begin{aligned}(9k+q)(2k-q) &= 18k^2 - 9kq + 2kq - q^2 \\ &= 18k^2 - 7kq - q^2\end{aligned}$$

□

4. Expand $(3x-9)(2x+1)$

Solution FOIL the expression to get

$$\begin{aligned}(3x-9)(2x+1) &= (3x)(2x)+(3x)(1)+(-9)(2x)+(-9)(1)\\ &= 6x^2+3x-18x-9\\ &= 6x^2-15x-9.\end{aligned}$$

□

5. Simplify: $\dfrac{t^2-4t-21}{t+3}$

Solution Factor the top of the fraction to get

$$t^2-4t-21=(t-7)(t+3)$$

so the fraction is now

$$\frac{(t-7)\cancel{(t+3)}}{\cancel{(t+3)}}=t-7, t\neq 3$$

When cancelling, don't forget that t cannot equal anything where the denominator is zero so we need that extra condition for it to be the same expression.

□

6. Combine into a single fraction: $\dfrac{12}{x^2+2x}+\dfrac{4}{x}+\dfrac{2}{x+2}$

Solution We need to find a common denominator. The first step to help us do this is to completely factor all of the denominators to get:

$$\frac{12}{x(x+2)}+\frac{4}{x}+\frac{2}{x+2}$$

Now it is easy to see that the common denominator is $x(x+2)$ so we get

$$\begin{aligned}\frac{12}{x(x+2)} + \frac{4(x+2)}{x(x+2)} + \frac{2x}{x(x+2)} &= \frac{12 + 4(x+2) + 2x}{x(x+2)} \\ &= \frac{12 + 4x + 8 + 2x}{x(x+2)} \\ &= \frac{6x + 20}{x(x+2)} \\ &= \frac{2(3x+10)}{x(x+2)}\end{aligned}$$

□

7. Combine into a single fraction: $\dfrac{5x+2}{x^2-1} + \dfrac{3}{x^2+x} - \dfrac{1}{x^2-x}$

Solution

$$\frac{5x+2}{x^2-1} + \frac{3}{x^2+x} - \frac{1}{x^2-x} = \frac{5x+2}{(x-1)(x+1)} + \frac{3}{x(x+1)} - \frac{1}{x(x-1)}$$
$$= \frac{x(5x+2)}{x(x-1)(x+1)} + \frac{3(x-1)}{x(x-1)(x+1)} - \frac{(x+1)}{x(x-1)(x+1)}$$

□

8. Simplify $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$

Solution To simplify this, you must remember the method of multiplying by a conjugate. The conjugate of $\sqrt{x}-\sqrt{x+1}$ is $\sqrt{x}+\sqrt{x+1}$. Using this method we get:

$$\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} = \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}}$$
$$= \frac{x+\sqrt{x(x+1)}+\sqrt{x(x+1)}+x+1}{x-(x+1)}$$
$$= \frac{2x+2\sqrt{x(x+1)}+1}{-1}$$
$$= -2x-2\sqrt{x(x+1)}-1$$

□

LECTURE 1: REVIEW OF ALGEBRA AND LINEAR FUNCTIONS

Functions

Before we get into the specifics of functions, we will review function inputs and linear functions (aka lines).

We typically use f, g, or h to denote a function and often define functions by providing a formula. For example, $f(x)$ denotes the value of f and x.

Examples:

1. If $f(x) = 2x + 5$, find the following:
   (a) $f(1)$
   (b) $f(2)$
   (c) $f(-1.5)$

   Solution

   (a)

$$\begin{aligned} f(1) &= 2(1) + 5 \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

   (b)

$$\begin{aligned} f(2) &= 2(2) + 5 \\ &= 4 + 5 \\ &= 9 \end{aligned}$$

   (c)

$$\begin{aligned} f(1.5) &= 2(1.5) + 5 \\ &= -3 + 5 \\ &= 2 \end{aligned}$$

□

2. If $f(x) = 1 - x^2$, find the following:

   (a) $f(1)$
   (b) $f(k)$
   (c) $f(1+h)$
   (d) $f(x+h) - f(x)$

Solution It might be helpful to put parenthesis around where you see the variable x and rewrite $f(x)$ as $f(x) = 1 - (x)^2$. Then where you see x replace it by what you see in the parenthesis in $f()$

(a)

$$\begin{aligned} f(1) &= 1 - (1)^2 \\ &= 0 \end{aligned}$$

(b)

$$\begin{aligned} f(k) &= 1 - (k)^2 \\ &= 1 - k^2 \end{aligned}$$

(c)

$$\begin{aligned} f(1+h) &= 1 - (1+h)^2 \\ &= 1 - (1 + 2h + h^2) \\ &= 1 - 1 - 2h - h^2 \\ &= -2h - h^2 \end{aligned}$$

(d)

$$\begin{aligned} f(x+h) &= 1 - (x+h)^2 \\ &= 1 - (x^2 + 2xh + h^2) \\ &= 1 - x^2 - 2xh - h^2 \end{aligned}$$

$$\begin{aligned} f(x+h) - f(x) &= (1 - x^2 - 2xh - h^2) - (1 - x^2) \\ &= \cancel{1} - \cancel{x^2} - 2xh - h^2 - \cancel{1} + \cancel{x^2} \\ &= 2xh - h^2 \end{aligned}$$

□

Recall: A function defined by the formula $f(x) = mx + b$, where m and b are real numbers, is called a *linear function* or a *line*.

A line is determined by any two distinct points on the line. It is also determined by its slope and a single point on the line.

Given two points, (x_1, y_1) and (x_2, y_2), on a line, the slope is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given a point, (x_0, y_0), and slope, m, the *point-slope equation* of the line is

$$y - y_0 = m(x - x_0)$$

This is because slope is constant so

$$\frac{y - y_0}{x - x_0} = m$$

Given a y-intercept, b, and a slope, m, the *slope-intercept equation* of the line is

$$y = mx + b$$

(Note: you can get this from using the point $(0, b)$ in point-slope form).

Examples:

1. Find the slope of the line through:
   (a) $(2, 3)$ and $(-5, -6)$
   (b) $(3, 0)$ and $(0, 5)$

   Solution

   (a) Thus we have

$$\begin{aligned} m &= \frac{-6-3}{-5-2} \\ &= \frac{-9}{-7} \\ &= \frac{9}{7} \end{aligned}$$

   (b) $m = \dfrac{5-0}{0-3} = \dfrac{5}{-3} = -\dfrac{5}{3}$

$\square$

2. Find the equation of a line given its properties:

   (a) Through the point $(2,2)$ and slope -1
   (b) Through the point $(0,5)$ and slope 0
   (c) y-intercept 3 and slope 2
   (d) Through the points $(1,2)$ and $(2,5)$

   Solution

   (a) Thus we have $y-2=-1(x-2)$ or $y=-x+4$
   (b) $y-5=0(x-0) \Leftrightarrow y=5$
   (c) So we have

   $$y=2x+3$$

   (d)

   $$m=\frac{5-2}{2-1}=3$$

   Thus a point-slope equation of the line is $y-2=3(x-1)$

□

3. Find the slope and y-intercept of each line:

   (a) $3y = -2x + 1$

   (b) $6 - 2y = 10x - 2$

Solution We want to write it in the slope-intercept form $y = mx + b$

(a)

$$3y = -2x + 1 \Leftrightarrow y = \frac{-2x+1}{3}$$
$$\Leftrightarrow y = \frac{-2}{3}x + \frac{1}{3}$$

So the slope is $-\frac{2}{3}$ and the y-intercept is $\frac{1}{3}$

(b)

$$6 - 2y = 10x - 2 \Leftrightarrow -2y = 10x - 8$$
$$\Leftrightarrow y = \frac{10x-8}{-2}$$
$$\Leftrightarrow y = \frac{10}{-2}x + \frac{8}{2}$$
$$\Leftrightarrow y = -5x + 4$$

So the slope is -5 and the y-intercept is 4

□

Recall:

- Not every line is a graph of a function! Vertical lines have infinite slope and are represented by $x = k$ where k is a real number.
- Every line can be represented in *standard form* $Ax + By + C = 0$, where at least one of A and B is not zero.
- Two lines are *parallel* if they have the same slope. Two lines are *perpendicular* if the product of their slopes is -1 (or one is vertical and the other is horizontal)

4. Write the equation of the line through the point $(3, -3)$ that is:

   (a) Parallel to the line $y = 2x + 5$

   (b) Perpendicular to the line $y = 2x + 5$

Solution

(a) The slope will be 2 since they are parallel so we can use point slope form to get:

$$y + 3 = 2(x - 3) \text{ or } y = 2x - 9$$

(b) The slope will be $-1/2$ so we get

$$\begin{aligned} y + 3 = -\frac{1}{2}(x - 3) \Leftrightarrow y &= -\frac{1}{2}x + \frac{3}{2} - 3 \\ &= -\frac{1}{2}x + \frac{3}{2} - \frac{6}{2} \\ &= -\frac{1}{2}x - \frac{3}{2} \end{aligned}$$

□

Lecture 2: Properties of Functions

Definition: A function takes elements in the *domain* and maps them to elements in the *target space*. The set of elements in the target space that the function maps to is called the *range* (so the range could be smaller than the target space itself.) When the target space of a function is the same as its range, the function is *onto*. The domain and target space we'll be considering in the course are sets of real numbers.

Note: If you have trouble seeing the domain and range, you can always graph the function and look at the x and y values that it attains.

Examples:

1. Find the domain of $f(x) = \sqrt{1 - x^2}$

   Solution You cannot take the square root of a negative number so we need

   $$1 - x^2 \geq 0$$

   which is equivalent to $-1 \leq x \leq 1$ i.e. $[-1, 1]$ (If you don't understand why this is without the graph, review the key-number method for solving inequalities from Math 140)

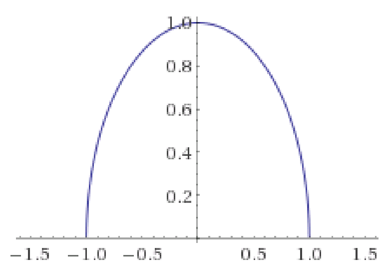


Figure 1: Graph of $\sqrt{1 - x^2}$

□

2. Find the domain and range of $f(x) = x^2 - 2$

   Solution The function has no restrictions on x so the domain is all real numbers, written $\mathbb{R}$. Further we know that $x^2 \geq 0$ for all x so $f(x) \geq -2$. Writing this in interval notation we get that the range of f is $[-2, \infty)$

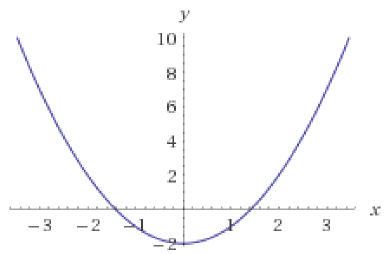


Figure 2: Graph of $x^2 - 2$

□

3. Find the domain and range of $g(x) = \dfrac{2}{x-1}$

   Solution Since you cannot divide by 0, we have that $x \neq 1$ so our domain is $(-\infty, 1) \cup (1, \infty)$. Now, can you think of any value that $f(x)$ cannot take? $f(x)$ will never equal 0 as the numerator is always 2 thus our range is $(-\infty, 0) \cup (0, \infty)$

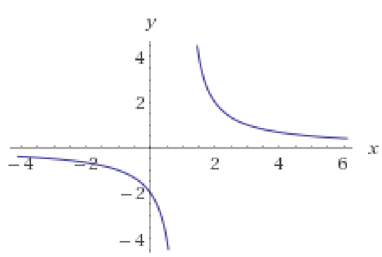


Figure 3: Graph of $\dfrac{2}{x-1}$

□

<u>Definition:</u> A function in which the formula depends on the input value is called a *piecewise function.*

<u>Example:</u>

$$f(x) = \begin{cases} 1 & x < -1, x > 1 \\ x + 2 & -1 \leq x < 0 \\ 2 - x & 0 \leq x \leq 1 \end{cases}$$

Definition: Given two functions f and g, if f maps x to a value y and g maps y to a value z then we define the *composition* of g and f as:

$$(g \circ f)(x) = g(f(x))$$

Note: $g \circ f$ is not the same as $f \circ g$
Examples:

1. If $f(x) = \dfrac{x-3}{2}$ and $g(x) = \sqrt{x}$, find $g \circ f$ and $f \circ g$

   Solution

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(\frac{x-3}{2}\right)\\ &= \sqrt{\frac{x-2}{2}}\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(\sqrt{x})\\ &= \frac{\sqrt{x}-3}{2}\end{aligned}$$

□

2. if $f(x) = x + 5$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$

   Solution

$$\begin{aligned}(f \circ g)(x) &= g(f(x))\\ &= g(x+5)\\ &= (x+5)^2\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= f(g(x))\\ &= f(x^2)\\ &= x^2 + 5\end{aligned}$$

□

3. Write $(x+2)^5$ as a composition of functions, $g \circ f$

Solution Find the inner function. In our case the inner function is $x+2$, so let

$$f(x) = x + 2$$

To find g, replace $x+2$ with x in the original function to get

$$g(x) = x^5$$

□

4. Find the domain of the function $f(x) = \sqrt{\dfrac{x^2+1}{x^2-3x+2}}$

Solution The expression inside the square root cannot be negative and the denominator of the fraction cannot be zero. This we need

$$\frac{x^2+1}{x^2-3x+2} \geq 0 \text{ and } x^2 - 3x + 2 \neq 0$$

Since x^2+1 is always positive, this is the same as

$$x^2 - 3x + 2 > 0 \text{ and } x^2 - 3x + 2 \neq 0 \Leftrightarrow x^2 - 3x + 2 > 0$$

Using the key-number method we get $x \in (-\infty, 1) \bigcup (2, \infty)$

□

Recall: A function assigns an element of the domain to *exactly one* element of the target space. Graphically, this is the same as passing the vertical line test. This is equivalent to saying if you plug in an x value, will you only get one y value.

5. Which of the following are functions:

   (a) $x^2 + y^2 = 1$

   (b) $xy + y + x = 1, x \neq 1$

   (c) $x = \dfrac{y}{y+1}$

Solution

(a)

$$\begin{aligned} x^2 + y^2 = 1 &\Leftrightarrow y^2 = 1 - x^2 \\ &\Leftrightarrow y = \pm\sqrt{1 - x^2} \end{aligned}$$

which gives two y values thus it is not a function. Another way to see this is by recognizing this is the equation for a circle, so it clearly won't pass the vertical line test.

(b)

$$\begin{aligned} xy + y + x = 1 &\Leftrightarrow xy + y = 1 - x \\ &\Leftrightarrow y(x+1) = 1 - x \\ &\Leftrightarrow y = \frac{1-x}{x+1} \end{aligned}$$

so it is a function (Note $x \neq -1$ so the denominator is never 0)

(c)

$$\begin{aligned} x = \frac{y}{y+1} &\Leftrightarrow x(y+1) = y \\ &\Leftrightarrow xy + x = y \\ &\Leftrightarrow xy - y = x \\ &\Leftrightarrow y(x-1) = x \\ &\Leftrightarrow y = \frac{x}{x-1} \end{aligned}$$

so it is a function.

$\square$

Lecture 3: Polynomial and Rational Functions

Definition: A *polynomial* is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. The *degree* of a polynomial is the highest degree (i.e. the highest exponent) of x that appears. The *leading term* of a polynomial is the term with the largest degree of x. The *leading coefficient* is the coefficient of the leading term. A *quadratic* is a polynomial of degree two. A *cubic* is a polynomial of degree three.

Note: A polynomial of degree one is a linear function

Reflections and Translations:

Given the graph of a function f:

- $-f(x)$ is the graph of f reflected across the x-axis
- $f(-x)$ is the graph of f reflected across the y-axis
- $f(x+h)$ is the graph of f translated left h units if h is positive or right h units if h is negative
- $f(x)+h$ is the graph of f translated up h units if h is positive or down h units if h is negative

Thus if a function g can be written as $af(b(x \pm c)) \pm d$, then you can graph g using the graph of f by using the above steps in a left to right order.

Note: To graph most quadratics, you must know how to complete the square.

Completing the Square:

$$\begin{aligned} ax^2 + abx + c &= a(x^2 + bx) + c \\ &= a\left(x^2 + bx + \left(\frac{b}{2}\right)^2\right) + c - a\left(\frac{b}{2}\right)^2 \\ &= a\left(x + \frac{b}{2}\right)^2 - a\left(\frac{b}{2}\right)^2 \end{aligned}$$

Example: Graph $1-\sqrt{1-(x+2)^2}$ using the graph of $\sqrt{1-x^2}$

Solution First notice that $\sqrt{1-x^2}$ is the top half of a circle of radius 1 centered at the origin so the graph looks like:

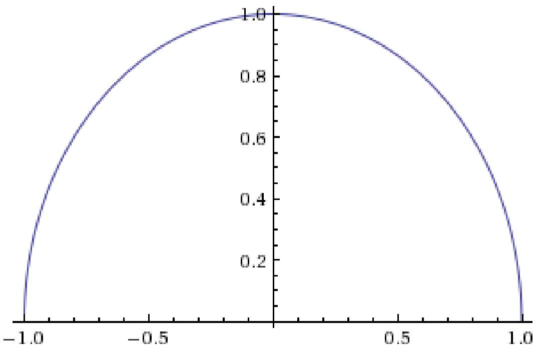


Figure 4

Next rewrite the function in the form given above to get $-\sqrt{1-(x+2)^2}+1$. Thus if $f(x)=\sqrt{1-x^2}$ then we have $-\sqrt{1-(x+2)^2}+1=-f(x+2)+1$. Moving from left to right we have the following graphing steps:

Step 1. Reflect across the x-axis

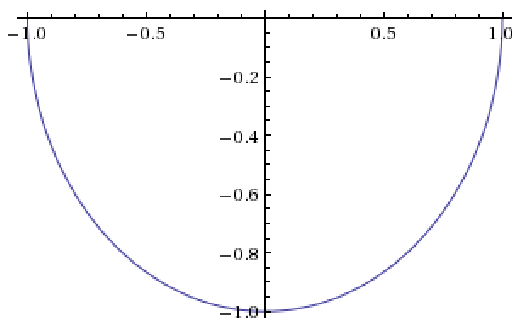


Figure 5

Step 2. Translate left 2 units (note that the y-axis occurs at $x=-3$ in Figure 5 and Figure 6)

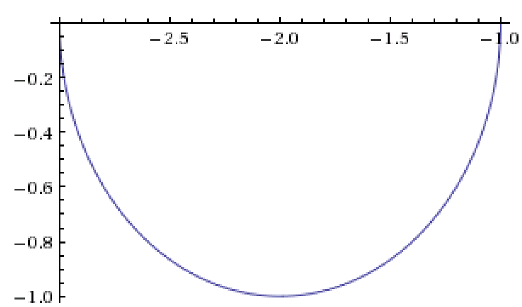


Figure 6

Step 3. Translate up 1 unit

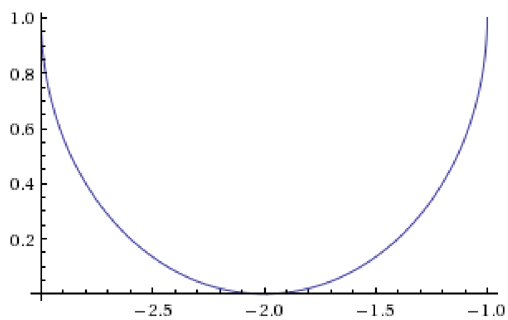


Figure 7

Figure 7 gives us the final graph.

□

Recall: In the graph of a polynomial of even degree, both ends go up or down. In the graph of a polynomial of odd degree, one end goes up and the other goes down.

Definition: A *rational function* is a function of the form $f(x) = \dfrac{g(x)}{h(x)}$ where g and h are polynomials.

Note: The domain of f does not include points where $h(x) = 0$

Graphing a Rational Function:

(i) Factor the numerator and denominator as much as possible

(ii) Find the x-intercept by setting y equal to 0 (i.e. find when the numerator is 0) and find the y-intercept by setting x equal to 0

(iii) Find the leading term of the numerator and the leading term of the denominator and form a new fraction with these two individual leading terms (this fraction is the leading term of the rational function). The form of the leading term indicates the horizontal asymptote:

$$\text{Leading Term} = a \Rightarrow \text{Horizontal Asymptote: } y = a$$

$$\text{Leading Term} = \frac{a}{bx^n} \Rightarrow \text{Horizontal Asymptote: } y = 0$$

$$\text{Leading Term} = \text{Other} \Rightarrow \text{Horizontal Asymptote: None}$$

(iv) Find the vertical asymptotes by finding what values of x make the denominator 0

(v) Plot the key x-values found in steps (ii) and (iv) and test the intervals to see if the function is positive or negative on each interval.

(vi) Sketch the graph

Examples:

1. Graph $f(x) = \dfrac{2x^3 - 8x^2}{2x - 6}$

   Solution

   (i)
   $$f(x) = \frac{2x^2(x-4)}{2(x-3)} = \frac{x^2(x-4)}{x-3}$$

   (ii)
   $$x^2(x-4) = 0 \Leftrightarrow x = 0, x = 4$$

   so our x-intercepts are at $(0,0)$ and $(4,0)$
   $$f(0) = \frac{0}{6}$$

   so our y-intercept is at $(0,0)$

   (iii) The leading term of the numerator is $2x^3$ and the leading term of the denominator is $2x$ giving us
   $$\text{Leading Term} = \frac{2x^3}{2x} = x^2$$

   Thus we have no horizontal asymptote.

   (iv) $x - 3 = 0 \Leftrightarrow x = 3$ so our vertical asymptote is $x = 3$

   (v) Our key values are $x = 0$, $x = 3$, and $x = 4$. Testing the intervals we have
   $$f(x) > 0 \text{ on } (-\infty, 0), (0,3), (4, \infty)$$
   $$f(x) < 0 \text{ on } (3,4)$$

   (vi)

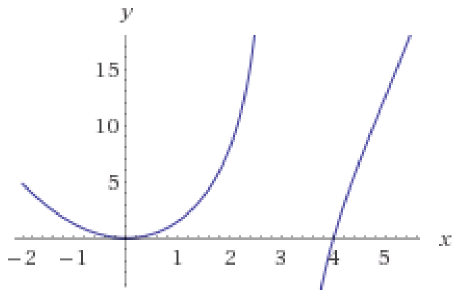


Figure 8

□

2. Find the asymptotes of the function $f(x) = \dfrac{4x^2 - 16x}{2x^2 - 11x + 12}$

Solution

$$\begin{aligned} f(x) &= \frac{4x^2 - 16x}{2x^2 - 11x + 12} \\ &= \frac{4x(x-4)}{(2x-3)(x-4)} \\ &= \frac{4x}{2x-3} \end{aligned}$$

The vertical asymptotes occur when

$$2x - 3 = 0 \Leftrightarrow x = \frac{3}{2}$$

There is no vertical asymptote at $x = 4$ because of the cancellation. In the graph, this means that there is a hole $x = 4$

Since the degree of the numerator and denominator is the same, the horizontal asymptotes is

$$y = \frac{4}{2} = 2$$

□

Lecture 4: Exponential and Logarithmic Functions

Definition: An *exponential function* is a function of the form $f(x) = b^x$ where $b > 0$, $b \neq 1$. A particularly useful number is called *Euler's number* and is denoted by e.

Rules of Exponents:

- $(b^n)^m = b^{nm}$
- $b^n b^m = b^{n+m}$
- $(\frac{a}{b})^n = \frac{a^n}{b^n}$
- $\frac{b^n}{b^m} = b^{n-m}$
- $(ab)^n = a^n b^n$
- $a^x = a^y \Leftrightarrow x = y$

Examples:

1. Solve for x: $3^{x-1} = 2^{x+4}$

   Solution

$$
\begin{aligned}
3^{x-1} = 2^{x+4} &\Leftrightarrow \ln 3^{x-1} = \ln 2^{x+4} \\
&\Leftrightarrow (x-1)\ln 3 = (x+4)\ln 2 \\
&\Leftrightarrow x\ln 3 - \ln 3 = x\ln 2 + 4\ln 2 \\
&\Leftrightarrow x\ln 3 - x\ln 2 = \ln 3 + 4\ln 2 \\
&\Leftrightarrow x(\ln 3 - \ln 2) = \ln 3 + 4\ln 2 \\
&\Leftrightarrow x\ln\frac{3}{2} = \ln 3 + \ln 2^4 \\
&\Leftrightarrow x = \frac{\ln(3(16))}{\ln 3/2} \\
&\Leftrightarrow x = \frac{\ln 48}{\ln 3/2}
\end{aligned}
$$

□

2. Solve for x: $2^{x^2-4x} = \left(\frac{1}{16}\right)^{x-4}$

Solution

$$\begin{aligned} 2^{x^2-4x} = \left(\frac{1}{16}\right)^{x-4} &\Leftrightarrow 2^{x^2-4x} = \left(2^{-4}\right)^{x-4} \\ &\Leftrightarrow 2^{x^2-4x} = 2^{-4(x-4)} \\ &\Leftrightarrow x^2 - 4x = -4(x-4) \\ &\Leftrightarrow x^2 - 4x = -4x + 16 \\ &\Leftrightarrow x^2 = 16 \\ &\Leftrightarrow x = \pm 4 \end{aligned}$$

□

Definition: For $b > 0$, $b \neq 1$, and $x > 0$, $y = \log_b x$, is defined to be such that $b^y = x$ (i.e. $\log_b x$ is the inverse of b^x). Similarly, the *natural logarithm* $\ln x$ is the inverse of e^x since $\ln x = \log_e x$.

Properties of Logarithms (For $b > 0$, $b \neq 1$, and $x > 0$):

- $\log_b a^x = x \log_b a$
- $\log_b ac = \log_b a + \log_b c$
- $\log_b \frac{a}{c} = \log_b a - \log_b c$
- $\log_b b^x = x$
- $\ln e^x = x$
- $b^{\log_b x} = x$
- $e^{\ln x} = x$
- $\log_a x = \frac{\log_b x}{\log_b a}$

Definition: A function f is called a *logarithmic function* with base a if $f(x) = \log_a x$ for $a > 0$, $a \neq 1$

Examples:

1. Simplify $\log_9 \frac{1}{3}$

   Solution $3^2 = 9 \Leftrightarrow 3 = 9^{1/2} \Leftrightarrow \frac{1}{3} = \frac{1}{9^{1/2}} = 9^{-1/2}$. Thus $\log_9 \frac{1}{3} = \log_9 9^{-1/2} = -\frac{1}{2}$

   □

2. Simplify $\log_9 \sqrt{3}$

   Solution Remember $\sqrt{3} = 3^{1/2}$ so $3 = 9^{1/2} \Leftrightarrow \sqrt{3} = (9^{1/2})^{1/2} = 9^{1/4}$. Thus $\log_9 \sqrt{3} = \log_9 9^{1/4} = \frac{1}{4}$

   □

3. Write $\ln\left(\frac{9\sqrt[3]{5}}{\sqrt[4]{3}}\right)$ as a sum or difference of multiples of logarithms.

   Solution

$$\begin{aligned}\ln\left(\frac{9\sqrt[3]{5}}{\sqrt[4]{3}}\right) &= \ln\left(\frac{3^3}{3^{1/4}}\right) + \ln\left(5^{1/3}\right)\\ &= \frac{11}{12}\ln 3 + \frac{1}{3}\ln 5\end{aligned}$$

   □

4. Solve for x: $\ln(1-x) - \ln 6 = -\ln 2 - x$

   Solution

$$\begin{aligned}\ln(1-x) - \ln 6 = -\ln(2-x) &\Leftrightarrow \ln(1-x) + \ln(2-x) = \ln 6\\ &\Leftrightarrow \ln\left[(1-x)(2-x)\right] = \ln 6\\ &\Rightarrow (1-x)(2-x) = 6\\ &\Leftrightarrow 2 - 3x + x^2 = 6\\ &\Leftrightarrow x^2 - 3x - 4 = 0\\ &\Leftrightarrow (x-4)(x+1) = 0\\ &\Leftrightarrow x = 4, x = -1\end{aligned}$$

   Now the last thing we have to check is if the values are defined. Notice that when $x = 4$, $\ln(1-x)$ is undefined (you cannot take the log of anything ≤ 0) so our answer is $x = -1$.

   □

5. Solve for x: $\log_3(x^2+17) - \log_3(x+5) = 1$

   Solution

$$\begin{aligned} \log_3(x^2+17) - \log_3(x+5) = 1 &\Leftrightarrow \log_3\left(\frac{x^2+17}{x+5}\right) = 1 \\ &\Rightarrow \frac{x^2+17}{x+5} = 3 \\ &\Leftrightarrow x^2+17 = 3(x+5) \\ &\Leftrightarrow x^2+17 = 3x+15 \\ &\Leftrightarrow x^2-3x+2 = 0 \\ &\Leftrightarrow x = 1,2 \end{aligned}$$

   These are both valid so they are our answers.

□

6. Solve for x: $\ln x + \ln(3x) = -1$

   Solution

$$\begin{aligned} \ln x + \ln(3x) = -1 &\Leftrightarrow \ln(3x^2) = -1 \\ &\Rightarrow 3x^2 = \frac{1}{e} \\ &\Leftrightarrow x^2 = \frac{1}{3e} \\ &\Leftrightarrow x = \pm\frac{1}{\sqrt{3e}} \end{aligned}$$

   $-\frac{1}{\sqrt{3e}}$ isn't valid so the answer is $\frac{1}{\sqrt{3e}}$

□

Compound Interest/Exponential Growth and Decay

If an amount is compounded m times per year, the amount can be expressed with the equation

$$P(t) = P(1+\frac{r}{m})^{tm}$$

If the amount is compounded continuously, the equation becomes

$$P(t) = Pe^{rt}$$

Note: If you took Math 140, the equation you're used to may be

$$N(t) = N_0 e^{kt}$$

Compounded continuously is the same as the function for exponential growth and decay.

Examples:

1. A bank account starts with \$4000 and is compounded continuously. 6 years later it has \$6000. How much money is in the account after 12 years? When will there be \$8000 in the account?

Solution We are given that $P = 4000$ and $P(6) = 6000$. Putting this in our equation we have

$$\begin{aligned} 6000 = 4000e^{6r} \Leftrightarrow e^{6r} &= \frac{3}{2} \\ \Leftrightarrow 6r &= \ln\frac{3}{2} \\ \Leftrightarrow r &= \frac{1}{6}\ln\frac{3}{2} \end{aligned}$$

So our general equation is

$$P(t) = 4000e^{1/6\ln(3/2)}$$

So after 12 years we have

$$\begin{aligned} P(12) &= 4000e^{12/6\ln(3/2)} \\ &= 4000e^{2\ln(3/2)} \\ &= 4000e^{\ln(3/2)^2} \\ &= 4000\left(\frac{9}{4}\right) \\ &= 9000 \end{aligned}$$

To find when the amount is \$8000, we have to solve for t in the equation

$$8000 = 4000e^{\frac{t}{6}\ln\frac{3}{2}}$$

We have

$$\begin{aligned} 8000 = 4000e^{t/6\ln(3/2)} &\Leftrightarrow 2 = e^{t/6\ln(3/2)} \\ &\Leftrightarrow \ln 2 = \frac{t}{6}\ln\frac{3}{2} \\ &\Leftrightarrow \frac{t}{6} = \frac{\ln 2}{\ln 3/2} \\ &\Leftrightarrow t = \frac{6\ln 2}{\ln 3/2}\text{years} \end{aligned}$$

□

2. Since 1960, the growth in world population (in millions) is well approximated by the following exponential function: $f(t) = 3100e^{0.0166t}$, where t is the number of years since 1960. The world population in 2000 was about 6115 million. How well does the above function approximate it? When should we expect the population to reach 10 billion?

Solution $t = 2000 - 1960 = 40$ so we have

$$f(40) = 3100e^{4(0.166)} \approx 6022$$

So the function approximates it pretty well. For the second question we need to find t such that

$$3100e^{0.0166t} = 10000$$

So we have

$$\begin{aligned} 3100e^{0.0166t} = 10000 &\Leftrightarrow e^{0.0166t} = \frac{10000}{3100} \\ &\Leftrightarrow e^{0.0166t} = \frac{100}{31} \\ &\Leftrightarrow 0.0166t = \ln\left(\frac{100}{31}\right) \\ &\Leftrightarrow t = \frac{\ln\left(100/31\right)}{0.0166} \approx 70.5 \end{aligned}$$

this means that we should expect the population to reach 10 billion around June of 2030.

□

Lecture 5: Trigonometric Functions

Trigonometry deals with the study of triangles. It is typically regarded as functions assigning values to angles. The following is a review of facts that you should have learned in previous courses.

Definition: An *angle* is a geometric figure formed by two rays sharing a common endpoint. We often consider angles formed by the positive x-axis and rays starting at the origin and passing through points on a circle of radius r.

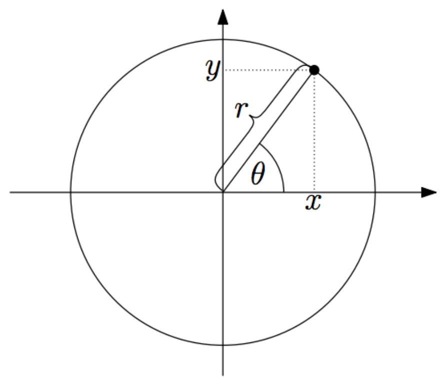


Figure 9

The size of an angle corresponds to the length of the circular arch bounded by its rays using the following formula:

$$\theta = \frac{\text{length of arc}}{\text{radius}}$$

where θ is in radian measure.

Definitions:

Use the acronym SOHCAHTOA to remember what parts of the right triangle determine the function's value. The acronym indicates the following relations:

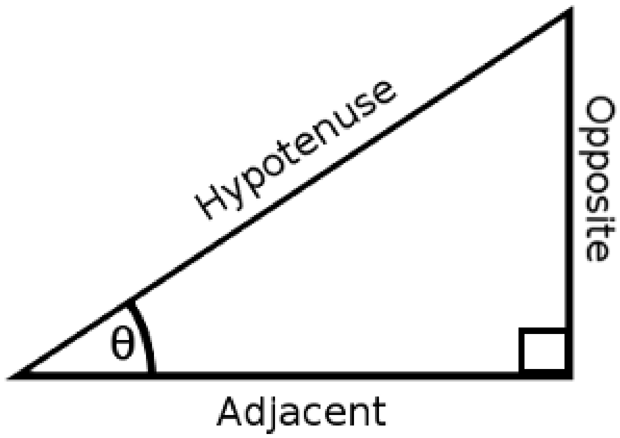


Figure 10

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

alongside these functions we also have:

- $\csc\theta = \dfrac{1}{\sin\theta}$

- $\sec\theta = \dfrac{1}{\cos\theta}$

- $\cot\theta = \dfrac{1}{\tan\theta}$

Note:

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

A common tool used to evaluate trigonometric functions is called the *unit circle*.

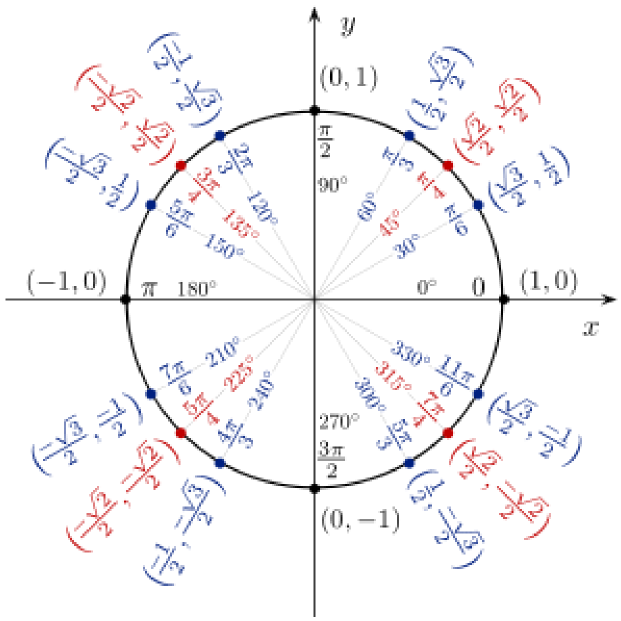


Figure 11: The Unit Circle

Unit indicates that the radius is 1. The picture indicates that the x-coordinate is the value for $\cos\theta$ and the y-coordinate is the value for $\sin\theta$. Starting in the first quadrant and moving counter-clockwise, you can use the acronym ASTC (All Students Take Calculus) to see which values are positive in what quadrant.

From the unit circle, you can also see an important conversion factor: π radians = 180°

<u>Note:</u> Radians are dimensionless so we often don't write the word radian.

<u>Useful Angles:</u>

- $\frac{\pi}{2} = 90°$
- $\pi = 180°$
- $\frac{3\pi}{2} = 270°$
- $2\pi = 360°$
- $\frac{\pi}{4} = 45°$

- $\frac{\pi}{6} = 30°$

- $\frac{\pi}{3} = 60°$

Examples:

1. Convert 30° to radian measure.

   Solution Multiplying by our conversion factor we have

$$30° \cdot \frac{\pi \text{ rad}}{180°} = \frac{30\pi}{180} \text{ rad} = \frac{\pi}{6}\text{radians}$$

□

2. Convert $\frac{3\pi}{2}$ to degrees

   Solution

$$\frac{3\cancel{\pi}}{2} \cdot \frac{180°}{\cancel{\pi}} = \frac{540°}{2} = 270°$$

□

Identities:

We have three main trigonometric identities:

(1) $\sin^2\theta + \cos^2\theta = 1$ (This comes from the pythagorean theorem on the unit circle)

(2) $\tan^2\theta + 1 = \sec^2\theta$ (This comes from dividing (1) by $\cos^2\theta$)

(3) $1 + \cot^2\theta = \csc^2\theta$ (Note: This comes from dividing (1) by $\sin^2\theta$)

And we have many other identities including:

- $\sin(2\theta) = 2\sin\theta\cos\theta$

- $$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$

Definition: An *even function* is a function such that $f(-x) = f(x)$. An *odd function* is a function such that $f(-x) = -f(x)$.

Note:

- $\cos(-\theta) = \cos(\theta)$ (i.e cosine is an even function)
- $\sin(-\theta) = -\sin(\theta)$ (i.e. sine is an odd function)
- $\tan(-\theta) = -\tan(\theta)$ (i.e. tangent is an odd function)

Examples:

1. Simplify $\sin\theta\csc\theta\tan\theta$

Solution

$$\begin{aligned}\sin\theta\csc\theta\tan\theta &= \cancel{\sin\theta}\cdot\frac{1}{\cancel{\sin\theta}}\cdot\frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\end{aligned}$$

□

2. Simplify $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$

Solution

$$\begin{aligned}(\sec\theta+\tan\theta)(\sec\theta-\tan\theta) &= \sec^2\theta-\cancel{\sec\theta\tan\theta}+\cancel{\sec\theta\tan\theta}-\tan^2\theta\\ &= \sec^2\theta-\tan^2\theta\\ &= 1\end{aligned}$$

□

3. Simplify $\dfrac{\cot^2\theta(\sec^2\theta-1)}{\sec^2\theta-\tan^2\theta+1}$

Solution

$$\begin{aligned}\frac{\cot^2\theta(\sec^2\theta-1)}{\sec^2\theta-\tan^2\theta+1} &= \frac{\cot^2\theta(\tan^2\theta)}{(\cancel{\tan^2\theta}+1)-\cancel{\tan^2\theta}+1}\\ &= \frac{\cot^2\theta\tan^2\theta}{2}\\ &= \frac{1/\cancel{\tan^2\theta}\cdot\cancel{\tan^2\theta}}{2}\\ &= \frac{1}{2}\end{aligned}$$

□

4. If θ is acute and $\tan\theta = \frac{3}{4}$ find:

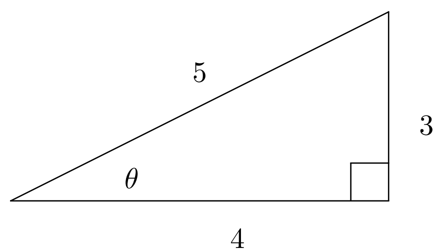
   (a) $\cos(-\theta)$

   (b) $\sin(4\pi + \theta)$

Solution

$$\tan\theta = \frac{3}{4} = \frac{\text{opposite}}{\text{adjacent}}$$

Using the pythagorean theorem we can make the following triangle:



(a)

$$\begin{aligned}\cos(-\theta) &= \cos\theta \\ &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{4}{5}\end{aligned}$$

(b)

$$\begin{aligned}\sin(4\pi + \theta) &= \sin(2\pi + (2\pi + \theta)) \\ &= \sin(2\pi + \theta) \\ &= \sin\theta \\ &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{3}{5}\end{aligned}$$

□

Graphs and Periods

Here are the graphs for the three main trigonometric functions.

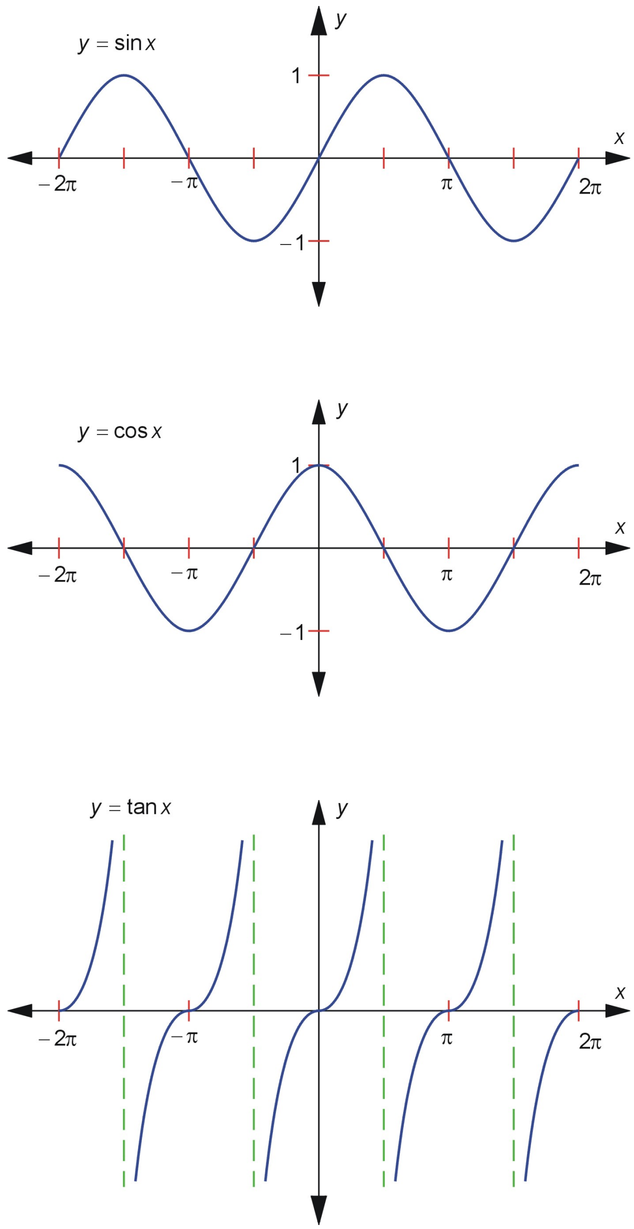


Figure 12

From the graphs we can easily see the following properties:

- Sine has range $[-1, 1]$

- Cosine has range $[-1, 1]$

- Tangent has range $(-\infty, \infty)$

Definition: A function f is called *periodic* if there is an $a > 0$ such that $f(x + a) = f(x)$ for all x in the domain of f. The smallest such a is called the *period* of f. (i.e. the graph repeats itself after a units) Thus from the graphs we can also see:

- Sine has period 2π
  $\Rightarrow \sin(\theta + 2\pi) = \sin\theta$

- Cosine has period 2π
  $\Rightarrow \cos(\theta + 2\pi) = \cos\theta$

- Tangent has period π
  $\Rightarrow \tan(\theta + \pi) = \tan\theta$

Note: You could also see this by using the fact that one full rotation around the unit circle is 2π and a half rotation is π

Similarly:

- Secant has period 2π
  $\Rightarrow \sec(\theta + 2\pi) = \sec\theta$

- Cosecant has period 2π
  $\Rightarrow \csc(\theta + 2\pi) = \csc\theta$

- Cotangent has period π
  $\Rightarrow \cot(\theta + \pi) = \cot\theta$

Examples:

1. Find all values of x such that $\cos x = \dfrac{1}{2}$

   Solution We know that

$$x = \frac{\pi}{3}$$

satisfies the equation and from the unit circle, we also have

$$x = \frac{5\pi}{3}$$

satisfies it as well. Since cosine has period 2π, that means that adding any integer multiple of 2π to these values will also give us a solution. This our solutions are

$$x = \frac{\pi}{3} + 2\pi k \text{ or } \frac{5\pi}{3} + 2\pi k$$

where k is an integer.

□

2. Find all solutions to $2\sin^2(\theta) - 3\sin(\theta) + 1 = 0$

Solution

$$\begin{aligned} 2\sin^2(\theta) - 3\sin(\theta) + 1 = 0 &\Leftrightarrow (2\sin\theta - 1)(\sin\theta - 1) = 0 \\ &\Leftrightarrow 2\sin\theta - 1 = 0 \text{ or } \sin\theta - 1 = 0 \\ &\Leftrightarrow 2\sin\theta = 1 \text{ or } \sin\theta = 1 \\ &\Leftrightarrow \sin\theta = \frac{1}{2} \text{ or } \sin\theta = 1 \\ &\Leftrightarrow \theta = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n \text{ or } \theta = \frac{\pi}{2} + 2\pi n \end{aligned}$$

□

3. Find $\tan(\pi - \theta)$ if $\tan\theta = 3$

Solution

$$\begin{aligned} \tan(\pi - \theta) &= \tan(-\theta) \\ &= -\tan\theta \\ &= -3 \end{aligned}$$

□

Lecture 6: Limits of Functions

Definitions:

- The *(two-sided) limit* of f as x approaches a, written

  $$\lim_{x \to a} f(x)$$

  is the value that f approaches as x gets closer to a.

- The *limit from the left* of f as x approaches a, written

  $$\lim_{x \to a^-} f(x)$$

  is the value that f approaches as x gets closer to a for values of x less than a (i.e. to the left of a).

- The *limit from the right* of f as x approaches a, written

  $$\lim_{x \to a^+} f(x)$$

  is the value that f approaches as x gets closer to a for values of x greater than a (i.e. to the right of a).

Note: When evaluating a limit, it doesn't matter what happens when $x = a$. You'll see later that this only matters when dealing with a concept known as continuity.

For "nice" functions $\lim_{x \to a} f(x) = f(a)$

Definition: We say that $\lim_{x \to a} f(x)$ does not exist if at least one of the following is true:

(i) $f(x)$ becomes arbitrarily large (positively or negatively) as x approaches a.
For positive values we write

$$\lim_{x \to a} f(x) = \infty$$

For negative values we write

$$\lim_{x \to a} f(x) = -\infty$$

(ii) The limit from one side is $-\infty$ and the limit from the other is ∞

(iii) The one-sided limits exist but are not equal.

Properties of Limits:

- $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
- $\lim_{x\to a} p(x) = p(a)$, where p is a polynomial
- $\lim_{x\to a}(f(x))^k = (\lim_{x\to a} f(x))^k$
- $\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)}$
- $\lim_{x\to a}(\log_b f(x)) = \log_b(\lim_{x\to a} f(x))$
- $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ if it exists
- $\lim_{x\to 0^+} \frac{1}{x^n} = \infty$
- $\lim_{x\to\infty} x^n e^{-x} = 0$
- $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$
- $\lim_{x\to 0^+} x^n \ln x = 0$
- $\lim_{x\to a} f(x) = \lim_{t\to 0} f(t+a)$
- $\lim_{x\to 0^+} f(x) = \lim_{t\to\infty} f\left(\frac{1}{t}\right)$

- $\lim_{x \to 0^-} f(x) = \lim_{t \to -\infty} f\left(\frac{1}{t}\right)$

To find the limit of a rational function, we have tricks:

- For $\lim_{x \to \infty} \frac{p(x)}{q(x)}$, divide everything by the largest power of x in $q(x)$.

- To find $\lim_{x \to 0} \frac{p(x)}{q(x)}$, divide everything by the smallest power of x in $q(x)$.

For $\lim_{x \to \infty} \frac{p(x)}{q(x)}$, you will notice the following pattern:

- If the degree of p is larger than the degree of q then the limit is $\pm\infty$
- If the degree of q is larger than the degree of p then the limit is 0
- If the degrees are equal, the limit is the leading term

You may use this pattern to check your answers but **not** as justification for an answer.

Examples:

1. Find $\lim_{x \to 3}(4x - 5)$

   Solution

$$\begin{aligned}\lim_{x \to 3}(4x - 5) &= 4(3) - 5 \\ &= 7\end{aligned}$$

□

2. Find $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$

   Solution

$$\begin{aligned}\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} &= \lim_{x \to 3} \frac{\cancel{(x - 3)}(x + 2)}{\cancel{x - 3}} \\ &= \lim_{x \to 3}(x + 2) \\ &= 3 + 2 \\ &= 5\end{aligned}$$

□

3. Find $\lim_{n \to \infty} \frac{n^2}{n^2 + 1}$

Solution

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{n^2+1} &= \lim_{n\to\infty} \frac{n^2/n^2}{n^2/n^2 + 1/n^2} \\
&= \lim_{n\to\infty} \frac{1}{1 + 1/n^2} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}$$

□

4. Find $\lim_{n\to\infty} \frac{n}{n^2+1}$

Solution

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n^2+1} &= \lim_{n\to\infty} \frac{n/n^2}{n^2/n^2 + 1/n^2} \\ &= \lim_{n\to\infty} \frac{1/n}{1 + 1/n^2} \\ &= \frac{0}{1+0} \\ &= 0\end{aligned}$$

□

5. Find $\lim_{x\to\infty} \frac{x^2+8}{6x^2-x}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2+8}{6x^2-x} &= \lim_{x\to\infty} \frac{x^2/x^2 + 8/x^2}{6/x^2 - x/x^2} \\ &= \lim_{x\to\infty} \frac{1 + 8/x^2}{6 - 1/x} \\ &= \frac{1+0}{6-0} \\ &= \frac{1}{6}\end{aligned}$$

□

6. Find $\lim_{x\to 0} \frac{x^2+8x}{6x^2-x}$

Solution

$$\begin{aligned}\lim_{x\to 0} \frac{x^2+8x}{6x^2-x} &= \lim_{x\to 0} \frac{x^2/x + 8x/x}{6x^2/x - x/x} \\ &= \lim_{x\to 0} \frac{x+8}{6x-1} \\ &= \frac{8}{-1} \\ &= -8\end{aligned}$$

□

7. Find $\lim_{n\to\infty} \frac{n^2}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n^2/n}{n/n + 1/n}$$
$$= \lim_{n\to\infty} \frac{n}{1 + 1/n}$$

□

8. Find $\lim_{x\to 0} \frac{3x^6 + x}{x^5 - 3x^2}$

Solution

$$\lim_{x\to 0} \frac{3x^6 + x}{x^5 - 3x^2} = \lim_{x\to 0} \frac{3x^6/x^2 + x/x^2}{x^5/x^2 - 3x^2/x^2}$$
$$= \lim_{x\to 0} \frac{3x^4 + 1/x}{x^3 - 3}$$

DNE

□

9. Find $\lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3}$

Solution

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x\to 3} \frac{(x-2)\cancel{(x-3)}}{\cancel{x-3}}$$
$$= \lim_{x\to 3}(x-2)$$
$$= 1$$

□

10. Find $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$

Solution Recall: $|x-2| = \begin{cases} -(x-2) & x > 2 \\ x-2 & x \le 2 \end{cases}$

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-\cancel{(x-2)}}{\cancel{(x-2)}}$$
$$= \lim_{x\to 2^-}(-1)$$
$$= -1$$

□

11. Find $\lim_{x\to\infty} \cos x$

Solution The graph of $\cos x$ oscillates so the limit DNE.

□

12. Find $\lim_{x\to\infty} (x^{17} - 3)e^{-x}$

Solution

$$\begin{aligned}\lim_{x\to\infty} (x^{17} - 3)e^{-x} &= \lim_{x\to\infty} (x^{17}e^{-x}) - \lim_{x\to\infty} 3e^{-x} \\ &= 0 - 0 \\ &= 0\end{aligned}$$

□

13. Given the graph below, find the following limits:

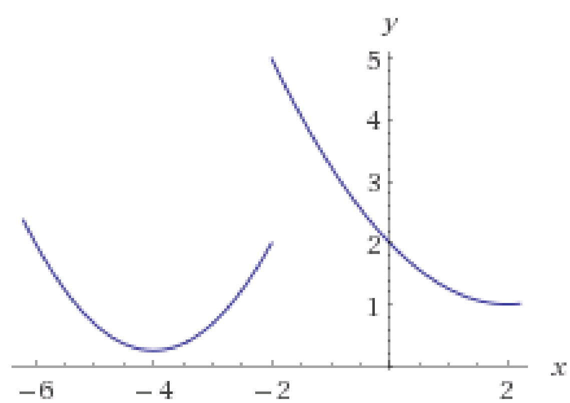


Figure 13

(a) $\lim_{x\to 2} f(x)$

Solution Looking at the graph we see that as we approach 2 from both the left and the right, the value of $f(x)$ approaches 1. Thus the limit is 1

□

(b) $\lim_{x\to -2} f(x)$

Solution Looking at the graph, we notice that the graph approaches two values as x gets closer to -2 so the limit does not exist.

□

(c) $\lim_{x \to -2^+} f(x)$

Solution We are approaching -2 from the right so ignoring the graph to the left of $x = -2$ we see that the graph is approaching the value 5 so the limit is 5.

□

(d) $\lim_{x \to -2^-} f(x)$

Solution Now we are approaching from the left so ignore the graph to the right of $x = -2$. The graph is approaching 2 thus the limit is 2.

□

Lecture 7: Continuity

Definitions:

- A function f is *continuous* at $x = a$ if $\lim_{x \to a} f(x) = f(a)$
- If f is not continuous at a we say that f is *discontinuous* at a
- A function f is *continuous from the left* at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$
- A function f is *continuous from the right* at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$
- A function f is continuous on (a, b) if it is continuous at every point of (a, b)
- A function f is continuous on $[a, b]$ if it is continuous on (a, b), from the right at a, and from the left at b.

Some Common Functions:

- Polynomials are continuous for all x
- Rational functions $\frac{p(x)}{q(x)}$ are continuous for all x such that $q(x) \neq 0$
- $\sqrt{ax+b}$, a and b real numbers, is continuous for all x such that $ax + b \geq 0$
- a^x, where $a > 0$ is continuous for all x
- $\log_b x$, where $b > 0$ and $b \neq 1$, is continuous for all $x > 0$
- $\sin x$ and $\cos x$ are continuous for all x
- $\tan x = \frac{\sin x}{\cos x}$ is continuous for all x such that $\cos x \neq 0$

Properties of Continuous Functions:

- If a function f is continuous at $x = a$, and g is continuous at $y = f(a)$, then $g \circ f$ is continuous at $x = a$
- If g is continuous everywhere, then $\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)$

Examples:

1. Let $f(x) = \sqrt{x-1}$ and $g(x) = e^x$. Where is $h(x) = (g \circ f)(x)$ continuous?

   Solution $g(x)$ is continuous for all x and $f(x)$ is continuous for $x \geq 1$. Thus h is continuous for $x \geq 1$. Since $f(x)$ is not defined for $x < 1$ we have that h is not continuous for $x < 1$

   $\square$

2. Find $\lim_{x\to\infty} e^{2x^2+7x-3/x^2-5x+1}$

   Solution

$$\lim_{x\to\infty} e^{2x^2+7x-3/x^2-5x+1} = e^{\lim_{x\to\infty} 2x^2+7x-3/x^2-5x+1}$$

   Evaluating the exponential limit we have:

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2+7x-3}{x^2-5x+1} &= \lim_{x\to\infty} \frac{2x^2/x^2 + 7x/x^2 - 3/x^2}{x^2/x^2 - 5x/x^2 + 1/x^2} \\ &= \lim_{x\to\infty} \frac{2 + 7/x - 3/x^2}{1 - 5/x + 1/x^2} \\ &= \frac{2+0-0}{1-0+0} \\ &= 2\end{aligned}$$

   Thus we have

$$\lim_{x\to\infty} e^{2x^2+7x-3/x^2-5x+1} = e^2$$

   $\square$

3. Where is the following function continuous?

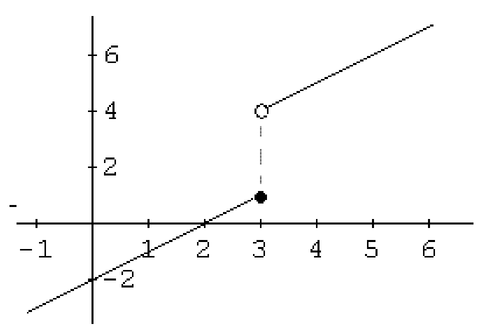


Figure 14

Solution There is one jump at $x = 3$ so the function is continuous for $x \neq 3$

$\square$

4. Where is the following function continuous?

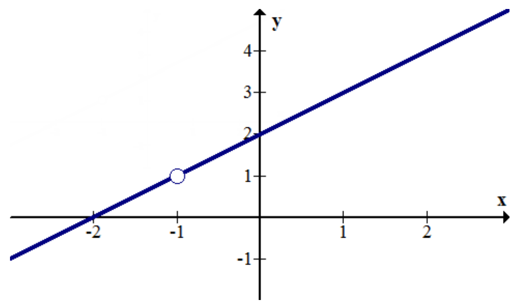


Figure 15

Solution There is one hole at $x = -1$ so the function is continuous for $x \neq 1$

□

5. Is $h(x) = \dfrac{x^2+1}{x^3+1}$ continuous at $x = -1$?

Solution Notice that at $x = -1$ the denominator is 0, thus $f(-1)$ is undefined which means that f cannot be continuous at $x = -1$

□

6. Is the $f(x)$ continuous at $x = 3$? At $x = -3$? Where $f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9} & x \neq 3 \\ \dfrac{9}{2} & x = 3 \end{cases}$

Solution

$$\begin{aligned}
\lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{x^3-27}{x^2-9} \\
&= \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}(x+3)} \\
&= \lim_{x\to 3} \frac{x^2+3x+9}{x+3} \\
&= \frac{3^2+3(3)+9}{3+3} \\
&= \frac{27}{6} \\
&= \frac{9}{2} \\
&= f(3)
\end{aligned}$$

Thus the function is continuous at $x = 3$. However $f(-3)$ is undefined so it is not continuous at $x = -3$.

□

7. Determine where $f(x)$ is continuous if $f(x) = \begin{cases} 0 & x < 0 \\ x^2 - 5x & 0 \leq x \leq 5 \\ 5 & x > 5 \end{cases}$

Solution Notice that on the interiors of all the intervals present, each piece is continuous. Thus we know our function is continuous on $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$. We need to check what happens at $x = 0$ and $x = 5$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x^2 - 5x) \\ &= (0)^2 - 5(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^2 - 5(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 5^-} f(x) &= \lim_{x \to 5^-} (x^2 - 5x) \\ &= (5)^2 - 5(5) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 5^+} f(x) &= \lim_{x \to 5^+} 5 \\ &= 5 \end{aligned}$$

$$f(5) = 0$$

This gives us that it is continuous at $x = 0$ but not at $x = 5$. Thus we have that $f(x)$ is continuous for $(-\infty, 0] \cup [0, 5) \cup (5, \infty)$

□

8. Determine where $f(x)$ is continuous if $f(x) = \begin{cases} 3x - 5 & x \neq 1 \\ 2 & x = 1 \end{cases}$

Solution $3x - 5$ and 2 are polynomials so they are continuous individually. Therefore, the only place to check is at $x = 1$. We have

$$\begin{aligned} \lim_{x\to 1} f(x) &= \lim_{x\to 1}(3x - 5) \\ &= 3(1) - 5 \\ &= -2 \neq f(1) \\ &= 2 \end{aligned}$$

so the function is discontinuous at $x = 1$

□

9. For what value of c is $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & x < 1 \\ cx - 2 & x \geq 1 \end{cases}$ continuous?

Solution We want the function to match up for $x = 1$.

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}(cx - 2) \\ &= c(1) - 2 \\ &= c - 2 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{x^3 - 1}{x - 1} \\ &= \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x - 1}} \\ &= \lim_{x\to 1^-} x^2 + x + 1 \\ &= (1)^2 + 1 + 1 \\ &= 3 \end{aligned}$$

We want them equal so we want $c - 2 = 3 \Leftrightarrow c = 5$

□

Lecture 8: Rates of Change

Definitions:

- The *average rate of change* of f over $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

  It tells how the function would change if the rate of change remained constant.

- If we know the values of the function for each x, then we can ask what the average rate of change is over a smaller and smaller interval. In other words, if the interval is $[a, b]$, we can ask what happens if b becomes closer to a. The *instantaneous rate of change* is

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Examples:

1. Suppose that after class you drive from UH to Hawaii Kai (Distance around 9.5 miles).You leave at 1:30pm and get there at 2pm. What was your average speed?

   Solution Let f denote the distance you traveled at time t, then

$$\begin{aligned} \text{Average Speed} &= \frac{f(14) - f(13.5)}{14 - 13.5} \\ &= \frac{9.5}{0.5} \\ &= 19 \text{ mph} \end{aligned}$$

□

2. Suppose that after class you drive from UH to Hawaii Kai. You leave at 1:30pm, pass by Kahala Mall at 1:36pm and get to Hawaii Kai at 2pm. What was your average speed on the time interval [13.5,13.6] and [13.6,14]?

Solution

$$\begin{aligned}\text{Average Speed on } [13.5, 13.6] &= \frac{f(13.6) - f(13.5)}{13.6 - 13.5} \\ &= \frac{3}{0.1} \\ &= 30 \text{ mph}\end{aligned}$$

$$\begin{aligned}\text{Average Speed on } [13.6, 14] &= \frac{f(14) - f(13.6)}{14 - 13.6} \\ &= \frac{6.5}{0.4} \\ &= 16.25 \text{ mph}\end{aligned}$$

□

3. Supposed the distance traveled over the time interval $[0, 10]$ is described by the function $f(t) = 4t^2$. What is the instantaneous speed at $t = 1$?

Solution

$$\begin{aligned}\text{Instantaneous Speed} &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{4(1+h)^2 - 4}{h} \\ &= \lim_{h\to 0} \frac{4(1+2h+h^2) - 4}{h} \\ &= \lim_{h\to 0} \frac{\cancel{4} + 8h + 4h^2 - \cancel{4}}{h} \\ &= \lim_{h\to 0} \frac{8h + 4h^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(8+4h)}{\cancel{h}} \\ &= \lim_{h\to 0} (8+4h) \\ &= 8\end{aligned}$$

□

4. It is estimated that t days after the flu begins to spread in town, the percentage of the population infection by the flu is approximated by the function $f(t) = t^2 + t$ for $0 \leq t \leq 5$. What is the average rate of change for f with respect to t from 1 to 4 days? What is the instantaneous rate of change of f with respect to t at $t = 3$? How can we interpret it?

Solution

$$\begin{aligned}\text{Average Rate} &= \frac{f(4) - f(1)}{4 - 1} \\ &= \frac{20 - 2}{4 - 1} \\ &= \frac{18}{3} \\ &= 6 \text{ percent per day}\end{aligned}$$

$$\begin{aligned}\text{Instantaneous Rate} &= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{(3 + h)^2 + h - (3^3 + 3)}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 7h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(h + 7)}{\cancel{h}} \\ &= \lim_{h \to 0} (h + 7) \\ &= 7\end{aligned}$$

So right after 3 days, the flue was spreading at a rate of 7% per day

□

Lecture 9: Definition of the Derivative

Definition: The *derivative* of a function $f(x)$, written $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$

provided that the limit exists. Thus the derivative is the instantaneous rate of change of f at x. The derivative can also be considered as the slope of the tangent line to f at x.

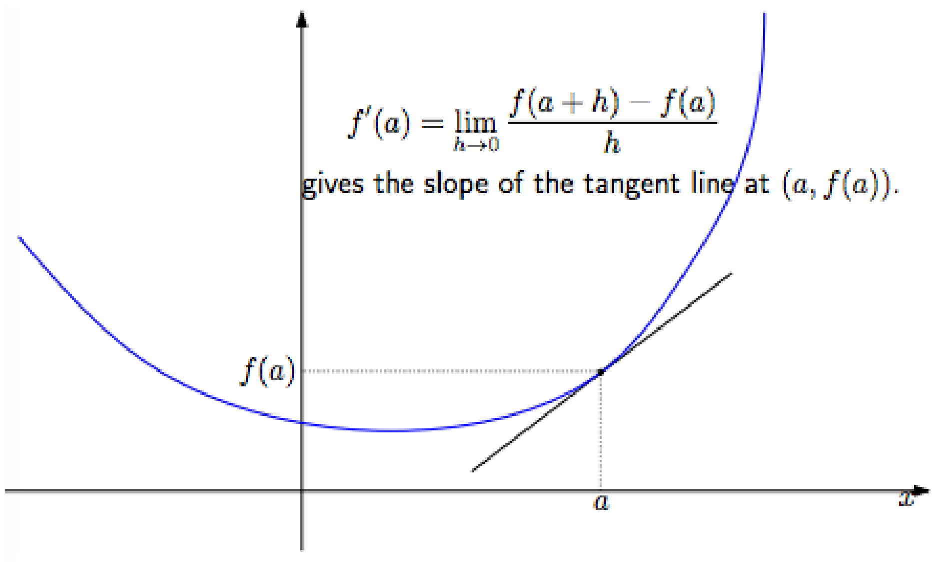


Figure 16

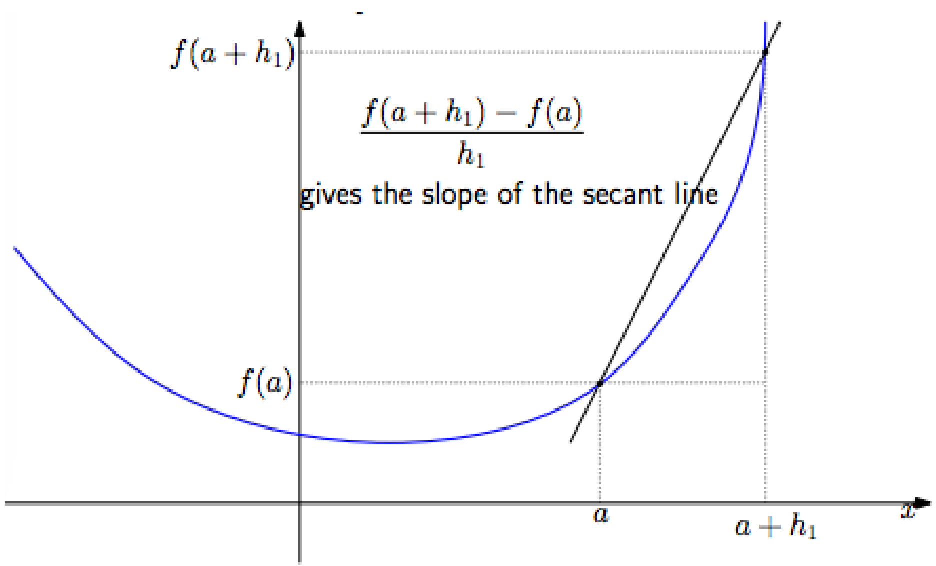


Figure 17

Examples:

1. Find the derivative of $f(x) = x^2$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x+h) \\
&= 2x
\end{aligned}$$

□

2. Find the derivative of $f(x) = 5x^2 - 3x + 7$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(5(x+h)^2 - 3(x+h) + 7) - (5x^2 - 3x + 7)}{h} \\
&= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) - 3x - 3h + 7 - 5x^2 + 3x - 7}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{3x} - 3h + \cancel{7} - \cancel{5x^2} + \cancel{3x} - \cancel{7}}{h} \\
&= \lim_{h\to 0} \frac{10xh + 5h^2 - 3h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(10x + 5h - 3)}{\cancel{h}} \\
&= \lim_{h\to 0} (10x + 5h - 3) \\
&= 10x + 0 - 3 \\
&= 10x - 3
\end{aligned}$$

□

3. Find the equation of the tangent line to the graph of $f(x) = \frac{5}{x}$ at $(2, f(2))$

Solution

$$\begin{aligned}
\text{slope} &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{5/(2+h) - 5/2}{h} \\
&= \lim_{h\to 0} \frac{2(2+h)(5/(2+h) - 5/2)}{2(2+h)h} \\
&= \lim_{h\to 0} \frac{10 - 5(2+h)}{2(2+h)h} \\
&= \lim_{h\to 0} \frac{\cancel{10} - \cancel{10} - 5h}{2(2+h)h} \\
&= \lim_{h\to 0} \frac{-5\cancel{h}}{\cancel{h}(4+2h)} \\
&= \lim_{h\to 0} \frac{-5}{4+2h} \\
&= -\frac{5}{4}
\end{aligned}$$

Thus using point-slope form we have

$$y - \frac{5}{2} = -\frac{5}{4}(x-2) \Leftrightarrow y = -\frac{5}{4}x + 5$$

□

Note: A derivative of a function f at a point x does not exist if:

- f is discontinuous at x
- f has a sharp corner at x
- f has a vertical tangent line at x

Examples:

1. Show that for $f(x) = |x|$, $f'(0)$ does not exist.

   Solution Graphically, there is a corner at $x = 0$ but we can show this algebraically.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h} \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+} \frac{|h|}{h} &= \lim_{h\to 0^+} \frac{h}{h} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^-} \frac{|h|}{h} &= \lim_{h\to 0^-} \frac{-h}{h} \\ &= -1 \end{aligned}$$

   Thus the limit does not exist.

   □

2. For a human, the cumulative intake of food during a meal can be described as

$$I(t) = 27 + 72t - 1.5t^2$$

   where t is the number of minutes since the meal began, and $I(t)$ is the amount (in grams) that the person has eaten. What is he instantaneous rate of change of the intake of food 5 minutes into the meal? 24 minutes into the meal?

   Solution

$$\begin{aligned} I'(t) &= \lim_{h\to 0} \frac{I(t+h) - I(t)}{h} \\ &= \lim_{h\to 0} \frac{(27 + 72(t+h) - 1.5(t+h)^2) - (27 + 72t - 1.5t^2)}{h} \\ &= \lim_{h\to 0} \frac{72h - 1.5((t+h)^2 - t^2)}{h} \\ &= \lim_{h\to 0} \frac{72h - 1.5h(2t+h)}{h} \\ &= \lim_{h\to 0} (72 - 1.5(2t+h)) \\ &= 72 - 3t \end{aligned}$$

Thus

$$I'(5) = 72 - 3 \cdot 5 = 58 \text{ g/min and } I'(24) = 72 - 3 \cdot 24 = 0 \text{ g/min}$$

□

Lecture 10: Graphical Differentiation

Recall: $f'(x)$ is the slope of the tangent line to the curve f at x.

When determining what the graph of a derivative looks like, locate the points at which the derivative is 0 (i.e. smooth turning points), where the derivative does not exists (corners or discontinuities), where the derivative is positive (i.e. f is increasing), and where it is negative (i.e. f is decreasing). It is also useful to remember that the slope of a straight line is constant and the slope of a horizontal line is 0. Then using that information, sketch the graph.

Examples:

1.

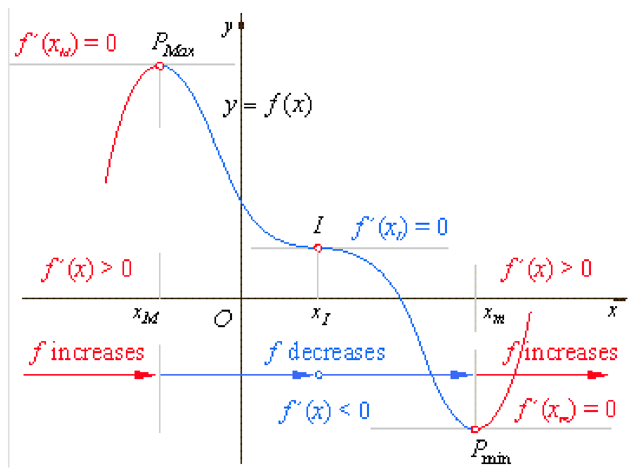


Figure 18

2.

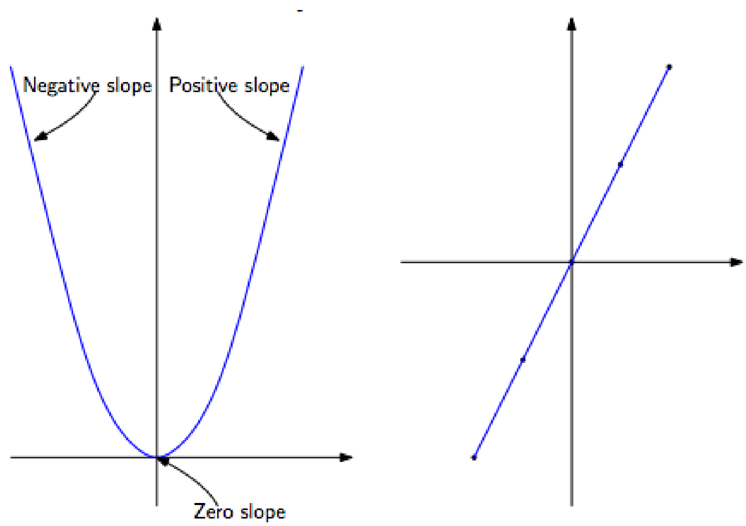


Figure 19

3.

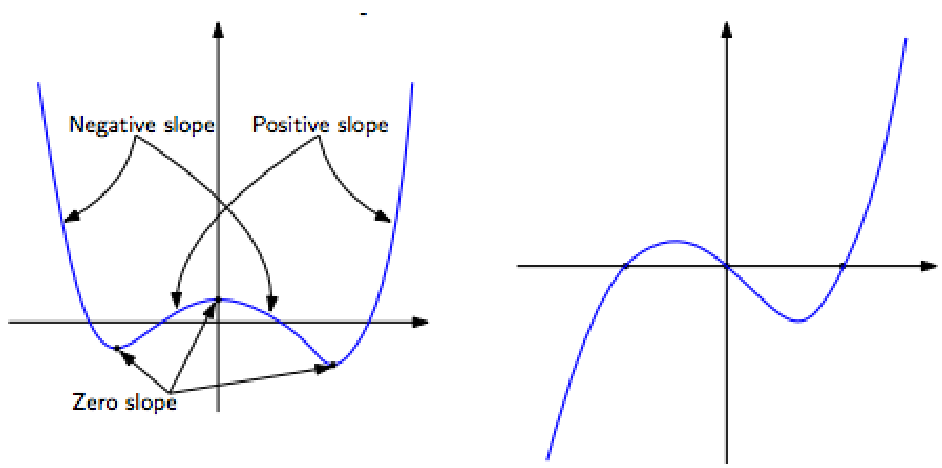


Figure 20

4. Given the graph of f, sketch the graph of f'.

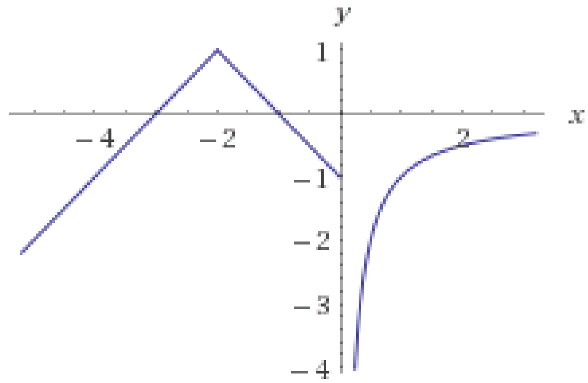


Figure 21

Solution Notice that the derivative will be undefined at $x = -2$ (there is a corner in the graph) and at $x = 0$ (there is a discontinuity). The graph of f is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$ so $f'(x) > 0$ on those intervals. f is decreasing on $(-2, 0)$ so $f'(x) < 0$ on those intervals. Lastly, the know that the slope of a line is constant and we can use rise over run to calculate it. On $(-\infty, -2)$ the slope is 1 (thus $f'(x) = 1$ on this interval) and on $(-2, 0)$ the slope is -1 (thus $f'(x) = -1$ on this interval). Putting this all together we get:

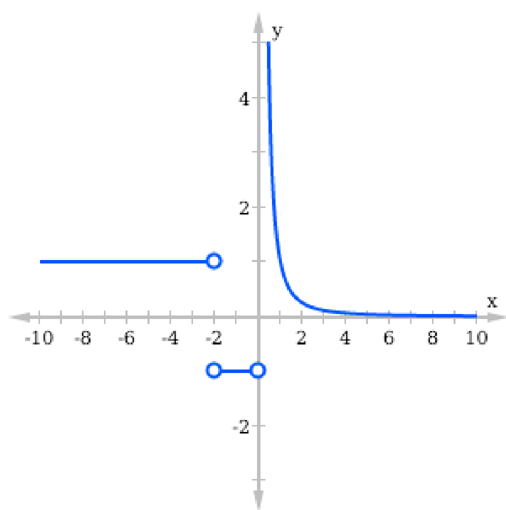


Figure 22

□

Lecture 11: Techniques for Finding Derivatives

Notation: The derivative of a function f at a point x can be written in many different ways including:

- $f'(x)$
- $\frac{df}{dx}(x)$
- $\frac{d}{dx}[f(x)]$
- $D_x[f(x)]$

Properties of Derivatives:

1. $\frac{d}{dx}(k) = 0$, where k is a constant (i.e a variable independent from x)

   Proof. Let $f(x) = k$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{k-k}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

□

2. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Proof. For any positive integer n we have

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

So we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(\not{x} + h - \not{x})((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1})}{h} \\ &= \lim_{h\to 0} \frac{\not{h}((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1})}{\not{h}} \\ &= x^{n-1} + x^{n-1} + \ldots + x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

□

3. $\dfrac{d}{dx}(kf(x)) = kf'(x)$

Proof. Let $g(x) = kf(x)$

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} \\ &= \lim_{h\to 0} \frac{k[f(x+h) - f(x)]}{h} \\ &= k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= kf'(x) \end{aligned}$$

□

4. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Proof.

$$\begin{aligned}
\frac{d}{dx}(f(x) \pm g(x)) &= \lim_{h\to 0} \frac{(f(x+h) \pm g(x+h)) - (f(x) \pm g(x))}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x)) \pm (g(x+h) - g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) \pm g'(x)
\end{aligned}$$

□

Examples:

1. Find the following:

   (a) $\dfrac{d}{dx}(\sqrt{17})$

   (b) Find $\dfrac{d}{dx}\left(\sin\left(\dfrac{11^{\sqrt{73}}}{\pi^e - 1}\right)\right)$

   Solution

   (a) $\sqrt{17}$ is a constant so $\dfrac{d}{dx}(\sqrt{17}) = 0$

   (b) $\sin\left(\dfrac{11^{\sqrt{73}}}{\pi^e - 1}\right)$ is a constant so $\dfrac{d}{dx}\left(\sin\left(\dfrac{11^{\sqrt{73}}}{\pi^e - 1}\right)\right) = 0$

□

2. Find the following:

(a) $\frac{d}{dx}(x^3)$

(b) $\frac{d}{dx}\left(\frac{1}{x}\right)$

(c) $\frac{d}{dx}\left(x^{61}\right)$

(d) $\frac{d}{dx}\left(\frac{1}{x^{13}}\right)$

(e) $\frac{d}{dx}(\sqrt{x})$

(f) $\frac{d}{dx}\left(\frac{1}{\sqrt[7]{x}}\right)$

Solution

(a)

$$\begin{aligned}\frac{d}{dx}(x^3) &= 3x^{3-1}\\ &= 3x^2\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{d}{dx}(x^{-1})\\ &= (-1)x^{-1-1}\\ &= -x^{-2}\\ &= -\frac{1}{x^2}\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dx}\left(x^{61}\right) &= 61x^{61-1}\\ &= 61x^{60}\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^{13}}\right) &= \frac{d}{dx}(x^{-13})\\ &= (-13)x^{-13-1}\\ &= -13x^{-14}\\ &= -\frac{13}{x^{14}}\end{aligned}$$

(e)

$$\begin{aligned}\frac{d}{dx}(\sqrt{x}) &= \frac{d}{dx}(x^{1/2})\\ &= \left(\frac{1}{2}\right)x^{1/2-1}\\ &= \frac{1}{2}x^{-1/2}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

(f)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\sqrt[7]{x}}\right) &= \frac{d}{dx}\left(x^{-1/7}\right)\\ &= -\frac{1}{7}x^{-1/7-1}\\ &= -\frac{1}{7}x^{-8/7}\\ &= -\frac{1}{7x^{8/7}}\end{aligned}$$

□

3. Find $\frac{d}{dx}(7x^{11})$

Solution

$$\begin{aligned}\frac{d}{dx}(7x^{11}) &= 7\frac{d}{dx}x^{11}\\ &= 7(11x^{11-1})\\ &= 77x^{10}\end{aligned}$$

□

4. $\frac{d}{dx}(x^2 + x^3)$

Solution

$$\begin{aligned}\frac{d}{dx}(x^2 + x^3) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) \\ &= 2x^{2-1} + 3x^{3-1} \\ &= 2x + 3x^2\end{aligned}$$

□

5. $\frac{d}{dx}(5x^2 + x^3 - 7x^4)$

Solution

$$\begin{aligned}\frac{d}{dx}(5x^2 + x^3 - 7x^4) &= \frac{d}{dx}(5x^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(7x^4) \\ &= 5\frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) - 7\frac{d}{dx}(x^4) \\ &= 5(2x^{2-1}) + 3x^{3-1} - 7(4x^{4-1}) \\ &= 10x + 3x^2 - 28x^3\end{aligned}$$

□

6. Find $\frac{d}{dx}\left(\frac{3x^3 - 4x}{\sqrt{x}}\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\frac{3x^3 - 4x}{\sqrt{x}}\right) &= \frac{d}{dx}\left(\frac{3x^3}{x^{1/2}} - \frac{4x}{x^{1/2}}\right) \\ &= 3\frac{d}{dx}\left(\frac{x^3}{x^{1/2}}\right) - 4\frac{d}{dx}\left(\frac{x}{x^{1/2}}\right) \\ &= 3\frac{d}{dx}(x^{3-1/2}) - 4\frac{d}{dx}(x^{1-1/2}) \\ &= 3\frac{d}{dx}(x^{5/2}) - 4\frac{d}{dx}(x^{1/2}) \\ &= 3\left(\frac{5}{2}x^{5/2-1}\right) - 4\left(\frac{1}{2}x^{1/2-1}\right) \\ &= \frac{15}{2}x^{3/2} - 2x^{-1/2} \\ &= \frac{15}{2}x^{3/2} - \frac{2}{\sqrt{x}}\end{aligned}$$

□

Lecture 12: Derivatives of Products and Quotients

Product and Quotient Rules:

1. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Proof.

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)(g(x+h) - g(x))}{h} + \frac{g(x)(f(x+h) - f(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

□

2. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Proof.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \lim_{h\to 0} \frac{f(x+h)/g(x+h) - f(x)/g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)g(x)(f(x+h)/g(x+h) - f(x)/g(x))}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{(\cancel{g(x+h)}g(x)f(x+h))/\cancel{g(x+h)} - (g(x+h)\cancel{g(x)}f(x))/\cancel{g(x)})}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{(f(x+h)g(x) - f(x)g(x) - (f(x)g(x+h) - f(x)g(x))}{g(x+h)g(x)g} \\
&= \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \left(\frac{g(x)(f(x+h) - f(x))}{g(x+h)g(x)h} - \frac{f(x)(g(x+h) - g(x))}{g(x+h)g(x)h}\right) \\
&= \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x))}{g(x+h)g(x)h} - \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{g(x)}{g(x+h)g(x)} \cdot \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{f(x)}{g(x+h)g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x)}{g(x+h)g(x)} \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{f(x)}{g(x+h)g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{g(x)}{g(x)g(x)} f'(x) - \frac{f(x)}{g(x)g(x)} g'(x) \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

□

Examples:

1. Let $f(x) = 3x(x-10)^2 + 40$. Find $f'(x)$

   Solution

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(3x(x-10)^2) + \frac{d}{dx}(40) \\
&= \frac{d}{dx}(3x)(x-10)^2 + (3x)\frac{d}{dx}((x-10)^2) \\
&= 3\frac{d}{dx}(x)(x-10)^2 + (3x)\frac{d}{dx}(x^2 - 20x + 100) \\
&= 3(1x^{1-1})(x-10)^2 + (3x)\left(\frac{d}{dx}(x^2) - \frac{d}{dx}(20x) + \frac{d}{dx}(100)\right) \\
&= 3x(x-10)^2 + 3x\left(\frac{d}{dx}(x^2) - 20\frac{d}{dx}(x)\right) \\
&= 3x(x-10)^2 + 3x(2x^{2-1} - 20(1x^{1-1})) \\
&= 3x(x-10)^2 + 3x(2x-20)
\end{aligned}$$

□

2. Let $f(x) = \dfrac{(3x^2+1)(2x-1)}{5x+4}$. Find $f'(x)$

   Solution

$$\begin{aligned}
f'(x) &= \frac{(5x+4)(d/dx((3x^2+1)) - ((3x^2+1)(2x-1))(d/dx(5x+4))}{(5x+4)^2} \\
&= \frac{(5x+4)((d/dx(3x^2+1))(2x-1) + (3x^2+1)(d/dx(2x-1))) - (3x^2+1)(2x-1)(5)}{(5x+4)^2} \\
&= \frac{(5x+4)(6x(2x-1) + 2(3x^2+1)) - 5(3x^2+1)(2x-1)}{(5x+4)^2} \\
&= \frac{(5x+4)(12x^2 - 6x + 6x^2 + 2) - 5(6x^3 - 3x^2 + 2x - 1)}{(5x+4)^2} \\
&= \frac{(5x+4)(18x^2 - 6x + 2) - 30x^3 + 15x^2 - 10x + 5}{(5x+4)^2} \\
&= \frac{90x^3 - 30x^2 + 10x + 72x^2 - 24x + 8 - 30x^3 + 15x^2 - 10x + 5}{(5x+4)^2} \\
&= \frac{60x^3 + 57x^2 - 24x + 13}{(5x+4)^2}
\end{aligned}$$

□

3. Find $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{x+1}\right) &= \frac{(d/dx(x))(x+1) - x(d/dx(x+1))}{(x+1)^2} \\ &= \frac{(1)(x+1) - x(1)}{(x+1)^2} \\ &= \frac{\cancel{x} + 1 - \cancel{x}}{(x+1)^2} \\ &= \frac{1}{(x+1)^2}\end{aligned}$$

□

4. Find $\dfrac{d}{dx}\left(\dfrac{x^2-1}{3x^4}\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2-1}{3x^4}\right) &= \frac{(d/dx(x^2-1))(3x^4) - (x^2-1)(d/dx(3x^4))}{(3x^4)^2} \\ &= \frac{(2x)(3x^4) - (x^2-1)(12x^3)}{9x^8} \\ &= \frac{6x^5 - (12x^5 - 12x^3)}{9x^8} \\ &= \frac{-6x^5 + 12x^3}{9x^8} \\ &= \frac{\cancel{3x^3}(4 - 2x^2)}{\cancel{3x^3}(3x^5)} \\ &= \frac{4 - 2x^2}{3x^5}\end{aligned}$$

□

Lecture 13: The Chain Rule

Recall: Given two functions f and g, the *composition* $g \circ f$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Examples:

1. Write $h(x) = e^{x^2-2}$ as a composition $g \circ f$.

   Solution Let $f(x) = x^2 - 2$, then $g(x) = e^x$

   □

2. Write $h(x) = \frac{1}{x^2+1}$ as a composition $g \circ f$

   Solution Let $f(x) = x^2 + 1$, then $g(x) = \frac{1}{x}$

   □

Chain Rule:

If $f(x) = v(u(x))$, then $f'(x) = v'(u(x))u'(x)$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{v(u(x+h)) - v(u(x))}{h} \\
&= \lim_{h\to 0} \frac{v(u(x+h)) - v(u(x))}{u(x+h) - u(x)} \frac{u(x+h) - u(x)}{h} \\
&= \lim_{h\to 0} \frac{v(u(x+h)) - v(u(x))}{u(x+h) - u(x)} \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= \lim_{b\to u(x)} \frac{v(b) - v(u(x))}{b - u(x)} \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= v'(u(x))u'(x)
\end{aligned}$$

□

Examples:

1. Find the derivative of $f(x) = \sqrt{13x^2 - 5x + 8}$

   Solution $f(x) = (13x^2 - 5x + 8)^{1/2}$ Thus

$$f(x) = v(u(x)), \text{ where } u(x) = 13x^2 - 5x + 8 \text{ and } v(x) = x^{1/2}$$

   Using the chain rule we have

$$\begin{aligned} f'(x) &= \frac{1}{2}(13x^2 - 5x + 8)^{-1/2}(d/dx(13x^2 - 5x + 8)) \\ &= \frac{26x - 5}{2\sqrt{13x^2 - 5x + 8}} \end{aligned}$$

$\square$

2. Find the derivative of $f(x) = \dfrac{(x^3 + 4)^5}{(1 - 2x^2)^3}$

   Solution This will be a combination of quotient rule along with the chain rule.

$$\begin{aligned} f'(x) &= \frac{(1 - 2x^2)^3[d/dx(x^3 + 4)^5] - (x^3 + 4)^5[d/dx((1 - 2x^2)^3)]}{[(1 - 2x^2)^3]^2} \\ &= \frac{(1 - 2x^2)^3(5(x^3 + 4)^4(3x^2) - (x^3 + 4)^5(3(1 - 2x^2)^2(-4x)}{(1 - 2x^2)^6} \\ &= \frac{(1 - 2x^2)^3(15x^2)(x^3 + 4)^4 - (x^3 + 4)^5(-12x(1 - 2x^2)^2)}{(1 - 2x^2)^6} \\ &= \frac{\cancel{(1 - 2x^2)^2}[(1 - 2x^2)(15x^2)(x^3 + 4)^4 - (x^3 + 4)^5(-12x)}{\cancel{(1 - 2x^2)^2}(1 - 2x^2)^4} \\ &= \frac{(x^3 + 4)^4[(1 - 2x^2)(15x^2) - (x^3 + 4)(-12x)]}{(1 - 2x^2)^4} \\ &= \frac{(x^3 + 4)^4(3x)[(1 - 2x^2)(5x) - (x^3 + 4)(-4)]}{(1 - 2x^2)^4} \\ &= \frac{(x^3 + 4)^4(3x)(5x - 10x^3 + 4x^3 + 16)}{(1 - 2x^2)^4} \\ &= \frac{(3x)(x^3 + 4)^4(-6x^3 + 5x + 16)}{(1 - 2x^2)^4} \end{aligned}$$

$\square$

3. Find the derivative of $f(x) = \dfrac{x^2 + 4x}{(3x^3 + 2)^4}$

Solution First notice that

$$f(x) = \frac{x^2 + 4x}{v(u(x))} \text{ where } u(x) = 3x^3 + 2 \text{ and } v(x) = x^4$$

$$\begin{aligned}
f'(x) &= \frac{(3x^3 + 2)^4(d/dx(x^2 + 4x)) - (x^2 + 4x)(d/dx((3x^3 + 2)^4)}{((3x^3 + 2)^4)^2} \\
&= \frac{(3x^3 + 2)^4(2x + 4) - (x^2 + 4x)(v'(u(x))(u'(x)))}{(3x^3 + 2)^8} \\
&= \frac{(3x^3 + 2)^4(2x + 4) - (x^2 + 4x)(4(3x^3 + 2)^3(9x^2)}{(3x^3 + 2)^8} \\
&= \frac{(3x^3 + 2)^3((3x^3 + 2)(2x + 4) - 36x^2(x^2 + 4x))}{(3x^3 + 2)^8} \\
&= \frac{(3x^3 + 2)(2x + 4) - 36x^3(x^2 + 4x)}{(3x^3 + 2)^5}
\end{aligned}$$

□

Lecture 14: Derivatives of the Exponential and Logarithmic Functions

More Properties of Derivatives:

1. $\frac{d}{dx}(e^x) = e^x$

   Proof.
$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x\end{aligned}$$
□

2. $\frac{d}{dx}(a^x) = (\ln a)a^x$

   Proof.
$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{(\ln a)x})\\ &= e^{(\ln a)x}\cdot\frac{d}{dx}((\ln a)x)\\ &= e^{(\ln a)x}(\ln a)\\ &= a^x\ln a\end{aligned}$$
□

3. $\frac{d}{dx}(\log_a x) = \frac{d}{dx}(\log_a|x|) = \frac{1}{(\ln a)x}$

   Proof.
$$\frac{d}{dx}(a^{\log_a x}) = (\ln a)a^{\log_a x}\cdot\frac{d}{dx}(\log_a x)$$

but

$$\frac{d}{dx}(a^{\log_a x}) = \frac{d}{dx}(x) = 1$$

Thus

$$1 = (\ln a)a^{\log_a x} \cdot \frac{d}{dx}(\log_a x) \Leftrightarrow \frac{d}{dx}(\log_a x) = \frac{1}{(\ln a)x}$$

□

4. $\frac{d}{dx}(\ln x) = \frac{d}{dx}(\ln|x|) = \frac{1}{x}$

Examples:

1. Find the derivative of $f(x) = \log_a(-x)$

Solution

$$\begin{aligned} f'(x) &= \frac{d/dx(-x)}{(\ln a)(-x)} \\ &= \frac{-1}{(\ln a(-x)} \\ &= \frac{1}{(\ln a)x} \end{aligned}$$

□

2. Find the derivative of $f(x) = e^{1/(3+2x^2)}$

Solution

$$\begin{aligned} f'(x) &= e^{1/(3+2x^2)} \frac{d}{dx}\left(\frac{1}{3+2x^2}\right) \\ &= e^{1/(3+2x^2)} \frac{d}{dx}((3+2x^2)^{-1}) \\ &= e^{1/(3+2x^2)} \left(-(3+2x^2)^{-2} \frac{d}{dx}(3+2x^2)\right) \\ &= -e^{1/(3+2x^2)}(3+2x^2)^{-2}(4x) \\ &= -e^{1/(3+2x^2)} \frac{4x}{(3+2x^2)^2} \\ &= -\frac{4xe^{1/(3+2x^2)}}{(3+2x^2)^2} \end{aligned}$$

$\square$

3. Find the derivative of $f(x) = \ln\left((3x+1)^4(x^4+5x+7)\right)$

   Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(4\ln(3x+1) - \ln(x^4+5x+7)\right) \\ &= 4\frac{d/dx(3x+1)}{3x+1} - \frac{d/dx(x^4+5x+7)}{x^4+5x+7} \\ &= \frac{12}{3x+1} - \frac{4x^3+5}{x^4+5x+7} \end{aligned}$$

$\square$

4. Find the derivative of $f(x) = e^x \ln x$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^x)\ln x + e^x \frac{d}{dx}(\ln x) \\ &= \frac{e^x}{x} + e^x \ln x \end{aligned}$$

□

5. Find the derivative of $f(x) = x^2 e^{-x}$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2)e^{-x} + x^2 \frac{d}{dx}(e^{-x}) \\ &= 2xe^{-x} - e^{-x}x^2 \end{aligned}$$

□

6. Find the derivative of $f(x) = x^4 + 4^x$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^4) + \frac{d}{dx}(4^x) \\ &= 4x^3 + (\ln 4)4^x \end{aligned}$$

□

7. Find the derivative of $f(x) = x^2 \ln(2x) + x\ln(3x) + 4\ln x$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2)\ln(2x) + x^2\frac{d}{dx}(\ln(2x)) + \frac{d}{dx}(x)\ln(3x) + x\frac{d}{dx}(\ln(3x)) + 4\frac{d}{dx}(\ln x) \\ &= \frac{2x^2}{2x} + 2x\ln(2x) + \frac{3x}{3x} + \ln(3x) + \frac{4}{x} \\ &= x + 2x\ln(2x) + 1 + \ln(3x) + \frac{4}{x} \end{aligned}$$

□

8. Find the derivative of $f(x) = \log_2 x^2$

Solution By chain rule with $u(x) = x^2$ and $v(x) = \log_2 x$ we have,

$$\begin{aligned} f'(x) &= \frac{1}{(\ln 2)x^2} \frac{d}{dx}(x^2) \\ &= \frac{2}{x \ln 2} \end{aligned}$$

□

9. The population of a certain collection of rare Brazilian ants can be described as follows:

$$P(t) = (t + 100)\ln(t + 2)$$

where t denotes the time in days. How fast does the population grow on the second day and on the eighth day?

Solution

$$\begin{aligned} P'(t) &= \frac{d}{dx}(t + 100)\ln(t + 2) + (t + 100)\frac{d}{dx}(\ln(t + 2)) \\ &= \ln(t + 2) + \frac{t + 100}{t + 2} \end{aligned}$$

$$\begin{aligned} P'(2) &= \ln(2 + 2) + \frac{2 + 100}{2 + 2} \\ &= \ln 4 + \frac{102}{4} \\ &= \ln 4 + \frac{51}{2} \end{aligned}$$

$$\begin{aligned} P'(8) &= \ln(8 + 2) + \frac{8 + 100}{8 + 2} \\ &= \ln 10 + \frac{110}{10} \\ &= \ln 10 + \ln 11 \\ &= \ln 110 \end{aligned}$$

□

Lecture 15: Derivatives of Trigonometric Functions

Trigonometric Identities and Limits

- $\sin^2 x + \cos^2 x = 1$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

More Properties of Derivatives:

- $\frac{d}{dx}(\sin x) = \cos x$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= (\sin x)\cdot 0 + (\cos x)\cdot 1 \\
&= \cos x
\end{aligned}$$

□

- $\frac{d}{dx}(\cos x) = -\sin x$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) \\
&= \cos\left(\frac{\pi}{2} - x\right)\cdot\left(\frac{\pi}{2} - x\right) \\
&= -\cos\left(\frac{\pi}{2} - x\right) \\
&= -\left(\cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{pi}{2}\right)\sin x\right) \\
&= -(0\cdot\cos x + 1\cdot\sin x) \\
&= -\sin x
\end{aligned}$$

□

- $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{d/dx(\sin x)\cos x - \sin x(d/dx(\cos x))}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

□

- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{d/dx(\cos x)\sin x - \cos x(d/dx(\sin x))}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\sec^2 x
\end{aligned}$$

□

- $\frac{d}{dx}(\sec x) = \sec x \tan x$

Proof.

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}((\cos x)^{-1}) \\ &= -(\cos x)^{-2}\frac{d}{dx}(\cos x) \\ &= -(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{\sin x}{\cos x \cos x} \\ &= \sec x \tan x\end{aligned}$$

□

- $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Proof.

$$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx}((\sin x)^{-1}) \\ &= -(\sin x)^{-2}\frac{d}{dx}(\sin x) \\ &= -(\sin x)^{-2}\cos x \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\frac{\cos x}{\sin x \sin x} \\ &= -\csc x \cot x\end{aligned}$$

□

<u>Examples:</u>

1. Evaluate $\lim_{h\to 0} \frac{\sin\left(\pi/3 + h\right) - \sin \pi/3}{h}$

   Solution Notice that the limit is the exact definition for $\frac{d}{dx}(\sin x)$ evaluated at $\frac{\pi}{3}$. $(\sin x)' = \cos x$ so the limit is $\cos\frac{\pi}{3} = \frac{1}{2}$

□

2. Find the derivative of $f(x) = x\tan^3(2x)$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x)\tan^3(2x) + x\frac{d}{dx}(\tan^3(2x)) \\ &= \tan^3(2x) + x(3\tan^2(2x))(\sec^2(2x))(2) \\ &= \tan^3(2x) + 6x\tan^2(2x)\sec^2(2x) \end{aligned}$$

□

3. Find the derivative of $f(x) = \dfrac{\cos x}{1+\sin x}$

Solution

$$\begin{aligned} f'(x) &= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - 1}{(1+\sin x)^2} \\ &= \frac{-(\sin x + 1)}{(1+\sin x)^2} \\ &= \frac{-1}{1+\sin x} \end{aligned}$$

□

4. Find the derivative of $f(x) = e^{\sin x}$

Solution From the chain rule with $u(x) = \sin x$ and $v(x) = e^x$ we have $f'(x) = e^{\sin x}\cos x$

□

5. Find the derivative of $f(x) = \tan(\sin x)$

Solution Using chain rule with $u(x) = \sin x$ and $v(x) = \tan x$ we get that $f'(x) = \sec^2(\sin x)\cos x$

□

6. Let $f(x) = \ln\left|\tan^2 x\right|$. Find $f'(x)$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2\ln|\tan x|) \\ &= 2\left(\frac{1}{\tan x}\right)\frac{d}{dx}(\tan x) \\ &= 2\left(\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}\right) \\ &= \frac{2}{\sin x\cos x} \end{aligned}$$

$\square$

Lecture 16: Increasing and Decreasing Functions

Definition: If $f(x_1) < f(x_2)$ for $x_1 < x_2$ in the interval (a,b), then f is *increasing* on (a,b). If $f(x_1) > f(x_2)$ for $x_1 < x_2$ in the interval (a,b), then f is *decreasing* on (a,b)

Note: If we change the strict inequalities to non-strict inequalities, we say *non-decreasing* and *non-increasing* instead.

Example:

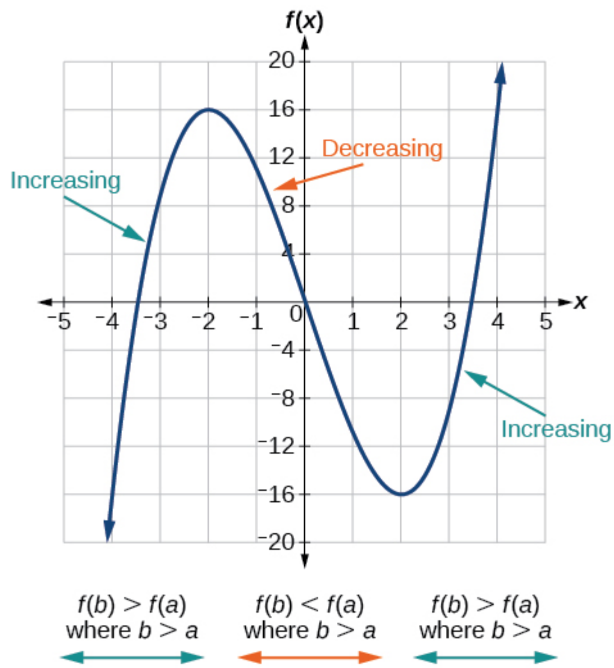


Figure 23

Relationship With Derivatives and Monotonicity

- If $f'(x) > 0$ on (a,b), then f is increasing on (a,b)
- If $f'(x) < 0$ on (a,b), then f is decreasing on (a,b)
- If $f'(x) = 0$ on (a,b), then f is constant on (a,b)

Definition: A number x a which $f'(x) = 0$ or $f'(x)$ is undefined is called a *critical number*. If x is a critical number and $f(x)$ is defined, then $f(x)$ is called a *critical value*.

LECTURE 16: INCREASING AND DECREASING FUNCTIONS

The First Derivative Test

To find the intervals of increase/decrease of a function complete the following steps:

(i) Find all critical numbers.

(ii) Plot these critical numbers on a number line.

(iii) Determine the sign of $f'(x)$ on each interval

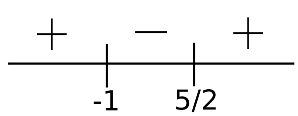
Examples:

1. Let $f(x) = 4x^3 - 9x^2 - 30x + 6$. Find critical numbers of f and determine the intervals on which f is increasing and on which f is decreasing.

   Solution

$$
\begin{aligned}
f'(x) = 0 &\Leftrightarrow 12x^2 - 18x - 30 = 0 \\
&\Leftrightarrow 6(2x^2 - 3x - 5) = 0 \\
&\Leftrightarrow x = \frac{5}{2}, -1
\end{aligned}
$$

   Plotting these on a number line we get:



   Thus we have that f is increasing on $(-\infty, -1) \cup \left(\frac{5}{2}, \infty\right)$ and decreasing on $\left(-1, \frac{5}{2}\right)$

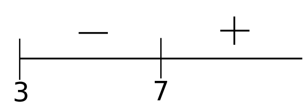
□

2. Let $f(x) = x - 4\ln(3x - 9)$. Find critical numbers of f and determine the intervals on which f is increasing and on which f is decreasing.

Solution

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 1 - 4\left(\frac{1}{3x-9}\right)\frac{d}{dx}(3x-9) = 0 \\ &\Leftrightarrow 1 - \frac{12}{3x-9} = 0 \\ &\Leftrightarrow 1 - \frac{4}{x-3} = 0 \\ &\Leftrightarrow \frac{x-3-4}{x-3} = 0 \\ &\Leftrightarrow x - 7 = 0 \\ &\Leftrightarrow x = 7 \end{aligned}$$

Plotting these on a number line we get:



Note that the number line starts at 3 because the function is not defined for $x \leq 3$. Thus we have f is increasing on $(7, \infty)$ and decreasing on $(3, 7)$.

□

3. Determine where $f(x) = \ln(x^2 + 1)$ is increasing and decreasing.

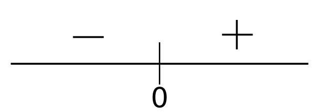
Solution

$$f'(x) = \frac{1}{x^2+1}\left(\frac{d}{dx}(x^2+1)\right) = \frac{2x}{x^2+1}$$

$f'(x)$ is never undefined and

$$f'(x) = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$$

So our only critical number is 0. Plotting it on a number line and testing the intervals we get



Thus f is decreasing on $(-\infty, 0)$ and f is increasing on $(0.\infty)$

□

4. Determine where $f(x) = x^2 e^x$ is increasing and decreasing.

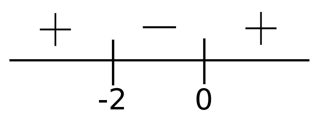
Solution

$$f'(x) = x^2 e^x + 2xe^x = e^x(x^2 + 2x)$$

$f'(x)$ is never undefined.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow x^2 + 2x = 0 \\ &\Leftrightarrow x(x+2) = 0 \\ &\Leftrightarrow x = 0, x = -2 \end{aligned}$$

So our critical numbers are 0 and -2. Testing our intervals we have



Thus f is increasing on $(-\infty, -2)$, $(0, \infty)$ and f is decreasing on $(-2, 0)$.

□

Lecture 17: Relative (or Local) Extrema

In many cases, it is important to find where a function attains its largest or smallest value. For example, for an equation describing the number of flue cases during an epidemic, you might want to know when the epidemic is expected to peak.

Definition: Let c be in the domain of f. $f(c)$ is a *relative/local maximum* if there is an interval (a,b) containing c such that $f(x) \geq f(x)$ for all x in (a,b). $f(c)$ is a *relative/local minimum* if $f(c) \leq f(x)$ for all x in (a,b). The word *extremum* (pl. extrema) means either a relative max or a relative min. If a function has a relative extremum at c then we mean $f(x)$ is either a relative max or relative min.

Theorem (First Derivative Test Continued):

- If $f(c)$ is a relative extremum, then c is a critical number or an endpoint of the domain.
- If $f'(x)$ switches from positive to negative at c then $f(c)$ is a relative maximum.
- If $f'(x)$ switches from negative to positive at c then $f(c)$ is a relative minimum.

Examples:

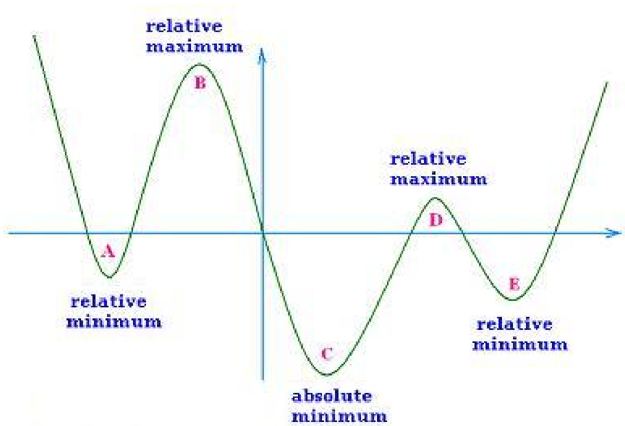
1. 

Figure 24

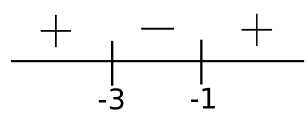
2. Let $f(x) = x^3 + 6x^2 + 9x - 8$. Find the values of all relative extrema and the corresponding critical numbers.

Solution

$$f'(x) = 3x^2 + 12x + 9 \text{ is never undefined}$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 3x^2 + 12x + 9 = 0 \\ &\Leftrightarrow 3(x^2 + 4x + 3) = 0 \\ &\Leftrightarrow 3(x+3)(x+1) = 0 \\ &\Leftrightarrow x = -3, -1 \end{aligned}$$

Plotting these on a number line and testing the intervals we get:



So there is a relative maximum at $x = -3$ and a relative minimum at $x = -1$

$$\begin{aligned} f(-3) &= (-3)^3 + 6(-3)^2 + 9(-3) - 8 \\ &= -27 - 26 + 9 - 27 + 8 \\ &= -8 \end{aligned}$$

$$\begin{aligned} f(-1) &= (-1)^3 + 6(-1)^2 - 9 - 8 \\ &= -1 + 6 - 9 - 8 \\ &= -12 \end{aligned}$$

Thus $f(-3) = -8$ is a relative maximum and $f(-1) = -12$ is a relative minimum.

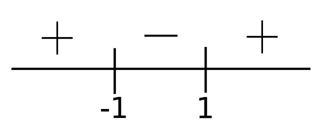
□

3. Find all relative extrema of $f(x) = \dfrac{x^3}{3} - x$

Solution

$$f'(x) = x^2 - 1 = (x+1)(x-1)$$

$f'(x)$ is never undefined and $f'(x) = 0 \Leftrightarrow x = \pm 1$ so we have two critical points. Testing the intervals we get



Thus there is a relative maximum at $x = -1$ and a relative minimum at $x = 1$

$$\begin{aligned} f(-1) &= \frac{(-1)^3}{3} - (-1) \\ &= -\frac{1}{3} + 1 \\ &= -\frac{1}{3} + \frac{3}{3} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} f(1) &= \frac{(1)^3}{3} - (1) \\ &= \frac{1}{3} - 1 \\ &= \frac{1}{3} - \frac{3}{3} \\ &= -\frac{2}{3} \end{aligned}$$

So the relative maximum is $\frac{2}{3}$ and the relative minimum is $-\frac{2}{3}$

□

4. Find all relative extrema of $f(x) = \sin^2\theta$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$

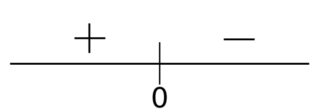
Solution

$$f'(x) = 2\sin\theta\cos\theta$$

On the given interval

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow \sin\theta = 0, \cos\theta = 0 \\ &\Rightarrow x = 0 \ \left(\text{the other values are outside of } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \end{aligned}$$

So we have one critical point of 0. Testing the intervals we get



Thus there is a relative max at $x = 0$ and since $f(0) = 0$, the relative max is 0.

□

5. Find all relative extrema of $f(x) = \dfrac{x}{x^2+4}$

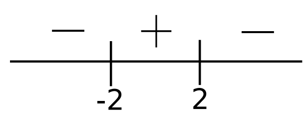
Solution

$$\begin{aligned} f'(x) &= \frac{(x^2+4)(d/dx(x)) - x(d/dx(x^2+4))}{(x^2+4)^2} \\ &= \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} \\ &= \frac{x^2+4-2x^2}{(x^2+4)^2} \\ &= \frac{-x^2+4}{(x^2+4)^2} \end{aligned}$$

It is never undefined because the denominator is always positive.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow -x^2+4 = 0 \\ &\Leftrightarrow x^2 - 4 = 0 \\ &\Leftrightarrow (x-2)(x+2) = 0 \\ &\Leftrightarrow x = \pm 2 \end{aligned}$$

Thus we have two critical values. Testing the intervals we have



Thus we have a relative minimum at $x = -2$ and a relative maximum at $x = 2$

$$\begin{aligned} f(-2) &= \frac{-2}{(-2)^2+4} \\ &= -\frac{2}{4+4} \\ &= -\frac{2}{8} \\ &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} f(2) &= \frac{2}{(2)^2+4} \\ &= \frac{2}{4+4} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

So the relative minimum is $-\frac{1}{4}$ and the relative maximum is $\frac{1}{4}$

□

Lecture 18: Higher Derivatives, Convexity/Concavity

Given a function f, we know that $f'(x)$ gives the instantaneous rate of change of f at x. We could also ask how fast $f'(x)$ changes. For this, we would have to take the derivative of $f'(x)$, called the *second derivative*. Similarly, we could compute the third, fourth, fifth, etc. derivatives of f.

Notation:

- Second Derivative → $f''(x)$ or $\dfrac{d^2 f}{dx^2}$
- Third Derivative → $f'''(x)$ or $\dfrac{d^3 f}{dx^3}$
- Nth Derivative → $f^{(n)}(x)$ or $\dfrac{d^n f}{dx^n}$

Examples:

1. Suppose $f(x) = 30 - x\ln x$. Find $f'''(x)$.

   Solution

$$\begin{aligned} f'(x) &= -\left(\left(\frac{d}{dx}(x)\right)\ln x + x\left(\frac{d}{dx}(\ln x)\right)\right) \\ &= -(\ln x + 1) \\ &= -\ln x - 1 \end{aligned}$$

$$\begin{aligned} f''(x) &= -\frac{1}{x} \\ &= -x^{-1} \end{aligned}$$

$$\begin{aligned} f'''(x) &= x^{-2} \\ &= \frac{1}{x^2} \end{aligned}$$

□

2. Suppose $f(x) = \sin(x^3)$. Find $f''(x)$.

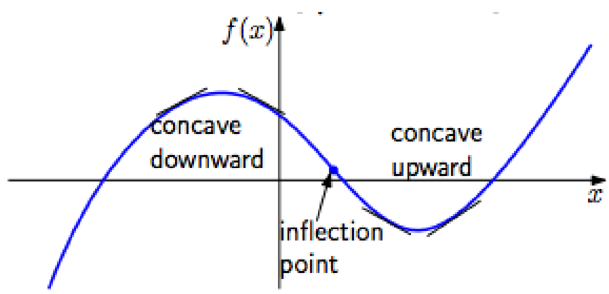
Solution

$$\begin{aligned} f'(x) &= \cos(x^3)\frac{d}{dx}(x^3) \\ &= 3x^2\cos(x^3) \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{d}{dx}(3x^2)\cos(x^3) + 3x^2\frac{d}{dx}(\cos(x^3)) \\ &= 6x\cos(x^3) + 3x^2(-\sin(x^3)\frac{d}{dx}(x^3)) \\ &= 6x\cos(x^3) + 3x^2(-\sin(x^3)(3x^2)) \\ &= 6x\cos(x^3) - 9x^4\sin(x^3) \end{aligned}$$

□

Definition: If $f''(x) > 0$ on (a,b), then f is *concave up* (i.e. the graph has a $\cup$ shape) on (a,b). If $f''(x) < 0$ on (a,b), then f is *concave down* (i.e. the graph has a $\cap$ shape) on (a,b). A point at which f changes concavity is called an *inflection point*.



Second Derivative Test For Concavity

(i) Find $f''(x)$

(ii) Find x values at which $f''(x) = 0$ or is undefined (Important Numbers)

(iii) Plot the x value and test the intervals for $f''(x)$

Examples:

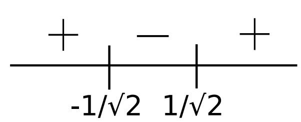
1. Let $f(x) = 2e^{-x^2}$. Find concavity intervals and inflection points, if any.

Solution

$$\begin{aligned} f'(x) &= 2e^{-x^2}\frac{d}{dx}(-x^2) \\ &= 2e^{-x^2}(-2x) \\ &= -4xe^{-x^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= \left(\frac{d}{dx}(-4x)\right)e^{-x^2} - 4x\left(\frac{d}{dx}(e^{-x^2})\right) \\ &= -4e^{-x^2} - 4xe^{-x^2}\left(\frac{d}{dx}(-x^2)\right) \\ &= -4e^{-x^2} - 4xe^{-x^2}(-2x) \\ &= -4e^{-x^2} + 8x^2e^{-x^2} \\ &= -4e^{-x^2}(1-2x^2) \end{aligned}$$

Thus our important points are $x = \pm\frac{1}{\sqrt{2}}$. Testing the intervals we get



Thus f is concave up on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

□

2. Find the intervals on which $f(x) = x^3 - 3x^2 + x - 2$ is concave up and concave down.

Solution

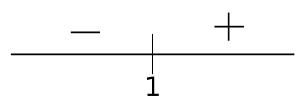
$$f'(x) = 3x^2 - 6x + 1$$

$$\begin{aligned} f''(x) &= 6x - 6 \\ &= 6(x-1) \end{aligned}$$

$f''(x)$ is defined for all x values and

$$f''(x) = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$$

Testing the intervals we get



Thus f is concave down on $(-\infty, 0)$ and f is concave up on $(0, \infty)$. (Note: this shows that there is an inflection points at $x = 0$)

□

The Second Derivative Test For Relative Extrema

(i) Find critical points, c, of f using the First Derivative Test

(ii) Plug the critical points into $f''(x)$

(iii) If $f''(c) > 0$, then $f(c)$ is a relative minimum. If $f''(c) < 0$, then $f(c)$ is a relative maximum. Otherwise, the test in inconclusive.

Examples:

1. Let $f(x) = 3x^3 - 3x^2 + 1$. Find all relative extrema.

Solution

$$\begin{aligned} f'(x) &= 9x^2 - 6x \\ &= 3x(3x-2) \\ &\text{so critical numbers are } x = 0, \frac{2}{3} \end{aligned}$$

$$f''(x) = 18x - 6$$

$$f''(0) = -9$$

$$f''\left(\frac{2}{3}\right) = 6$$

Thus $f(0) = 1$ is a relative maximum and

$$\begin{aligned} f\left(\frac{2}{3}\right) &= 3\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right)^2 + 1 \\ &= 3\left(\frac{8}{27}\right) - 3\left(\frac{4}{9}\right) + 1 \\ &= \frac{8}{9} - \frac{4}{3} + 1 \\ &= \frac{8}{9} - \frac{12}{9} + \frac{9}{9} \\ &= \frac{8-12+9}{9} \\ &= \frac{5}{9} \end{aligned}$$

gives the relative minimum.

□

2. Use the 2nd Derivative Test for Relative Extrema on $f(x) = 3x^5 - 5x^3 + 3$

Solution

$$\begin{aligned} f'(x) &= 15x^4 - 15x^2 \\ &= 15x^2(x^2 - 1) \\ &= 15x^2(x-1)(x+1) \\ &\text{so our critical points are 0 and } \pm 1 \end{aligned}$$

$$\begin{aligned} f''(x) &= 60x^3 - 30x \\ &= 30x(2x^2 - 1) \end{aligned}$$

$$f''(0) = 0$$

$$f''(-1) = -30$$

$$f''(1) = 30$$

Thus the test in inconclusive for $x = 0$, (using the First Derivative Test will tell us that there is no relative extremum at this point), there is a relative max of $f(-1) = 5$, and there is a relative min of $f(1) = 1$.

□

Lecture 19: Curve Sketching

Steps To Sketch a Curve

(i) Find the domain of f

(ii) Find the x and y intercepts

(iii) Take the one sided limits at the end points of the domain.

(iv) Using the limits in step 3, find vertical and horizontal asymptotes; if $\lim_{x\to\pm\infty} f(x) = c$, then $y = c$ is a horizontal asymptote. If $\lim_{x\to c} f(x) = \pm\infty$, then $x = c$ is a vertical asymptote. Alternatively, vertical asymptotes occur when a function is undefined.

(v) Check for symmetry. If $f(-x) = f(x)$, then the graph has y-axis symmetry. If $f(-x) = -f(x)$, then the graph has origin symmetry.

(vi) Use the First Derivative Test to find critical points, relative extrema, and intervals of increase/decrease.

(vii) Use the Second Derivative Test to find inflection points and intervals of concavity.

(viii) Graph using the information found in the previous steps.

Examples:

1. Sketch the graph of $f(x) = -3x^3 + 6x^2 - 4x - 1$

   Solution

   (i) No restrictions so the domain is $(-\infty, \infty)$

   (ii) y-intercept: $f(0) = -1$

   x-intercepts:

$$f(x) = 0 \Leftrightarrow -3x^3 + 6x^2 - 4x - 1 = 0 \Leftrightarrow \text{NOPE! TOO HARD}$$

   (iii)

$$\lim_{x \to -\infty} f(x) = \infty \text{ and } \lim_{x \to \infty} f(x) = -\infty$$

   (iv)

$$\begin{aligned} f(-x) &= -3(-x)^3 + 6(-x)^2 - 4(-1) - 1 \\ &= 3x^3 + 6x^2 + 4 - 1 \end{aligned}$$

   So no symmetry.

   (v)

$$f'(x) = -9x^2 + 12x - 4$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow -9x^2 + 12x + 4 = 0 \\ &\Leftrightarrow -(9x^2 - 12x - 4) = 0 \\ &\Leftrightarrow -(3x - 2)^2 = 0 \\ &\Leftrightarrow x = \frac{2}{3} \end{aligned}$$

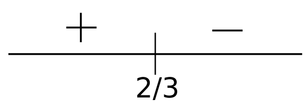
   Note that since $f'(x)$ is always negative, f is always decreasing.

$$
\begin{aligned}
f\left(\frac{2}{3}\right) &= -3\left(\frac{2}{3}\right)^3 + 6\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 1 \\
&= -3\left(\frac{8}{27}\right) + 6\left(\frac{4}{9}\right) - 4\left(\frac{2}{3}\right) - 1 \\
&= -\frac{8}{9} + \frac{\cancel{8}}{\cancel{3}} - \frac{\cancel{8}}{\cancel{3}} - 1 \\
&= -\frac{8}{9} - \frac{9}{9} \\
&= -\frac{17}{9}
\end{aligned}
$$

(vi)

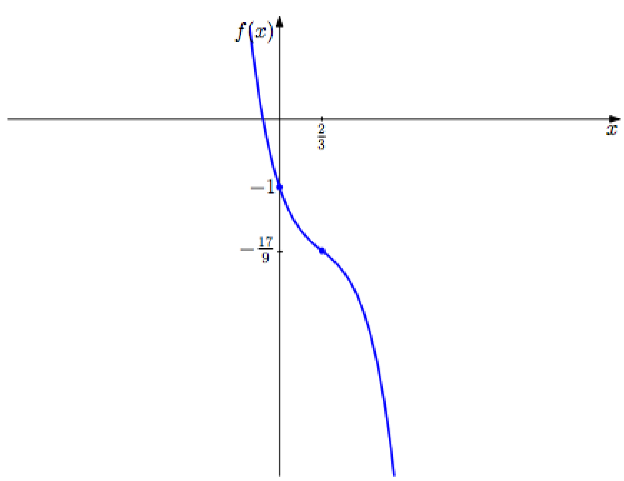
$$f''(x) = -18x + 12 = -6(3x - 2)$$

Thus $f''(x) = 0 \Leftrightarrow x = \frac{2}{3}$. Testing the intervals we get



So f is concave up on $\left(-\infty, \frac{2}{3}\right)$ and concave down on $\left(\frac{2}{3}, \infty\right)$

(vii) Putting this together we have



□

2. Sketch $f(x) = \dfrac{-x+4}{x+2}$

Solution

(i) f is not defined at $x = -2$ so the domain is $(-\infty, -2) \bigcup (-2, \infty)$

(ii) <u>y-intercept:</u> $f(0) = 2$

<u>x-intercept:</u>

$$\begin{aligned} f(x) = 0 &\Leftrightarrow \frac{-x+4}{x+2} = 0 \\ &\Leftrightarrow -x+4 = 0 \\ &\Leftrightarrow x = 4 \end{aligned}$$

(iii)

$$\begin{aligned} \lim_{x\to-\pm\infty} f(x) &= \lim_{x\to-\pm\infty} \frac{-x+4}{x+2} \\ &= \lim_{x\to-\pm\infty} \frac{-1+4/x}{1+2/x} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to-2^-} f(x) &= \lim_{x\to-2^-} \frac{-x+4}{x+2} \\ &= -\infty \end{aligned}$$

$$\begin{aligned} \lim_{x\to-2^+} f(x) &= \lim_{x\to-2^+} \frac{-x+4}{x+2} \\ &= \infty \end{aligned}$$

(iv) From the previous step we have that there is a vertical asymptote of $x = -2$ and a horizontal asymptote of $y = -1$

(v)

$$\begin{aligned} f(-x) &= \frac{-(-x)+4}{-x+2} \\ &= \frac{x+4}{-x+2} \end{aligned}$$

Thus there are no symmetries.

(vi)

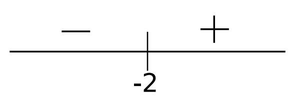
$$\begin{aligned} f'(x) &= \frac{(x+2)(-1) - (-x+4)(1)}{(x+2)^2} \\ &= \frac{-\cancel{x} - 2 + \cancel{x} - 4}{(x+2)^2} \\ &= -\frac{6}{(x+2)^2} \end{aligned}$$

Thus the critical number is -2 which is not in the domain. Notice that for every x value in the domain, $f'(x) < 0$. Thus f is decreasing everywhere in its domain.

(vii)

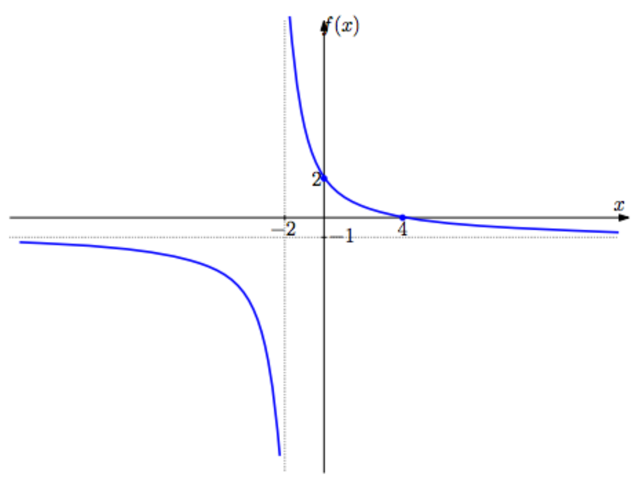
$$\begin{aligned} f''(x) &= \frac{d}{dx}(-6(x+2)^{-2}) \\ &= -6(-2)(x+2)^{-3} \\ &= \frac{12}{(x+2)^3} \end{aligned}$$

It is never zero but doesn't exist at $x = -2$. Testing the intervals we have:



Thus f is concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$.

(viii) Putting this together we get



□

3. Sketch $f(x) = \dfrac{1}{x^2 + 4}$

Solution

(i) There are no restrictions so the domain is $(-\infty, \infty)$

(ii) <u>y-intercept:</u> $f(0) = \dfrac{1}{4}$

No x-intercept.

(iii)

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{1}{x^2+4}\\ &= \lim_{x\to-\infty} \frac{1/x^2}{1+4/x^2}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1}{x^2+4}\\ &= \lim_{x\to\infty} \frac{1/x^2}{1+4/x^2}\\ &= 0\end{aligned}$$

(iv) From the previous step we have no vertical asymptote and a horizontal asymptote of $x = 0$

(v)

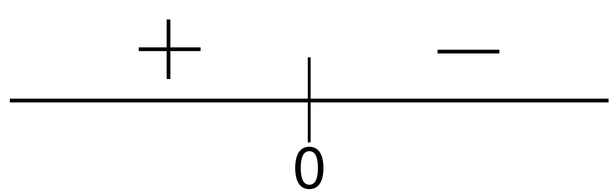
$$\begin{aligned}f(-x) &= \frac{1}{(-x)^2+4}\\ &= \frac{1}{x^2+4}\\ &= f(x)\end{aligned}$$

Thus the graph is symmetrical about the y-axis.

(vi)

$$\begin{aligned}f'(x) &= \frac{d}{dx}((x^2+4)^{-1})\\ &= -(x^2+4)^{-2}(2x)\\ &= -\frac{2x}{(x^2+4)^2}\end{aligned}$$

$f'(x)$ is never undefined and is 0 when $x = 0$. Testing the intervals we have



Thus f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. This says that $f(0) = \dfrac{1}{4}$ is a relative maximum.

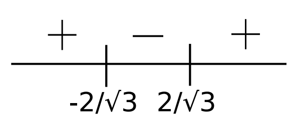
(vii)

$$\begin{aligned} f''(x) &= -\frac{(x^2+4)^2(2) - (2x)(2(x^2+4)(2x))}{((x^2+4)^2)^2} \\ &= -\frac{2(x^4+8x^2+16) - 8x^2(x^2+4)}{(x^2+4)^4} \\ &= -\frac{2x^4+16x^2+32-8x^4-32x^2}{(x^2+4)^4} \\ &= -\frac{-6x^4-32x^2+32}{(x^2+4)^4} \\ &= -\frac{-2(3x^4+16x^2-16)}{(x^2+4)^4} \\ &= -\frac{-2(x^2+4)(3x^2-4)}{(x^2+4)^4} \\ &= \frac{-2(3x^2-4)}{(x^2+4)^3} \end{aligned}$$

It is never undefined and

$$f''(x) = 0 \Leftrightarrow 3x^2 - 4 = 0 \Leftrightarrow x = \pm\frac{2}{\sqrt{3}}$$

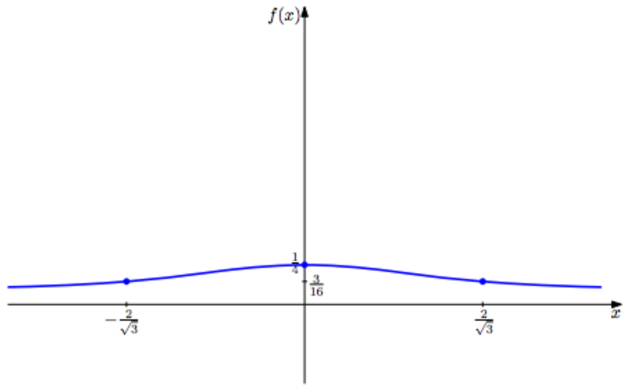
Testing our intervals we get



So f is concave up on $\left(-\infty, -\dfrac{2}{\sqrt{3}}\right) \cup \left(\dfrac{2}{\sqrt{3}}\right)$ and concave down on $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$. Thus there are inflection points at $x = \pm\dfrac{2}{\sqrt{3}}$

$$\begin{aligned} f\left(\pm\frac{2}{\sqrt{3}}\right) &= \frac{1}{(\pm 2/\sqrt{3})^2 + 4} \\ &= \frac{1}{4/3+4} \\ &= \frac{3}{4+12} \\ &= \frac{3}{16} \end{aligned}$$

(viii) Putting this all together we get



□

4. Graph $f(x) = \dfrac{4x}{x^2+1}$

Solution

(i) There are no restrictions so the domain is $(-\infty, \infty)$

(ii) $f(0) = 0$ so the y-intercept is 0. $f(x) = 0 \Leftrightarrow 4x = 0 \Leftrightarrow x = 0$ so the x-intercept is also 0.

(iii) $\lim_{x \to \pm\infty} f(x) = 0$

(iv) From (iii) we have a horizontal asymptote of $y = 0$

(v) $f(-x) = \dfrac{-4x}{(-x)^2+1} = -\dfrac{4x}{x^2+1} = -f(x)$ so there is origin symmetry.

(vi)

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2} \\ &= \frac{4x^2+4-8x^2}{(x^2+1)^2} \\ &= \frac{4-4x^2}{(x^2+1)^2} \\ &= \frac{-4(x^2-1)}{(x^2+1)^2} \\ &= \frac{-4(x-1)(x+1)}{(x^2+1)^2} \end{aligned}$$

So our critical points are ± 1. Testing the intervals we have f is increasing on $(-1,1)$ and decreasing on $(-\infty,-1),(1,\infty)$. Thus there is a relative max at $f(1)=2$ and a relative min at $f(-1)=-2$

(vii)

$$\begin{aligned}
f''(x) &= \frac{(x^2+1)^2(-8x)-(4-4x^2)(2(x^2+1)(2x)}{((x^2+1)^2)^2} \\
&= \frac{(x^4+2x^2+1)(-8x)-(4-4x^2)(4x^3+4x)}{(x^2+1)^4} \\
&= \frac{-8x^5-16x^3-8x-(16x^3+16x-16x^5-16x^3)}{(x^2+1)^4} \\
&= \frac{-8x^5-16x^3-8x-\cancel{16x^3}-16x+16x^5+\cancel{16x^3}}{(x^2+1)^4} \\
&= \frac{8x^5-16x^3-24x}{(x^2+1)^4} \\
&= \frac{8x(x^4-2x^2-3)}{(x^2+1)^4} \\
&= \frac{(8x)(x^2+1)(x^2+3)}{(x^2+1)^4} \\
&= \frac{8x(x^2-3)}{(x^2+1)^3}
\end{aligned}$$

$f''(x)$ is always defined at $f''(x)=0 \Leftrightarrow 8x(x^2-3)=0 \Leftrightarrow x=0,\pm\sqrt{3}$. Testing the intervals we get that f is concave up on $(-\sqrt{3},0)$, $(\sqrt{3},\infty)$ and concave down on $(-\infty,-\sqrt{3}),(0,\sqrt{3})$ giving us inflection points at $x=\pm\sqrt{3}$ and $x=0$.

(viii) Putting this all together we get:

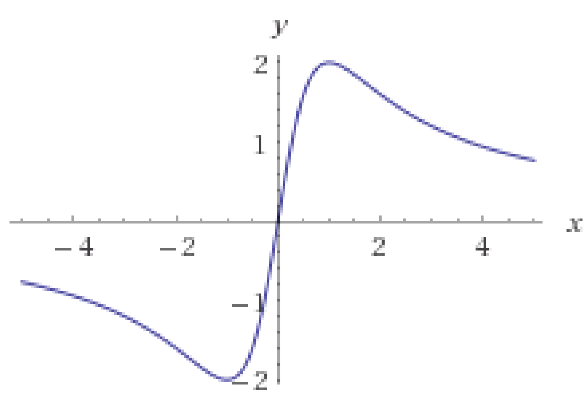


Figure 25

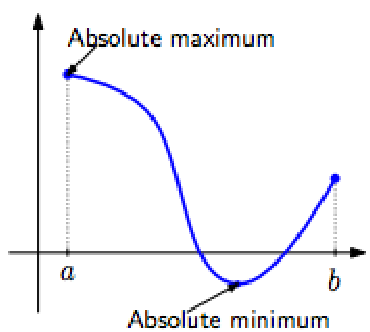
□

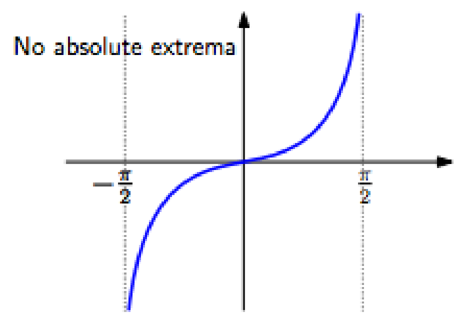
Lecture 20: Absolute Extrema

In many cases we want to find the largest or smallest value of a function. To find this, we need the following concepts.

Definition: If there exists a c in the domain of a function f such that $f(x) \leq f(x)$ for all x, then $f(c)$ is called the *absolute maximum* of f. If $f(x) \geq f(c)$ for all x then $f(c)$ is called the *absolute minimum* of f.

Theorem: Suppose that a function f is continuous on a closed interval $[a, b]$. Then f has both the absolute maximum and the absolute minimum on $[a, b]$.





Finding Absolute Extrema on a Closed Interval $[a, b]$

Step 1: Find all relative extrema.
Step 2: Find $f(a)$ and $f(b)$.
Step 3: Compare the values of the relative extrema with the values at the end points. The lowest is the absolute min and the highest is the absolute max.

Examples:

1. Find the absolute extrema of $f(x) = \dfrac{1-x}{3+x}$ on $[0, 3]$

   Solution

$$f'(x) = \frac{(3+x)(-1) - (1-x)}{(3+x)^2}$$

$$= \frac{-3 - \cancel{x} - 1 + \cancel{x}}{(3+x)^2}$$

$$= -\frac{4}{(3+x)^2}$$

The only critical number is −3 but its not in the restricted interval so there are no relative extrema in $[0, 3]$

$$f(0) = \frac{1}{3} \text{ and } f(3) = -\frac{1}{3}$$

Thus the absolute maximum is $\frac{1}{3}$ at $x = 0$ and the absolute minimum is $-\frac{1}{3}$ at $x = 3$

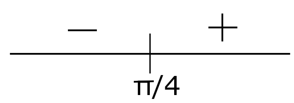
□

2. Find absolute extrema of $f(x) = \tan x - 2x$ on $\left[0, \frac{\pi}{3}\right]$

Solution

$$f'(x) = \sec^2 x - 2$$

For the restricted interval $\sec x = \pm\sqrt{2} \Rightarrow x = \frac{\pi}{4}$. Testing the intervals we have



Thus we have a relative minimum at $\frac{\pi}{4}$

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \tan\left(\frac{\pi}{4}\right) - 2\left(\frac{\pi}{4}\right) \\ &= 1 - \frac{\pi}{2} \\ &\approx -0.57 \end{aligned}$$

$$\begin{aligned} f(0) &= \tan(0) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= \tan\left(\frac{\pi}{3}\right) - \frac{2\pi}{3} \\ &= \sqrt{3} - \frac{2\pi}{3} \\ &\approx -0.36 \end{aligned}$$

Thus $f(0) = 0$ is the absolute maximum and the absolute minimum is $f\left(\frac{\pi}{4}\right) = -0.57$

□

3. Find the absolute extrema of $f(x) = 2x^3 + 3x^2 - 12x + 4$ on $[-4, 2]$

Solution $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$ so our critical points are -2 and 1. Using the first derivative test we get that there is a relative max at $x = -2$ and a relative min at $x = 1$. $f(-2) = 24$, $f(1) = -3$, $f(-4) = -28$, and $f(2) = 8$. The lowest of these values is -28 so our absolute minimum is $f(-4) = -28$. The largest is 24 so our absolute maximum is $f(-2) = 24$.

□

Theorem: Suppose that a function f is continuous on an interval I and has exactly one critical number $x = c$ inside I. Then if f has a relative minimum at c, the $f(x)$ is an absolute maximum of f on I and if f has a relative minimum at c, then $f(c)$ is the absolute minimum.

LECTURE 20: ABSOLUTE EXTREMA

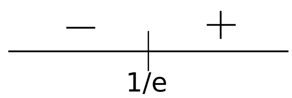
Examples:

1. Let $f(x) = x \ln x$. Find the absolute minimum of f on its domain The domain is $(0, \infty)$.

$$f'(x) = \ln x + 1$$

$$f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = \frac{1}{e}$$

Testing the intervals we have



Thus there is a relative minimum at $x = \frac{1}{e}$ and since this is the only critical point, that means that it is also the absolute minimum.

$$f(e^{-1}) = e^{-1} \ln(e^{-1}) = -e^{-1}$$

2. Let $f(x) = \frac{\ln x}{x^2}$. Find the absolute extrema of f on $[1, 4]$

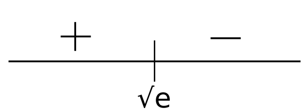
Solution

$$\begin{aligned} f'(x) &= \frac{x^2(1/x) - (\ln x)(x^2)}{(x^2)^2} \\ &= \frac{x - x^2 \ln x}{x^4} \\ &= \frac{x(x - x \ln x)}{x^4} \\ &= \frac{x - \ln x}{x^3} \end{aligned}$$

$f'(x)$ is undefined for $x = 0$ but that is not in the domain (or restricted interval)

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 1 - 2 \ln x = 0 \\ &\Leftrightarrow 1 = 2 \ln x \\ &\Leftrightarrow \ln x = \frac{1}{2} \\ &\Rightarrow x = e^{1/2} = \sqrt{e} \end{aligned}$$

Testing the intervals we have



So there is a relative maximum and thus an absolute maximum at $x = \sqrt{e}$

$$f(\sqrt{e}) = \frac{\ln e}{\sqrt{e}^2} = \frac{1}{e}$$

To find the absolute minimum we need to check the end points.

$$f(1) = \frac{\ln 1}{1^2} = 0 \text{ and } f(4) = \frac{\ln 4}{16} > 0$$

Thus the absolute minimum is $f(1) = 0$.

$\square$

Lecture 21: Applications of Extrema

Solving Applied Optimization Problems

(i) Determine which variable you want to make independent.

(ii) Determine the quantity that you want to be minimized or maximized. Write it as a function of your independent variable. (For the next steps we'll assume that your independent variable is called x and the function is $f(x)$.

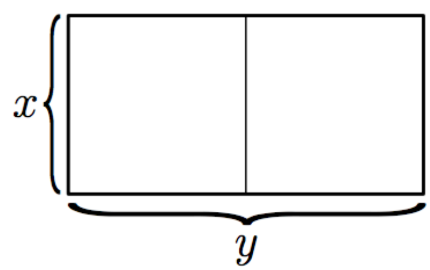
(iii) Find the domain of $f(x)$

(iv) Find the desired absolute extremum of f on it's domain.

Examples:

1. A farmer is constructing a rectangular pen with one additional fence across its width. Find the maximum area that can be enclosed with 2400 m of fencing.

   Solution Our picture looks like



   (i) The enclosed area is $A = xy$ and since we have 2400 m of fencing and choosing x as our independent variable we have

   $$3x + 2y = 2400 \Leftrightarrow y = \frac{1}{2}(2400 - 3x)$$

   (ii) Subbing this into our equation for area we have

   $$A(x) = \frac{x}{2}(2400 - 3x) = 1200x - \frac{3}{2}x^2$$

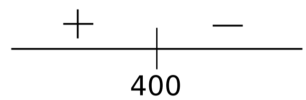
   (iii) x is a length so it must be positive and since the total amount of fencing is 2400 it cannot be bigger than that so we can consider the interval $(0, 2400)$.

(iv)

$$A'(x) = 1200 - 3x$$

$$A'(x) = 0 \Leftrightarrow 1200 - 3x = 0 \Leftrightarrow x = 400$$

Testing the intervals we have



Thus the absolute maximum occurs when $x = 400$ and

$$A(400) = 1200(400) - \frac{3}{2}(400)^2 = 240,000$$

So the maximum area is $240,000m^2$.

□

2. Suppose that if in a given year the population of some species is S, then the next years population is given by

$$f(S) = \frac{25S}{2+S}$$

where S is measured in thousands of individuals. So, every year we get a surplus of $f(S) - S$ salmon, which we can harvest without reducing the initial population. What should be the value of S so that our harvest is the largest possible?

Solution Our function for harvest is given by

$$h(S) = f(S) - S = \frac{25S}{2+S} - S$$

Since population cannot be negative, our interval is $[0, \infty)$

$$\begin{aligned} h'(S) &= \frac{(2+S)(25) - (25S)}{(2+S)^2} - 1 \\ &= \frac{50 + \cancel{25S} - \cancel{25S}}{(2+S)^2} - 1 \\ &= \frac{50}{(2+S)^2} - \frac{(2+S)^2}{(2+S)^2} \\ &= \frac{50 - (2+S)^2}{(2+S)^2} \end{aligned}$$

It is undefined at $S = -2$ but population can't be negative so we ignore this.

$$\begin{aligned} h'(S) = 0 &\Leftrightarrow 50 - (2+S)^2 = 0 \\ &\Leftrightarrow 50 = (2+S)^2 \\ &\Leftrightarrow 2 + S = \pm\sqrt{50} \\ &\Leftrightarrow S = \pm\sqrt{50} - 2 \end{aligned}$$

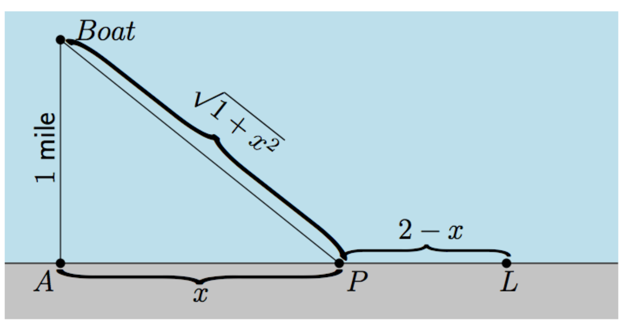
We can ignore the negative solution so our one critical point is $\sqrt{50} - 2$. Testing the intervals we see that there is a relative maximum thus it must be our absolute maximum.

$$h(\sqrt{50} - 2) = \frac{25(\sqrt{50} - 2)}{\sqrt{50}} - (\sqrt{50} - 2) \approx 12.8 \text{ salmon}$$

□

3. Homing pigeons may use more energy to fly over large bodies of water (since air pressure drops over water in the daytime). Suppose that a pigeon is released from a boat on a lake 1 mile away from the shore, and needs to reach its destination, L, along the shore which is 2 miles away from the point A on the shore which is closest to the boat. Assume that the pigeon needs 4/3 as much energy per mile to fly over water, and that it heads straight to a point P on the shore between A and L and then flies straight along the shore to L. Find the location of P which minimizes the energy spent by the pigeon.

Solution Our picture looks like the following



Let E be the amount of energy per mile used by the pigeon when flying over land. Then the total amount of energy used by the pigeon is

$$f(x) = E(2-x) + \frac{4}{3}E\sqrt{1+x^2} = E\left(2 - x + \frac{4}{3}\sqrt{1+x^2}\right)$$

Thus we are trying to find the absolute minimum of $f(x)$. From the problem we know that

x must in the interval $[0, 2]$

$$\begin{aligned} f'(x) &= E\left(-1 + \frac{4}{3}\cdot\frac{1}{2}(1+x^2)^{-1/2}(2x)\right) \\ &= E\left(-1 + \frac{4x}{3\sqrt{1+x^2}}\right) \\ &= E\left(-\frac{3\sqrt{1+x^2}}{3\sqrt{1+x^2}} + \frac{4x}{3\sqrt{1+x^2}}\right) \\ &= E\left(\frac{-3\sqrt{1+x^2}+4x}{3\sqrt{1+x^2}}\right) \end{aligned}$$

It is never undefined.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 4x - 3\sqrt{1+x^2} = 0 \\ &\Leftrightarrow 4x = 3\sqrt{1+x^2} \\ &\Leftrightarrow 16x^2 = 9(1+x^2) \\ &\Leftrightarrow 7x^2 = 9 \\ &\Leftrightarrow x^2 = \frac{9}{7} \\ &\Leftrightarrow x = \pm\frac{\sqrt{9}}{7} \\ &\Leftrightarrow x = \pm\frac{3}{\sqrt{7}} \end{aligned}$$

Keeping the only one in our interval gives us one critical number of $\frac{3}{\sqrt{7}}$. Testing the intervals we'll see that f has a relative minimum at $x = \frac{3}{\sqrt{7}}$ and thus that's where the absolute minimum is as well. $\frac{3}{\sqrt{7}} \approx 1.13$ so the optimal location of P is 1.13 miles from point A.

□

4. A box has a square base and its volume is 8m^3. Find the dimensions of the box that give it the smallest surface area.

Solution It may help to draw a picture of the box.

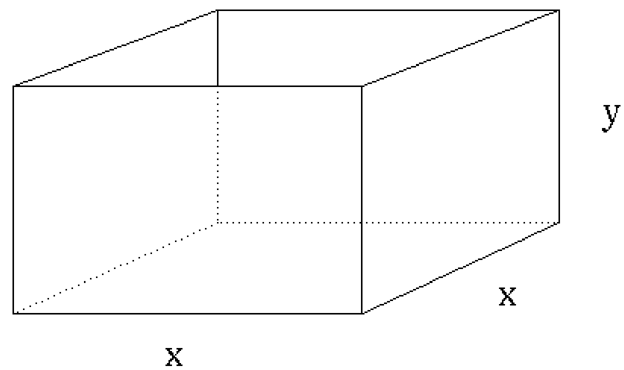


Figure 26

(i) Let's choose w to be the independent variable. (Choosing l will yield the same answer.)

(ii) We want to minimize surface area. The equation for surface area of the box is Surface Area=$2w^2 + 4lw$. Since our independent variable is w, we want to replace the l in the equation with an expression with w's in it. We are given that the volume of the box is 8 so we have $V = lw^2 = 8 \Rightarrow l = \frac{8}{w^2}$. Plugging this into our equation for surface area, we get Surface Area= $2w^2 + 4(\frac{8}{w^2}(w)) = 2w^2 + \frac{32}{w} = f(w)$

(iii) w is a length so it must be positive. Thus our domain of $f(w)$ is $(0, \infty)$

(iv) $f'(w) = 4w - \frac{32}{w^2} = \frac{4w^3 - 32}{w^2}$. It is undefined for $w = 0$ (this is not in the domain so we can ignore this x value). $f'(x) = 0 \Leftrightarrow 4w^3 - 32 = 0 \Leftrightarrow w^3 = 8 \Leftrightarrow w = 2$. Thus we have one critical point to plot on $(0, \infty)$. Testing the intervals, we find that there is a relative minimum at $x = 2$ and since this is the only relative extremum on our domain, it is the absolute minimum as well. Thus our surface area is minimized when $w = 2$. $l = \frac{8}{2^2} = 2$ so our dimensions are 2m x 2m x 2m (i.e. the box is a cube).

□

5. You want to enclose a field with 500ft of fencing. There is a building on one side (so you don't need fencing for one of the sides). Find the dimensions of the field that give the largest area.

Solution Our picture is:

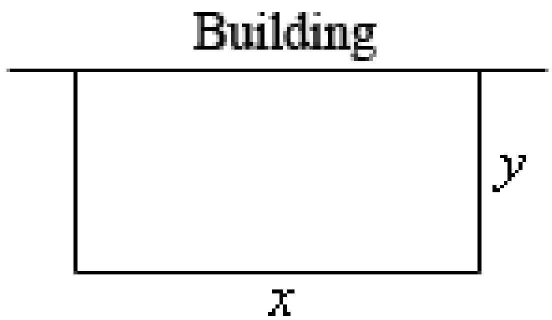


Figure 27

(i) Let's make x our independent variable.

(ii) We want to minimize area which is lw. We are given that we have 500ft of fencing so

$$500 = x + 2y \Rightarrow y = \frac{500 - x}{2}$$

Thus our equation becomes

$$f(x) = x(\frac{500 - x}{2}) = \frac{500x - x^2}{2}$$

(iii) Since x is a length it has to be positive and since our perimeter is 500, the max it can be is if we use all of it for that side i.e. 500 so our domain is $(0, 250)$. (Note: if you choose y to be the independent variable we have domain $(0, 250)$.

(iv) $f'(x) = \frac{500 - 2x}{2}$. It is never defined at $f'(x) = 0 \Leftrightarrow 500 - 2x = 0 \Leftrightarrow 2x = 500 \Leftrightarrow x = 250$. Testing the intervals we find that at $x = 250$ there is a relative max (and this is the absolute max as well). $y = \frac{500 - 250}{2} = \frac{250}{2} = 125$ so our dimensions are 250ft x 125ft.

□

Lecture 22: Implicit Differentiation

So far, we have been differentiating functions given to us explicitly. That is, the function had an independent variable x and dependent variable y where y could be written as $y = f(x)$. Sometimes, however, the relation between the variables are expressed in a more complicated way, i.e. implicitly.

Implicit Differentiation

We can use a method called implicit differentiation to find $\frac{dy}{dx}$ when you can't easily solve for y. To find $\frac{dy}{dx}$, differentiate the entire expression with respect to x then solve for $\frac{dy}{dx}$. This means that anytime you take the derivative of an expression with y in it, you must multiply by $\frac{dy}{dx}$.

Examples:

1. Find $\frac{dy}{dx}$ for $x + \ln y = x^2y^3$

   Solution

$$\begin{aligned} 1 + \frac{1}{x}\frac{dy}{dx} = 2xy^3 + 3x^2y^2\frac{dy}{dx} &\Leftrightarrow y + \frac{dy}{dx} - 3x^2y^3\frac{dy}{dx} = 2xy^4 \\ &\Leftrightarrow \frac{dy}{dx}(1 - 3x^2y^3) = 2xy^4 - y \\ &\Leftrightarrow \frac{dy}{dx} = \frac{2xy^4 - y}{1 - 3x^2y^3} \end{aligned}$$

□

2. Find $\frac{dy}{dx}$ for $4\sqrt{x} - 8\sqrt{y} = 6y^{3/2}$

   Solution

$$\begin{aligned} 4\sqrt{x} - 8\sqrt{y} = 6y^{3/2} &\Leftrightarrow \frac{2}{\sqrt{x}} - \frac{4}{\sqrt{y}}\frac{dy}{dx} = 9\sqrt{y}\frac{dy}{dx} \\ &\Leftrightarrow \frac{2\sqrt{y}}{\sqrt{x}} - 4\frac{dy}{dx} = 9y\frac{dy}{dx} \\ &\Leftrightarrow 4\frac{dy}{dx} + 9y\frac{dy}{dx} = \frac{2\sqrt{y}}{\sqrt{x}} \\ &\Leftrightarrow \frac{dy}{dx} = \frac{2\sqrt{y}}{\sqrt{x}(4 + 9y)} \end{aligned}$$

□

3. Find $\dfrac{dy}{dx}$ of $e^{xy} = e^{4x} - e^{5y}$

Solution Differentiating everything with respect to x we get

$$\begin{aligned} e^{xy}(y + x\frac{dy}{dx}) = e^{4x}(4) - e^{5y}(5\frac{dy}{dx}) &\Leftrightarrow e^{xy}y + xe^{xy}\frac{dy}{dx} = 4e^{4x} - 5e^{5y}\frac{dy}{dx} \\ &\Leftrightarrow \frac{dy}{dx}(xe^{xy} + 5e^{5y}) = 4x^{4x} - e^{xy}y \\ &\Leftrightarrow \frac{dy}{dx} = \frac{4e^{4x} - e^{xy}y}{xe^{xy} + 5e^{5y}} \end{aligned}$$

□

4. Find the equation of the line tangent to $3(x^2 + y^2)^2 = 25(x^2 - y^2)$ at $(2, 1)$

Solution

$$3(x^2 + y^2)^2 = 25(x^2 - y^2) \Leftrightarrow 6(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

We know that $x = 2$ and $y - 1$ so plugging this in we get

$$\begin{aligned} 6(4 + 1)\left(4 + 2\frac{dy}{dx}\right) = 25\left(4 - 2\frac{dy}{dx}\right) &\Leftrightarrow 30\left(4 + 2\frac{dy}{dx}\right) = 100 - 50\frac{dy}{dx} \\ &\Leftrightarrow 120 + 60\frac{dy}{dx} = 100 - 50\frac{dy}{dx} \\ &\Leftrightarrow 110\frac{dy}{dx} = -20 \\ &\Leftrightarrow \frac{dy}{dx} = -\frac{2}{11} \end{aligned}$$

Using point slope form we get

$$y - 1 = -\frac{2}{11}(x - 2)$$

□

5. Find the equation of the tangent line to $x^2 + \tan\left(\frac{\pi}{4}xy\right) = 2$ at $(1,1)$

Solution

$$\begin{aligned} x^2 + \tan\left(\frac{\pi}{4}xy\right) = 2 &\Leftrightarrow 2x + \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}xy\right)\left(y + x\frac{dy}{dx}\right) = 0 \\ &\Leftrightarrow y + x\frac{dy}{dx} = -\frac{8x}{\pi\sec^2((\pi/4)(xy))} \\ &\Leftrightarrow \frac{dy}{dx} = -\frac{8x}{\pi x\sec^2((\pi/4)(xy))} - \frac{y}{x} \end{aligned}$$

Plugging in $x = 1$ and $y = 1$ we have

$$\frac{dy}{dx} = -\frac{16}{\pi} - 1$$

So our equation is

$$y - 1 = -\left(\frac{16}{\pi} + 1\right)(x - 1)$$

□

6. Find the equation of the tangent line to $x^2 + xy + y^2 = 3$ at $(1,1)$

Solution Differentiating everything with respect to x gives

$$\begin{aligned} 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 &\Leftrightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y \\ &\Leftrightarrow \frac{dy}{dx}(x + 2y) = -2x - y \\ &\Leftrightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y} \end{aligned}$$

Plugging in our point $(1,1)$ we get

$$\frac{dy}{dx} = \frac{-3}{3} = -1$$

Now using point-slope form, we get that the equation of the tangent line is

$$y - 1 = -(x - 1) \Leftrightarrow y = -x + 2$$

□

Lecture 23: Related Rates

In the previous lecture, we considered equations involving two variables, x and y, and we assumed x was the independent variable and y was an implicut function of x. However, in many cases, we have equations involving two variables that rely on a third independent variable, usually time.

Related Rates

(i) Read the problem carefully and determine what equation is needed.

(ii) Differentiate the equation with respect to the independent variable, typically time (t)

(iii) Solve for the unknown rate using the given information.

Examples:

1. Consider a bacterial culture growing in a petri dish and suppose that the bacteria reproduce in such a way that they are always forming a disk of growing radius. Suppose that we know the area of the disk is increasing at the rate of 3 cm^2/day. Can we find the rate of change of the radius at the time when that radius is 4 cm?

   Solution We know that the area of a disk is given by $A = \pi r^2$ where the area and the radius of the disk depend on time. From the problem we know that $dA/dt = 3$ and we need to find dr/dt when $r = 4$. Differentiating both sides of the equation with respect to t we get

   $$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

   Plugging in our known info we have

   $$3 = 2\pi(4)\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt} = \frac{3}{8\pi}\text{cm/day}$$

   □

2. Suppose that x and y are two functions of t such that the following equation holds for all t:

   $$\sin(x+y) + (1+x)^2 + (1+y)^2 = 5$$

   and suppose that when $x = 0$ and $y = 0$ we have $dx/dt = -10$. What is the value of dy/dt?

   Solution Differentiating both sides with respect to t we have

   $$\cos(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) + 2(1+x)\frac{dx}{dt} + 2(1+y)\frac{dy}{dt} = 0$$

Plugging in the known information we have

$$\cos(0)\left(-10+\frac{dy}{dt}\right)+2(1)(-10)+2(1)\frac{dy}{dt}=0 \Leftrightarrow \frac{dy}{dt}=10$$

□

3. An ice cube that is 3 cm on each side is melting at a rate of 2 cm^3 per minute. How fast is the length of the side decreasing?

Solution Let x denote the length of the side, then the cube volume is $V = x^3$ and we are given that $dV/dt = -2$. We want to find dx/dt when $x = 3$. Differentiating both sides with respect to t we have

$$\frac{dV}{dt}=3x^2\frac{dx}{dt}$$

Plugging in our known values we have

$$-2=3^3\frac{dx}{dt} \Leftrightarrow \frac{dx}{dt}=-\frac{2}{27}$$

Thus the side is decreasing at a rate of 2/27 cm/minute.

□

4. Sociologists have found that crime rates are influenced by temperature. In a midwestern town of 100,000 people, the crime rate has been approximated by

$$C=\frac{1}{10}(T-60)^2+100$$

where C denotes the number of crimes per month and T is the average monthly temperature in degrees in Fahrenheit. Suppose that the average temperature in May was 76°, and by the end of May the temperature was rising at the rate 8° per month. What fast was the crime rate rising at the end of May?

Solution Differentiating both sides we get

$$\frac{dC}{dt}=\frac{1}{5}(T-60)\frac{dT}{dt}$$

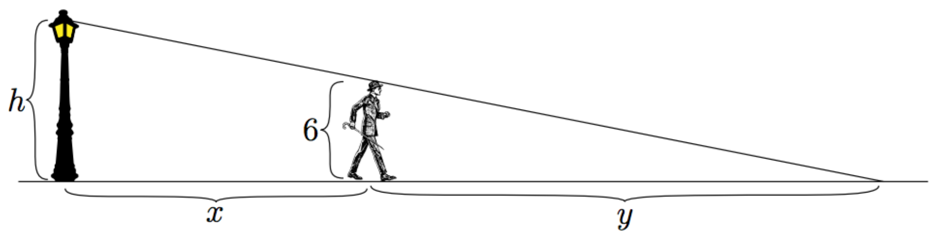
Plugging in we have

$$\frac{dC}{dt}=\frac{1}{5}(76-60)(8)=\frac{128}{5}\text{ crimes/month}$$

□

5. A man 6 feet tall is walking away from a lamp post at the rate of 50 feet per minute. When the man is 8 feet away from the post his shadow is 10 feet long. Find the rate at which the length of the shadow is increasing when he is 25 feet away from the post.

Solution Here is our picture:



From geometry we know that

$$\frac{h}{x+y} = \frac{6}{y}$$

We also know that when $x = 8$, then $y = 10$. Thus

$$\frac{h}{18} = \frac{6}{10} \Leftrightarrow h = \frac{54}{5}$$

This gives

$$54y = 30(x+y) \Leftrightarrow 24y = 30x$$

We are given that $dx/dt = 50$. Differentiating with respect to t we have

$$24\frac{dy}{dt} = 3 - \frac{dx}{dt}$$

Plugging in we have

$$24\frac{dy}{dt} = 1500 \Leftrightarrow \frac{dy}{dt} = 62.5 \text{ ft/min}$$

□

6. A ladder 20 ft long leans against a building. If the bottom of the ladder slides away from the building horizontally at a rate of 4 ft/sec, how fast is the latter sliding down when the top of the ladder is 8 ft from the ground?

Solution Here is our picture:

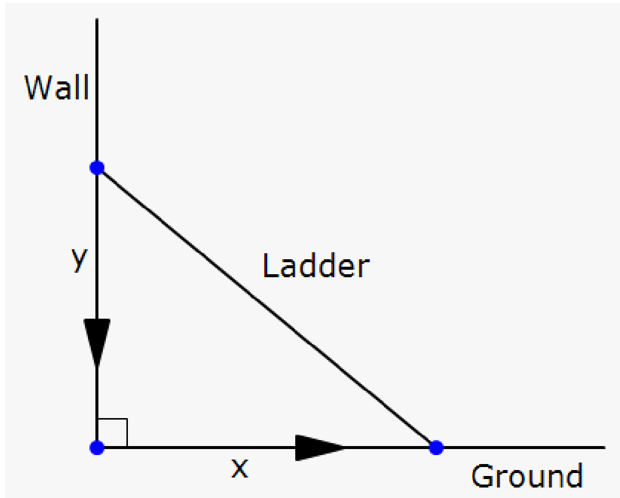


Figure 28

We have to relate the sides of the triangle that the ladder forms with the ground and the wall. Thus our equation is the Pythagorean Theorem

$$x^2 + y^2 = 20^2$$

Differentiating with respect to t we get

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

We are trying to find dy/dt and we know that $dx/dt = 4$ and $y = 8$. Lastly using the Pythagorean Theorem we have

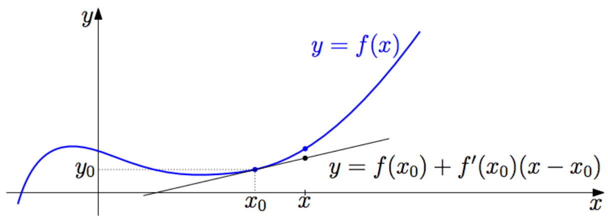
$$x^2 + 8^2 = 20^2 \Leftrightarrow x^2 = 336 \Rightarrow x = \sqrt{336}$$

Putting these into our equation we have

$$0 = 2\sqrt{336}(4) + 2(8)\frac{dy}{dt} \Leftrightarrow \frac{dy}{dt} = \frac{-8\sqrt{336}}{16} = \frac{-\sqrt{336}}{2}\,{}^{\text{ft}}\!/_{\text{sec}}$$

□

Lecture 24: Differentials and Linear Approximation



Notice that when x is close to x_0, then the value of $y = f(x)$ is close to the value on the tangent line.

Definition: Let f be a function which is differentiable at x_0. The *differential* of f at x_0 is defined as a linear function

$$dy = f'(x_0)dx$$

where dx denotes the argument of this function.

Examples:

1. Consider $f(x) = \sqrt{3x+4}$. Compute the differential of f at $x = 4$

   Solution

$$f'(x) = \frac{1}{2}(3x+4)^{-1/2}$$
$$= \frac{3}{2\sqrt{3x+4}}$$

$$f'(4) = \frac{3}{8}$$

$$dy = f'(4)dx$$
$$= \frac{3}{8}dx$$

□

2. Consider the function $y = \dfrac{2x-5}{x+1}$. Find its differential.

Solution

$$\begin{aligned} f'(x) &= \frac{(x+1)(2)-(2x-5)(1)}{(x+1)^2} \\ &= \frac{\cancel{2x}+2-\cancel{2x}+5}{(x+1)^2} \\ &= \frac{7}{(x+1)^2} \end{aligned}$$

$$\begin{aligned} dy &= f'(x)dx \\ &= \frac{7}{(x+1)^2}dx \end{aligned}$$

□

Linear Approximation

Let f be a function. We can approximate the value of the function at poins close to x_0 using the following formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

where x_0 is a value close to x that is easy to evaluate. (Note: $(x - x_0)$ is sometimes denoted Δx)

Examples:

1. Approximate $\ln(0.98)$

Solution Notice that $0.98 = 1 - 0.02$, so we can approximate it using $x_0 = 1$ and $f(x) = \ln x$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln x) \\ &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \ln(1-0.02) &\approx \ln 1 + f'(1)(-0.02) \\ &= 0 + (1)(-0.02) \\ &= -0.02 \end{aligned}$$

□

2. Approximate $\sqrt{17.02}$

   Solution $f(x) = \sqrt{x}$ and $x_0 = 16$

$$\begin{aligned}\sqrt{17.02} &= \sqrt{16 + 1.02} \\ &\approx f(16) + f'(16)(17.02 - 16) \\ &= \sqrt{16} + \frac{1}{2\sqrt{16}}(1.02) \\ &= 4 + \frac{1}{8}(1.02) \\ &= 4.1275\end{aligned}$$

□

3. Approximate $\sin\left(\frac{\pi}{4} + 0.02\right)$

   Solution $f(x) = \sin x \to f'(x) = \cos x,\ x_0 = \frac{\pi}{4}$

$$\begin{aligned}\sin\left(\frac{\pi}{4} + 0.02\right) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(\left(\frac{\pi}{4} + 0.02\right) - \frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} + \cos\left(\frac{\pi}{4}\right)(0.02) \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0.02)\end{aligned}$$

□

4. Approximate $f(3.06)$ and $f(2.9)$ given that $f(3) = 1$ and $f'(3) = 0.5$.

   Solution

$$\begin{aligned}f(3.06) &\approx f(3) + f'(3)(3.06 - 3) \\ &= 1 + (0.5)(.06) \\ &= 1.03\end{aligned}$$

$$\begin{aligned}f(2.9) &\approx f(3) + f'(3)(2.9 - 3) \\ &= 1 + 0.5(-0.1) \\ &= 0.95\end{aligned}$$

□

5. One hour after leaving France, a plane had travelled 975 miles and was flying at a rate of 1520 mi/h. Approximately how far was the plane from France 0.1 hours later?

   Solution Let $s(t)$ indicate the position that the plane is from France at time t. With the given information we have $s(1) = 975$ and $s'(1) = 1520$. Using linear approximation we have

$$s(1.1) \approx 975 + 1520(1.1 - 1) = 1127 \text{ miles}$$

□

6. Suppose that the concentration of a certain drug in the bloodstream t hours after being administered is described by the function

$$C(t) = \frac{5t}{9 + t^2}$$

   Approximately by how much does the concentration of the drug change from $t = 1$ to $t = 1.5$?

   Solution $t_0 = 1$, $\Delta t = 1.5 - 1 = 0.5$

$$\begin{aligned} C'(t) &= \frac{(9 + t^2)(5) - (5t)(2t)}{(9+^2)^2} \\ &= \frac{45 + 5t^2 - 10t^2}{(9 + t^2)^2} \\ &= \frac{45 - 5t^2}{(9 + t^2)^2} \end{aligned}$$

$$\begin{aligned} C'(1) &= \frac{40}{100} \\ &= \frac{2}{5} \end{aligned}$$

$$\begin{aligned} C(1.5) - C(1) &\approx \frac{2}{5} \cdot \frac{1}{2} \\ &- \frac{1}{5} \end{aligned}$$

□

7. Suppose that there's an oil slick in the shape of a circle. Approximately by how much does the area of the slick increase when its radius changes from 1.2 miles to 1.4 miles?

Solution Let A denote the area of the slick and r be the radius, then $A = \pi r^2$

$$
\begin{aligned}
A(1.4) - A(1.2) &\approx \frac{dA}{dr}(1.2) \cdot (1.4 - 1.2) \\
&= \frac{dA}{dr}(1.2) \cdot (0.2) \\
&= 2\pi \cdot 1.2 \cdot (0.2) \\
&= 0.48\pi
\end{aligned}
$$

□

Lecture 25: Antiderivatives

So far we've considered functions and found their rates of change. However, in practice the inverse problem occurs more often where we know the rate of change of a quantity and want to find the quantity itself.

Definition: Let f be a continuous function. If $F'(x) = f(x)$, then $F(x)$ is called an *antiderivative* of f.

Examples:

1. Find an antiderivative of $4x^3$.

   Solution We know that $\frac{d}{dx}(x^4) = 4x^3$ so the antiderivative of $4x^3$ is x^4.

   □

2. Find an antiderivative of e^x.

   Solution We know $\frac{d}{dx}(e^x) = e^x$ so e^x is an antiderivative of itself.

   □

Note that adding a constant doesn't change the derivative of a function so there are infinitely many antiderivatives of a function. In fact, if $F(x)$ and $G(x)$ are two antiderivatives of f then $F(x) - G(x) = C$ where C is a constant.

Definition: Let f be a continuous function. The family of all antiderivatives of f is called the *indefinite integral* of f written:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of f and C is a constant. $\int$ is called the *integral sign*, f is called the *integrand*, and x is called the *variable of integration*.

Examples:

1. $f(x) = 2$. Find the indefinite integral of f.

   Solution We know $\frac{d}{dx}(2x) = 2$ so

   $$\int 2\ dx = 2x + C$$

   □

2. Let $f(x) = 3x^2 - 2x$. Find the indefinite integral of f.

Solution Notice that $\frac{d}{dx}(x^3) = 3x^2$ and $\frac{d}{dx}(x^2) = 2x$. so $\frac{d}{dx}(x^3 - x^2) = 3x^2 - 2x$. Thus

$$\int (3x^2 - 2x)\ dx = x^3 - x^2 + C$$

□

<u>Some Integration Rules:</u>

- $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int kf(x)\ dx = k \int f(x)\ dx$, where k is a constant
- $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
- $\int e^x\ dx = e^x + C$
- $\int e^{kx}\ dx = \frac{e^{kx}}{k} + C$
- $\int a^x\ dx = \frac{a^x}{\ln a} + C$ If $a > 0,\ a \neq 1$
- $\int a^{kx}\ dx = \frac{a^{kx}}{k \ln a} + C$ If $a > 0,\ a \neq 1$
- $\int x^{-1}\ dx = \ln|x| + C$

Examples:

1. $\displaystyle\int \left(4x^7 - 2x^2 + \frac{12}{x^4}\right)\, dx$

Solution

$$\begin{aligned}\int \left(4x^7 - 2x^2 + \frac{12}{x^4}\right)\, dx &= 4\int x^7\, dx - 2\int x^2\, dx + 12\int \frac{1}{x^4}\, dx \\ &= 4\int x^7\, dx - 2\int x^2\, dx + 12\int x^{-4}\, dx \\ &= 4\cdot\frac{x^8}{8} - 2\cdot\frac{x^3}{3} + 12\cdot\frac{x^{-3}}{-3} + C \\ &= \frac{1}{2}x^8 - \frac{2}{3}x^3 - \frac{4}{x^3} + C\end{aligned}$$

□

2. $\displaystyle\int (\sqrt{x} - 1)\, dx$

Solution

$$\begin{aligned}\int x^{1/2}\, dx &= \int x^{1/2}\, dx - \int\, dx \\ &= \frac{2}{3}x^{3/2} - x + C\end{aligned}$$

□

3. $\displaystyle\int \frac{1+2t^3}{4t}\, dt$

Solution

$$\begin{aligned}\int \frac{1+2t^3}{4t}\, dt &= \int \left(\frac{1}{4t} + \frac{1}{2}t^2\right)\, dt \\ &= \frac{1}{4}\ln|t| + \frac{1}{6}t^3 + C\end{aligned}$$

□

4. $\displaystyle\int \left(\frac{9}{x} - 3e^{4x}\right) dx$

Solution

$$\begin{aligned}\int \left(\frac{9}{x} - 3e^{4x}\right) dx &= 9\int \frac{dx}{x} - 3\int e^{4x}\, dx \\ &= 9\ln|x| - \frac{3}{4}e^{4x} + C\end{aligned}$$

□

Integrals of Trigonometric Functions:

- $\int \sin x \ dx = -\cos x + C$
- $\int \cos x \ dx = \sin x + C$
- $\int \sec^2 x \ dx = \tan x + C$
- $\int \csc^2 x \ dx = -\cot x + C$
- $\int \sec x \tan x \ dx = \sec x + C$
- $\int \csc x \cot x \ dx = -\csc x + C$

5. $\int \left(3\cos x - \frac{17}{\cos^2 x}\right) \ dx$

Solution

$$\begin{aligned}\int \left(3\cos x - \frac{17}{\cos^2 x}\right) \ dx &= 3\int \cos x \ dx - 17\int \sec^2 x \ dx \\ &= 3\sin x - 17\tan x + C\end{aligned}$$

□

6. $\int \left(8x^{3/4} - \frac{2\sin x}{\cos^2 x}\right) \ dx$

Solution

$$\begin{aligned}\int \left(8x^{3/4} - \frac{2\sin x}{\cos^2 x}\right) \ dx &= 8\int x^{3/4} \ dx - 2\int \sec x \tan x \ dx \\ &= \frac{32}{7}x^{7/4} - 2\sec x + C\end{aligned}$$

□

Lecture 26: Substitution

When differentiating, we used the chain rule to find derivatives of composite functions. Reversing the chain rule gives us the following:

$$\frac{d}{dx}(g(f(x))) = g'(f(x))f'(x) \Rightarrow \int g'(f(x))f'(x)\ dx = g(f(x)) + C$$

This gives us an integration technique referred to as *substitution* or *u-substitution*.

u-Substitution

$$\int g'(f(x))f'(x)dx = g(f(x)) + C \text{ or } \int g(u)du = G(u) + C \text{ where } u = f(x) \text{ and } G'(u) = g(u)$$

Examples:

1. Find $\int (x^2 - 5)^4(2x)\ dx$

   Solution Let $u = x^2 - 5 \Rightarrow du = 2xdx$

$$\begin{aligned}\int (x^2-5)^4(2x)\ dx &= \int u^4\ du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}(x^2-5)^5 + C\end{aligned}$$

□

2. Find $\int \frac{2x+3}{3x^2+9x+7}\ dx$

   Solution Let $u = 3x^2 + 9x + 7 \Rightarrow du = (6x+9)dx = 3(2x+3)dx$

$$\begin{aligned}\int \frac{2x+3}{3x^2+9x+7}\ dx &= \int \frac{du}{3u} \\ &= \frac{1}{3}\int \frac{du}{u} \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln\left|3x^2+9x+7\right| + C\end{aligned}$$

□

3. Find $\int 3x^2 e^{2x^3}\ dx$

Solution $u = 2x^3 \Rightarrow du = 6x^2 dx = 2(3x^2)dx$

$$\begin{aligned}\int 3x^2 e^{2x^3}\ dx &= \int \frac{1}{2}e^u\ du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{2x^3} + C\end{aligned}$$

□

4. Find $\int 4r\sqrt{8-r}\ dr$

Solution $u = 8 - r \Rightarrow du = -dr \Rightarrow r = 8 - u$

$$\begin{aligned}\int 4r\sqrt{8-r}\ dr &= 4\int (8-u)\sqrt{u}\ (-du) \\ &= 4\int (u-8)\sqrt{u}\ du \\ &= 4\int (u^{3/2} - 8^{1/2})\ du \\ &= 4\left(\frac{2}{5}u^{5/2} - 8\cdot\frac{2}{3}u^{3/2}\right) + C \\ &= \frac{8}{5}(8-r)^{5/2} - \frac{64}{3}(8-r)^{3/2} + C\end{aligned}$$

□

5. Find $\int \frac{y^2+y}{2y^3+3y^2+1}\ dy$

Solution $u = 2y^3 + 3y^2 + 1 \Rightarrow du = (6y^2 + 6y)dy = 6(y^2 + y)dy$

$$\begin{aligned}\int \frac{y^2+y}{2y^3+3y^2+1}\ dy &= \int \frac{du}{6u} \\ &= \frac{1}{6}\ln|u| + C \\ &= \frac{1}{6}\ln\left|2y^3 + 3y^2 + 1\right| + C\end{aligned}$$

□

6. Find $\int \frac{x}{x^2+1}\ dx$.

Solution Let $u = x^2 + 1$, then $du = 2xdx$

$$\begin{aligned}\int \frac{x}{x^2+1}\ dx &= \int \frac{du}{2u} \\ &= \frac{1}{2}\int u^{-1}\ du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln\left|x^2+1\right| + C\end{aligned}$$

□

7. Find $\int \tan x\ dx$

Solution Let $u = \cos x \Rightarrow du = -\sin x dx$

$$\begin{aligned}\int \tan x\ dx &= \int \frac{-du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C\end{aligned}$$

□

8. Find $\int \cot x \; dx$

Solution Let $u = \sin x \Rightarrow du = \cos x dx$

$$\begin{aligned}\int \cot x \; dx &= \int \frac{\cos x}{\sin x} \; dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sin x| + C\end{aligned}$$

□

9. Find $\int \frac{dx}{x \ln x}$

Solution Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\begin{aligned}\int \frac{dx}{x \ln x} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\ln x| + C\end{aligned}$$

□

10. Find $\int x 8^{3x^2+1} \; dx$

Solution Let $u = 3x^2 + 1 \Rightarrow du = 6x dx$

$$\begin{aligned}\int x 8^{3x^2+1} \; dx &= \frac{1}{6} \int 8^u \; du \\ &= \frac{1}{6} \cdot \frac{8^u}{\ln 8} + C \\ &= \frac{8^{3x^2+1}}{6 \ln 8} + C\end{aligned}$$

□

11. Find $\int \sin^7 x \cos x \; dx$

Solution Let $u = \sin x \Rightarrow du = \cos x dx$

$$\begin{aligned}\int \sin^7 x \cos x \, dx &= \int u^7 \, du \\ &= \frac{1}{8}u^8 + C \\ &= \frac{1}{8}\sin^8 x + C\end{aligned}$$

□

12. An epidemic is growing in a region according to the rate

$$N'(t) = \frac{100t}{t^2+2}$$

where $N(t)$ is the number of people infected after t days. Find a formula for the number of people infected after t days, given that 37 people were infected at $t = 0$.

Solution $N(t)$ is some anti derivative of $\frac{100t}{t^2+2}$ and we know $N(0) = 37$. Let $u = t^2 + 2 \Rightarrow du = 2t \; dt$

$$\begin{aligned}\int \frac{100t}{t^2+2} \, dt &= \int \frac{50}{u} \, du \\ &= 50 \ln|u| + C \\ &= 50 \ln\left|t^2+2\right| + C\end{aligned}$$

$$\begin{aligned}N(0) = 37 &\Leftrightarrow 50 \ln 2 + C = 37 \\ &\Leftrightarrow C = 37 - 50 \ln 2 \\ &\Leftrightarrow N(t) = 50 \ln(t^2+2) + 37 - 50 \ln 2 \\ &\Leftrightarrow N(t) = 50 \ln\left(\frac{t^2}{2} + 1\right) + 37\end{aligned}$$

□

13. Find $\int \cos(x^2) 2x \; dx$.

Solution Let $u = x^2 \Rightarrow du = 2xdx$

$$\begin{aligned}\int \cos(x^2) 2x \, dx &= \int \cos u \, du \\ &= \sin u + C \\ &= \sin x^2 + C\end{aligned}$$

□

14. Find $\int \cos(x^2)6x\ dx$.

Solution Let $u = x^2 \Rightarrow du = 2xdx$

$$\begin{aligned}\int \cos(x^2)6x\ dx &= \int 3\cos x^2(2x)\ dx \\ &= 3\int \cos u\ du \\ &= 3\sin u + C \\ &= 3\sin(x^2) + C\end{aligned}$$

□

15. Find $\int x\sqrt{4-x}\ dx$

Solution Let $u = 4 - x \Rightarrow du = -dx, x = 4 - u$

$$\begin{aligned}\int x\sqrt{4-x}\ dx &= \int -(4-u)\sqrt{u}\ du \\ &= \int (-4u^{1/2} + u^{3/2})\ du \\ &= -4\int u^{1/2}du + \int u^{3/2}\ du \\ &= -4\left(\frac{2}{3}\right)u^{3/2} + \frac{2}{5}u^{5/2} + C \\ &= -\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + C\end{aligned}$$

□

16. Find $\int 4\cos(3x)\ dx$

Solution Let $u = 3x \Rightarrow du = 3dx$

$$\begin{aligned}\int 4\cos(3x)\ dx &= \frac{4}{3}\int \cos u\ du \\ &= \frac{4}{3}\sin u + C \\ &= \frac{4}{3}\sin(3x) + C\end{aligned}$$

$\square$

17. Find $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

Solution Let $u = \ln x \Rightarrow du = \dfrac{dx}{x}$

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du$$
$$= -\cos u + C$$
$$= -\cos(\ln x) + C$$

$\square$

18. Find $\int (3-x)^{10}\ dx$

Solution Let $u = 3 - x \Rightarrow du = -dx$

$$\begin{aligned}\int (3-x)^{10} dx &= \int u^{10}(-du)\\ &= -\int u^{10} du\\ &= \frac{u^{11}}{11} + C\\ &= \frac{1}{11}(3-x)^{11} + C\end{aligned}$$

□

19. Find $\int (2x+5)(x^2+5x)^7\ dx$

Solution Let $u = x^2 + 5x \Rightarrow du = (2x+5)\ dx$

$$\begin{aligned}\int (2x+5)(x^2+5x)^7\ dx &= \int u^7\ du\\ &= \frac{1}{8}u^8 + C\\ &= \frac{1}{8}(x^2+5x)^8 + C\end{aligned}$$

□

20. Find $\int x^2(3-10x^3)^4\ dx$

Solution Let $u = 3 - 10x^3 \Rightarrow du = 30x^2\ dx$

$$\begin{aligned}\int x^2(3-10x^3)^4\ dx &= \int \frac{1}{30}u^4\ du\\ &= \frac{1}{30}\int u^4\ du\\ &= \frac{1}{30(5)}u^5 + C\\ &= \frac{1}{150}(3-10x^3)^5 + C\end{aligned}$$

□

21. Find $\int \cos(3x)\sin^{10}(3x)\ dx$

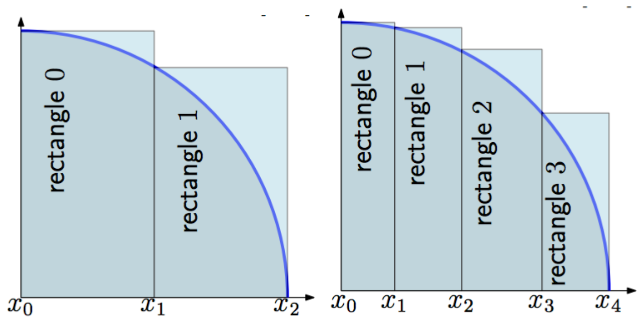
Solution Let $u = \sin(3x) \Rightarrow du = 3\cos(3x)\ dx$

$$\begin{aligned}\int \cos(3x)\sin^{10}(3x)\ dx &= \frac{1}{3}\int u^{10}\ du \\ &= \frac{1}{3(11)}u^{11} + C \\ &= \frac{1}{33}(\sin(3x))^{11} + C\end{aligned}$$

□

Lecture 27: Area and the Definite Integral

Consider the graph of the function $f(x) = \sqrt{9 - x^2}$ on the interval $[0, 3]$. Suppose we want to approximate the area under the graph and above the x-axis. We can try to do so using rectangles.



Approximating the Area Under the Curve:

Given a function f, we can approximate the area under the curve between $[a, b]$ (i.e the area between the curve and the x-axis from $x = a$ to $x = b$) using n rectangles:

(i) Find $\Delta x = \dfrac{b-a}{n}$

(ii) Label your x_i. $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = x_1 + \Delta x$, ... , $x_n = b$. Then find $f(x_i)$ for all i

(iii) If using left endpoints:

$$\text{Area} \approx \sum_{i=0}^{n-1} f(x_i)\Delta x$$

If you're using right endpoints then,

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

Lastly, if you're using midpoints then,

$$\text{Area} \approx \sum_{i=1}^{n} f(\frac{x_i + x_{i-1}}{2})\Delta x$$

These types of sums are called *Riemann Sums.*

Definition: The exact area under a curve is

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x$$

If a function is defined on the interval $[a, b]$, the *definite integral* of f from a to b is defined by:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x$$

Examples:

1. Approximate the area under the graph of $f(x) = -x^2 + 4$ and above the x-axis from $x = -2$ to $x = 2$. Use 4 intervals and left end points.

   Solution

   (i)

$$\Delta x = \frac{2-(-2)}{4} = \frac{4}{4} = 1$$

(ii)

$$x_0 = -2$$

$$\begin{aligned} x_1 &= -2 + 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} x_2 &= -1 + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= 0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(x_0) &= f(-2) \\ &= 4 - (-2)^2 \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(x_1) &= f(-1) \\ &= 4 - (-1)^2 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} f(x_2) &= f(0) \\ &= 4 - (0)^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(x_3) &= f(1) \\ &= 4 - (1)^2 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

(iii)

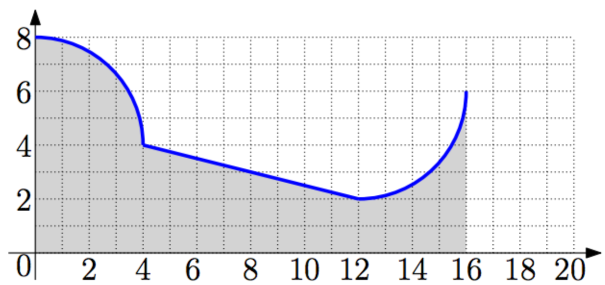
$$\begin{aligned} \text{Area } &\approx \sum_{i=0}^{n-1} f(x_i)\Delta x \\ &= f(-2) + f(-1) + f(0) + f(1) \\ &= 0 + 3 + 4 + 3 \\ &= 10 \end{aligned}$$

□

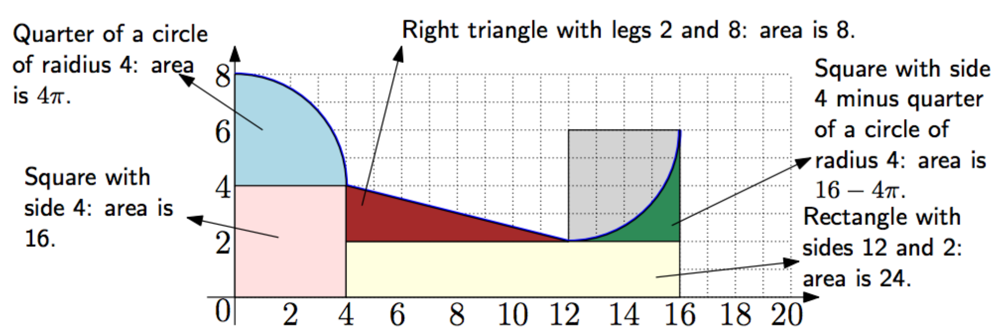
2. Find the exact value of the integral $\int_0^{20} f(x)\, dx$ where

$$f(x) = \begin{cases} 4 + \sqrt{16 - x^2} & 0 \leq x \leq 4 \\ 5 - \dfrac{x}{4} & 4 \leq x \leq 12 \\ 6 - \sqrt{16 - (x-12)^2} & 12 \leq x \leq 16 \end{cases}$$

Solution The graph looks like the following:



We can split it into simpler regions



□

3. Approximate the area under the curve $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ using left endpoints, right endpoints, and midpoints with 4 rectangles.

Solution Our graph looks like this:

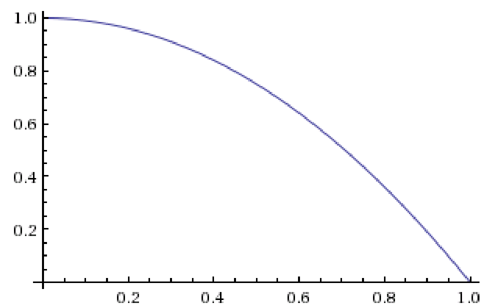


Figure 29

Left Endpoints:

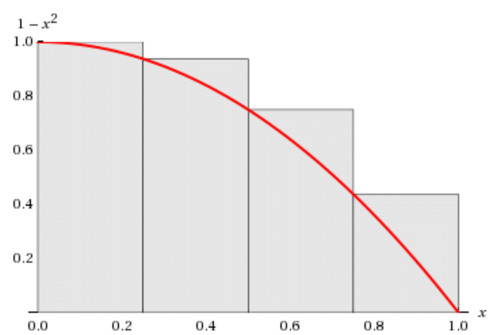


Figure 30

(i)

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

(ii)

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= 0 + \frac{1}{4} \\
&= \frac{1}{4} \\
x_2 &= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2} \\
x_3 &= \frac{1}{2} + \frac{1}{4} \\
&= \frac{3}{4} \\
x_4 &= 1
\end{aligned}
$$

$$\begin{aligned} f(x_0) &= 1 - (0)^2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(x_1) &= 1 - \left(\frac{1}{4}\right)^2 \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \end{aligned}$$

$$\begin{aligned} f(x_2) &= 1 - \left(\frac{1}{2}\right)^2 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} f(x_3) &= 1 - \left(\frac{3}{4}\right)^2 \\ &= 1 - \frac{9}{16} \\ &= \frac{7}{16} \end{aligned}$$

$$\begin{aligned} f(x_4) &= 1 - (1)^2 \\ &= 0 \end{aligned}$$

(iii)

$$\begin{aligned}\sum_{i=0}^{3} f(x_i)\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x\\ &= 1\left(\frac{1}{4}\right) + \frac{15}{16}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{4}\right) + \frac{7}{16}\left(\frac{1}{4}\right)\\ &= \frac{1}{4} + \frac{15}{64} + \frac{3}{16} + \frac{7}{64}\\ &= \frac{16}{64} + \frac{15}{64} + \frac{12}{64} + \frac{7}{64}\\ &= \frac{50}{64}\\ &= \frac{25}{32} \approx 0.78125\end{aligned}$$

Right Endpoints:

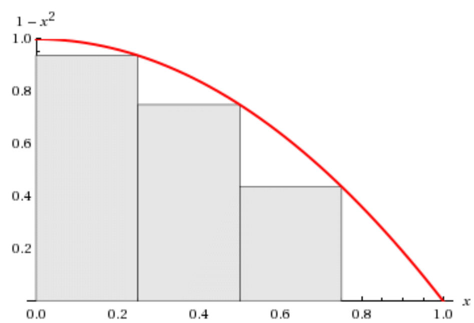


Figure 31

(i)

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

(ii) $x_0 = 0,\ x_1 = 0 + \frac{1}{4} = \frac{1}{4},\ x_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},\ x_3 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},\ x_4 = 1$

$$f(x_0) = 1 - (0)^2 = 1$$
$$f(x_1) = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$
$$f(x_2) = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$f(x_3) = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}$$
$$f(x_4) = 1 - (1)^2 = 0$$

(iii)

$$\begin{aligned}
\sum_{i=1}^{4} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= \frac{15}{16}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{4}\right) + \frac{7}{16}\left(\frac{1}{4}\right) + 0 \\
&= \frac{15}{64} + \frac{3}{16} + \frac{7}{64} \\
&= \frac{15}{64} + \frac{12}{64} + \frac{7}{64} \\
&= \frac{34}{64} \\
&= \frac{17}{32} \approx 0.53125
\end{aligned}$$

Midpoints:

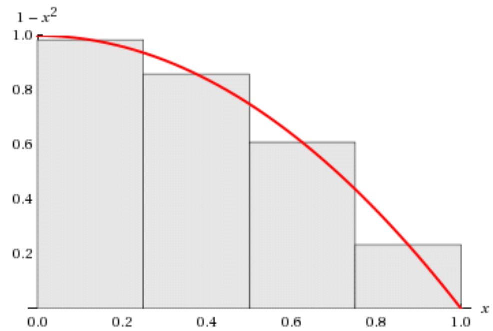


Figure 32

(i) $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$

(ii) $x_0 = 0,\ x_1 = 0 + \dfrac{1}{4} = \dfrac{1}{4},\ x_2 = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2},\ x_3 = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4},\ x_4 = 1$

Let $x_{i*} = \dfrac{x_i + x_{i-1}}{2}$, then

$$\text{Area } \approx \sum_{i=1}^{4} f(x_{i*})\Delta x$$

$$x_{1*} = \frac{0 + 1/4}{2} = \frac{1/4}{2} = \frac{1}{8}$$
$$x_{2*} = \frac{1/2 + 1/4}{2} = \frac{2/4 + 1/4}{2} = \frac{3/4}{2} = \frac{3}{8}$$
$$x_{3*} = \frac{3/4 + 1/2}{2} = \frac{\frac{3}{4} + \frac{2}{4}}{2} = \frac{\frac{5}{4}}{2} = \frac{5}{8}$$
$$x_{4*} = \frac{1 + 3/4}{2} = \frac{4/4 + 3/4}{2} = \frac{7/4}{2} = \frac{7}{8}$$
$$f(x_{1*}) = 1 - \left(\frac{1}{8}\right)^2 = 1 - \frac{1}{64} = \frac{64-1}{64} = \frac{63}{64}$$
$$f(x_{2*}) = 1 - \left(\frac{3}{8}\right)^2 = 1 - \frac{9}{64} = \frac{64-9}{64} = \frac{55}{64}$$
$$f(x_{3*}) = 1 - \left(\frac{5}{8}\right)^2 = 1 - \frac{25}{64} = \frac{64-25}{64} = \frac{39}{64}$$
$$f(x_{4*}) = 1 - \left(\frac{7}{8}\right)^2 = 1 - \frac{49}{64} = \frac{64-49}{64} = \frac{15}{64}$$

(iii)

$$\begin{aligned}
\sum_{i=1}^{4} f(x_{i*})\Delta x &= f(x_{1*})\Delta x + f(x_{2*})\Delta x + f(x_{3*})\Delta x + f(x_{4*})\Delta x \\
&= \left(\frac{63}{64}\right)\frac{1}{4} + \left(\frac{55}{64}\right)\frac{1}{4} + \left(\frac{39}{64}\right)\frac{1}{4} + \left(\frac{15}{64}\right)\frac{1}{4} \\
&= \frac{63}{256} + \frac{55}{256} + \frac{39}{256} + \frac{15}{256} \\
&= \frac{172}{256} \\
&= \frac{43}{64} \approx 0.671875
\end{aligned}$$

□

Lecture 28: The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus): Let f be a continuous function on the interval $[a, b]$ and let F be any antiderivative of f. Then:

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Examples:

1. Find $\int_0^1 x^2 \, dx$

   Solution

$$\begin{aligned} \int_0^1 x^2 \, dx &= \frac{x^3}{3}\Big|_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \end{aligned}$$

□

2. Find $\int_0^2 x^5 \, dx$

   Solution

$$\begin{aligned} \int_0^2 x^5 \, dx &= \frac{x^6}{6}\Big|_0^2 \\ &= \frac{2^6}{6} - 0 \\ &= \frac{32}{3} \end{aligned}$$

□

Properties of the Definite Integral:

- $\dfrac{d}{dx}\left(\int_a^x f(t)\ dt\right) = f(x)$
- $\int_a^a f(x)\ dx = 0$
- $\int_a^b k \cdot f(x)\ dx = k\int_a^b f(x)\ dx$, for any real constant k
- $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
- $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Examples:

1. Find $\int_0^{\pi/4} \sin x\ dx$

   Solution

$$\begin{aligned}\int_0^{\pi/4} \sin x\ dx &= (-\cos x)\Big|_0^{\pi/4} \\ &= -\cos\left(\frac{\pi}{4}\right) - (-\cos 0) \\ &= -\frac{\sqrt{2}}{2} + 1\end{aligned}$$

□

2. Find $\int_0^4 2(t^{1/2} - t)\ dt$

Solution

$$\begin{aligned}
\int_0^4 2(t^{1/2} - t)\ dt &= 2\int_0^4 (t^{1/2} - t)\ dt \\
&= 2\left(\frac{2}{3}t^{3/2} - \frac{1}{2}t^2\right)\Bigg|_0^4 \\
&= 2\left(\frac{2}{3}4^{3/2} - \frac{1}{2}4^2\right) - 0 \\
&= 2\left(\frac{2}{3}8 - 8\right) \\
&= -\frac{16}{3}
\end{aligned}$$

□

3. Find $\int_0^2 (x^2 + 1)\ dx$

Solution

$$\begin{aligned}
\int_0^2 (x^2 + 1)dx &= \left(\frac{x^3}{3} + x\right)\Bigg|_0^2 \\
&= \left(\frac{8}{3} + 2\right) - \left(\frac{0}{3} + 0\right) \\
&= \frac{8}{3} + \frac{6}{3} \\
&= \frac{14}{3}
\end{aligned}$$

□

4. Find $\int_1^2 \frac{2x^5 - x + 3}{x^2}\, dx$

Solution

$$\begin{aligned}\int_1^2 \frac{2x^5 - x + 3}{x^2} dx &= \int_1^2 (2x^3 - x^{-1} + 3x^{-2}) dx \\ &= \left(\frac{x^4}{2} - \ln|x| - \frac{3}{x}\right)\Bigg|_1^2 \\ &= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right) \\ &= 8 - \ln 2 - \frac{3}{2} - \frac{1}{2} + 3 \\ &= 9 - \ln 2\end{aligned}$$

□

When using u-substitution, you must change the bounds on your integral to be in terms of u.

Examples:

1. Find $\int_1^3 \frac{\sqrt{\ln x}}{x}\, dx$

Solution Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$x = 1 \Rightarrow u = \ln 1 = 0,\ x = 3 \Rightarrow u = \ln 3$

$$\begin{aligned}\int_1^3 \frac{\sqrt{\ln x}}{x}\, dx &= \int_0^{\ln 3} \sqrt{u}\, du \\ &= \left(\frac{2}{3} u^{3/2}\right)\Bigg|_0^{\ln 3} \\ &= \frac{2}{3} (\ln 3)^{3/2}\end{aligned}$$

□

2. Find $\int_0^1 \frac{e^{2z}}{\sqrt{1 + e^{2z}}}\, dz$

Solution Let $u = 1 + e^{2z} \Rightarrow du = 2e^{2z} dz$

$z = 0 \to u = 2,\ z = 1 \Rightarrow u = 1 + e^2$

$$\int_0^1 \frac{e^{2z}}{\sqrt{1+e^{2z}}}\ dz = \frac{1}{e}\int_2^{1+e^2} \frac{1}{\sqrt{u}}\ du$$
$$= \frac{1}{2}\int_2^{1+e^2} u^{-1/2}\ du$$
$$= \left(\frac{1}{2}2u^{1/2}\right)\Bigg|_0^{1+e^2}$$
$$= \sqrt{1+e^2}$$

□

3. Find $\displaystyle\int_1^2 \frac{x}{(x^2+2)^3}dx$

Solution Let $u = x^2 + 2 \Rightarrow du = 2xdx$
$x = 1 \Rightarrow u = 3,\ x = 2 \Rightarrow 6$

$$\int_1^2 \frac{x}{(x^2+2)^3}dx = \frac{1}{2}\int_3^6 \frac{1}{u^3}du$$
$$= \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)\Bigg|_3^6$$
$$= \frac{1}{2}\left(-\frac{1}{2u^2}\right)\Bigg|_3^6$$
$$= \frac{1}{2}\left[\left(-\frac{1}{2(6)^2}\right) - \left(-\frac{1}{2(3)^2}\right)\right]$$
$$= \frac{1}{2}\left[-\frac{1}{72} + \frac{1}{18}\right]$$
$$= \frac{1}{2}\left[-\frac{1}{72} + \frac{4}{72}\right]$$
$$= \frac{1}{2}\left[\frac{3}{72}\right]$$
$$= \frac{1}{2}\left[\frac{1}{24}\right]$$
$$= \frac{1}{48}$$

□

4. Find $\int_1^2 \left(e^{4x} - \frac{1}{(x+1)^2}\right)\, dx$

Solution

$$\begin{aligned}\int_1^2 \left(e^{4x} - \frac{1}{(x+1)^2}\right)\, dx &= \int_1^2 e^{4x}\, dx - \int_1^2 \frac{1}{(x+1)^2}\, dx \\ &= \frac{1}{4}\int_4^8 e^u\, du - \int_2^3 u^{-2}\, du \\ &= \left(\frac{1}{4}e^u\right)\Big|_4^8 - (-u^{-1})\Big|_2^3 \\ &= \frac{1}{4}(e^8 - e^4) - \frac{1}{5}\end{aligned}$$

□

Lecture 29: Total Area

Computing Total Area:

When finding the total area bounded by a function on the interval $[a, b]$ and the x-axis, you must consider when the function is negative and when the function is positive

(i) Find the x-intercepts of the function.

(ii) Plot the x-intercepts on a number line of the domain of f and test the intervals to see if your function is positive or negative on those intervals

(iii) Split your integral using the intervals found in step 2. If the function is negative on that interval, negate the integral. Finally, evaluate each integral and add them together. Alternatively, you could skip the step of testing your intervals for negative or positive and just add up the absolute value of the split integrals.

Examples:

1. Compute the area of the region bounded by the graph of $f(x) = 1 - e^{-x}$ and the lines $y = 0$, $x = -1$, $x = 2$.

   Solution

   (i)

$$\begin{aligned} f(x) = 0 &\Leftrightarrow 1 - e^{-x} = 0 \\ &\Leftrightarrow 1 = e^{-x} \\ &\Leftrightarrow x = 0 \end{aligned}$$

   Thus we must split the integral at 0.

   (ii) If we test the intervals we have that $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$.

   (iii)

$$\begin{aligned} \text{Area} &= \int_{-1}^{0} -f(x)\ dx + \int_{0}^{2} f(x)\ dx \\ &= \int_{-1}^{0} (e^{-x} - 1)\ dx + \int_{0}^{2} (1 - e^{-x})\ dx \\ &= (-e^{-x} - x)\Big|_{-1}^{0} + (x + e^{-x})\Big|_{0}^{2} \\ &= (-1 - 0) - (-e + 1) + (2 + e^{-2}) - (0 + 1) \\ &= -1 + e - 1 + 2 + e^{-2} - 1 \\ &= e - 1 + e^{-2} \end{aligned}$$

Alternative Method:

$$\begin{aligned}
\text{Area} &= \left|\int_{-1}^{0} f(x)\ dx\right| + \left|\int_{0}^{2} f(x)\ dx\right| \\
&= \left|\int_{-1}^{0} 1 - e^{-x}\ dx\right| + \left|\int_{0}^{2} 1 - e^{-x}\ dx\right| \\
&= \left|(x + e^{-x})\Big|_{-1}^{0}\right| + \left|(x + e^{-x})\Big|_{0}^{2}\right| \\
&= \left|(0+1) - (-1+e)\right| + \left|(2 + e^{-2}) - (0+1)\right| \\
&= |2 - e| + \left|1 + e^{-2}\right| \\
&= e - 2 + 1 + e^{-2} \\
&= 2 - 1 + e^{-2}
\end{aligned}$$

□

2. Find the total area bounded by the curve $f(x) = x$ and the $x - axis$ from $x = -3$ to $x = 1$

Solution

(i) $f(x) = 0 \Leftrightarrow x = 0$ so there is one x-intercept at $(0, 0)$.
(ii) Testing the intervals we find that $f(x) < 0$ on $[-3, 0)$ and $f(x) > 0$ on$(0, 1]$
(iii)

$$\begin{aligned}
\text{Area} &= -\int_{-3}^{0} x dx + \int_{0}^{1} x dx \\
&= -\left(\frac{x^2}{2}\right)\Bigg|_{-3}^{0} + \left(\frac{x^2}{2}\right)\Bigg|_{0}^{1} \\
&= -\left(\frac{(0)^2}{2} - \frac{(-3)^2}{2} -\right) + \left(\frac{(1)^2}{2} - \frac{(0)^2}{2}\right) \\
&= -\left(-\frac{9}{2}\right) + \frac{1}{2} \\
&= \frac{10}{2} \\
&= 5
\end{aligned}$$

Alternative Method:

$$
\begin{aligned}
\text{Area} &= \left|\int_{-3}^{0} x dx\right| + \left|\int_{0}^{1} x dx\right| \\
&= \left|\left(\frac{x^2}{2}\right)\Big|_{-3}^{0}\right| + \left|\left(\frac{x^2}{2}\right)\Big|_{0}^{1}\right| \\
&= \left|\left(\frac{(0)^2}{2} - \frac{(-3)^2}{2} -\right)\right| + \left|\left(\frac{(1)^2}{2} - \frac{(0)^2}{2}\right)\right| \\
&= \left|-\frac{9}{2}\right| + \left|\frac{1}{2}\right| \\
&= \frac{9}{2} + \frac{1}{2} \\
&= \frac{10}{2} \\
&= 5
\end{aligned}
$$

□

3. Find the area between the graph of $f(x) = x^2 - x - 2$ and the x-axis from $x = -2$ to $x = 3$

Solution

(i) $f(x) = 0 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow (x-2)(x+1) = 0 \Leftrightarrow x = 2, -1$

(ii) We get $f(x) > 0$ on $[-2, -1)$, $(2, 3]$ and $f(x) < 0$ on $(-1, 2)$

(iii)

$$\begin{aligned}
\text{Area} &= \int_{-2}^{-1}(x^2-x-2)dx + \left(-\int_{-1}^{2}(x^2-x-2)dx\right) + \int_{2}^{3}(x^2-x-2)dx \\
&= \left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Bigg|_{-2}^{-1} - \left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Bigg|_{-1}^{2} + \left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Bigg|_{2}^{3} \\
&= \left[\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}-2(-1)\right)-\left(\frac{(-2)^3}{3}-\frac{(-2)^2}{2}-2(-2)\right)\right] \\
&\quad -\left[\left(\frac{(2)^3}{3}-\frac{(2)^2}{2}-2(2)\right)-\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}-2(-1)\right)\right] \\
&\quad +\left[\left(\frac{(3)^3}{3}-\frac{(3)^2}{2}-2(3)\right)-\left(\frac{(2)^3}{3}-\frac{(2)^2}{2}-2(2)\right)\right] \\
&= \left[\left(\frac{-1}{3}-\frac{1}{2}+2\right)-\left(\frac{-8}{3}-\frac{4}{2}+4\right)\right]-\left[\left(\frac{8}{3}-\frac{4}{2}-4\right)-\left(\frac{-1}{3}-\frac{1}{2}+2\right)\right] \\
&\quad +\left[\left(\frac{27}{3}-\frac{9}{2}-6\right)-\left(\frac{8}{3}-\frac{4}{2}-4\right)\right] \\
&= \left[\left(\frac{-2}{6}-\frac{3}{6}+\frac{12}{6}\right)-\left(\frac{-16}{6}-\frac{12}{6}+\frac{24}{6}\right)\right]-\left[\left(\frac{16}{6}-\frac{12}{6}-\frac{24}{6}\right)-\left(\frac{-2}{6}-\frac{3}{6}+\frac{12}{6}\right)\right] \\
&\quad +\left[\left(\frac{54}{6}-\frac{27}{6}-\frac{36}{6}\right)-\left(\frac{16}{6}-\frac{12}{6}-\frac{24}{6}\right)\right] \\
&= \left[\frac{7}{6}-\frac{-4}{6}\right]-\left[\frac{-20}{6}-\frac{7}{6}\right]+\left[\frac{-9}{6}-\frac{-20}{6}\right] \\
&= \frac{11}{6}+\frac{27}{6}+\frac{11}{6} \\
&= \frac{49}{6}
\end{aligned}$$

□

4. Compute the area of the region bounded by the lines $y = 0$, $x = 0$, $x = 3$, and the graph of

$$f(x) = \begin{cases} x^2 - 1 & x < 2 \\ x + 1 & x \geq 212 \end{cases}$$

Solution We must split the integral at 2 because thats where the function splits. Futher

$$x^2 - 1 = 0 \Leftrightarrow x = \pm 1$$

We can ignore $x = -1$ since that is not in the interval $[0, 3]$

$$x + 1 = 0 \Leftrightarrow x = -1$$

This is not in the interval the piece is defined on nor is it in the interval $[0, 3]$. Testing the intervals we have f is negative on $[0, 1)$ and positive on the other intervals

$$\begin{aligned}
\text{Area} &= \int_0^1 -f(x)\ dx + \int_1^2 f(x)\ dx + \int_2^3 f(x)\ dx \\
&= \int_0^1 (1 - x^2)\ dx + \int_1^2 (x^2 - 1)\ dx + \int_2^3 (x + 1)\ dx \\
&= \left(x - \frac{x^3}{3}\right)\Bigg|_0^1 + \left(\frac{x^3}{3} - x\right)\Bigg|_1^2 + \left(\frac{x^2}{2} + x\right)\Bigg|_2^3 \\
&= \frac{2}{3} + \frac{4}{3} + \frac{7}{2} \\
&= \frac{11}{2}
\end{aligned}$$

□

5. A pollutant is entering a lake from a factory at the rate

$$P'(t) = 140t^{5/3}$$

where t is the number of years since the factory started introducing pollutants into the lake. Ecologists estimate that the lake can accept the total level of pollution of 4850 units before all the fish life in the lake ends. Can the factory operate for 4 years without killing all the fish?

Solution The total amount of pollutant released over 4 years is given by

$$\int_0^4 140t^{5/2}\ dt$$

$$\begin{aligned}
\int_0^4 140t^{5/2}\ dt &= \left(140 \cdot \frac{2}{7} t^{7/2}\right)\Bigg|_0^4 \\
&= 140 \cdot \frac{2}{7}(4)^{7/2} \\
&= 40(128) \\
&= 5120
\end{aligned}$$

□

Since $5120 > 4850$ it would kill all the fish.

6. An oil tanker hit a hidden rock at time $t = 0$ and started leaking oil into the ocean with the rate (given in barrels per hour) of

$$L'(t) = \frac{80\ln(t+1)}{t+1}$$

(a) Find the total number of barrels that the ship will leak on the first day

(b) Find the total number of barrels that the ship will leak on the second day

Solution

(a) The total amount of barrels that the ship will leak on the first day is given by

$$\int_0^{24} \frac{80\ln(t+1)}{t+1}\ dt$$

Let $u = \ln(t+1) \Rightarrow du = \dfrac{1}{t+1}dt$

$$\begin{aligned}\int_0^{24} \frac{80\ln(t+1)}{t+1}\ dt &= 80\int_0^{\ln 25} u\ du\\ &= 40u^2\Big|_0^{\ln 25}\\ &= 40(\ln(25))^2\\ &\approx 414.5 \text{ barrels}\end{aligned}$$

(b) The total amount of barrels that the ship will leak on the second day is given by

$$\int_{24}^{48} \frac{80\ln(t+1)}{t+1}\ dt$$

Let $u = \ln(t+1) \Rightarrow du = \dfrac{1}{t+1}dt$

$$\begin{aligned}\int_{24}^{48} \frac{80\ln(t+1)}{t+1}\ dt &= 80\int_{25}^{\ln 49} u\ du\\ &= 40u^2\Big|_{\ln 25}^{\ln 49}\\ &= 40((\ln 49)^2 - (\ln 25)^2)\\ &\approx 191.5 \text{ barrels}\end{aligned}$$

□

7. After a long study scientists conclude that a eucalyptus tree will grow at the rate $0.6 + \dfrac{4}{(t+1)^3}$ ft per year, where t is time (in years). Find the number of feet the tree will grow in the k-th

year.

Solution The number of feet the tree will grow in its k-th year is

$$\int_{k-1}^{k} \left(0.6 + \frac{4}{(t+1)^3}\right)\ dt$$

Let $u = t + 1 \Rightarrow du = dt$

$$\begin{aligned}
\int_{k-1}^{k} \left(0.6 + \frac{4}{(t+1)^3}\right)\ dt &= \int_{k-1}^{k} 0.6\ dt + 4\int_{k}^{k+1} u^{-3}\ du \\
&= 0.6t\Big|_{k-1}^{k} - 2u^{-2}\Big|_{k}^{k+1} \\
&= 0.6k - 0.6(k-1) - 2((k+1)^-2 - k^{-2}) \\
&= 0.6 + 2\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \\
&= 0.6 + 2\left(\frac{(k+1)^2 - k^2}{k^2(k+1)^2}\right) \\
&= 0.6 + 2\left(\frac{\cancel{k^2} + 2k + 1 - \cancel{k^2}}{k^2(k+1)^2}\right) \\
&= 0.6 + \frac{4k+2}{k^2(k+1)^2}\ \text{feet}
\end{aligned}$$

□

Lecture 30: The Area Between Two Curves

Area Between Two Curves

If f and g are both continuous and $f(x) \geq g(x)$ on $[a,b]$ then the area between f and g from $x = a$ to $x = b$ is given by

$$\int_a^b (f(x) - g(x))dx$$

Steps To Find Area:

(i) Find the points at which f and g intersect, if any. If the endpoints a and b were not given, the smallest intersection point is a and the largest is b.

(ii) Split the interval $[a,b]$ using the intersection points found above, if needed. For example, if f and g intersect at $x = c$ where c is between a and b, then the interval splits into $[a,c]$ and $[c,b]$.

(iii) For each interval, determine which function is larger (i.e. on top) by either plugging in a point or graphing them. Alternatively you can take the absolute value of each integral.

(iv) Evaluate the integrals to find the area.

Examples:

1. Find the area of the region bounded by the lines $x = 1$, $x = 2$, and the graphs of $f(x) = 2x$, $g(x) = x^2 - 1$.

   Solution

   (i)

$$\begin{aligned} 2x = x^2 - 1 &\Leftrightarrow x^2 - 2x - 1 = 0 \\ &\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} \\ &\Leftrightarrow x = \frac{2 \pm \sqrt{8}}{2} \\ &\Leftrightarrow x = \frac{2 \pm 2\sqrt{2}}{2} \\ &\Leftrightarrow x = 1 \pm \sqrt{2} \end{aligned}$$

   (ii) Both of the intersection points aren't in the interval $[1,2]$ so we do not need to split the interval.

   (iii) Testing the interval we see that $2x > x^2 - 1$ on $[1,2]$

(iv)

$$\begin{aligned}
\text{Area} &= \int_1^2 (2x - (x^2 - 1))\ dx \\
&= \int_1^2 (2x - x^2 + 1)\ dx \\
&= \left(x^2 - \frac{x^3}{3} + x\right)\Bigg|_1^2 \\
&= \left(2^2 - \frac{2^3}{3} + 2\right) - \left(1^1 - \frac{1^3}{3} + 1\right) \\
&= 4 - \frac{8}{3} + 2 - 1 + \frac{1}{3} - 1 \\
&= \frac{5}{3}
\end{aligned}$$

<u>Alternative Method:</u>

$$\begin{aligned}
\text{Area} &= \left|\int_1^2 (2x - (x^2 - 1))\ dx\right| \text{ or } \left|\int_1^2 ((x^2 - 1) - 2x)\ dx\right| \\
&= \left|\int_1^2 (2x - x^2 + 1)\ dx\right| \\
&= \left|\left(x^2 - \frac{x^3}{3} + x\right)\Bigg|_1^2\right| \\
&= \left|\left(2^2 - \frac{2^3}{3} + 2\right) - \left(1^1 - \frac{1^3}{3} + 1\right)\right| \\
&= \left|4 - \frac{8}{3} + 2 - 1 + \frac{1}{3} - 1\right| \\
&= \left|\frac{5}{3}\right| \\
&= \frac{5}{3}
\end{aligned}$$

□

2. Find the area of the region bounded by the lines $x = 1$, $x = 2$, and the graphs of $f(x) = x^2 - 2$, $g(x) = 1 - 3x$.

Solution

(i)

$$\begin{aligned} x^2 - 2 = 1 - 3x &\Leftrightarrow x^2 + 3x - 3 = 0 \\ &\Leftrightarrow x = \frac{-3 \pm \sqrt{9 - 4(1)(-3)}}{2(1)} \\ &\Leftrightarrow x = \frac{-3 \pm \sqrt{21}}{2} \end{aligned}$$

(ii) Both of these are outside of $[1,2]$ so we do not need to split the interval.

(iii) Testing we see that $x^2 - 2 > 1 - 3x$ on $[1,2]$.

(iv)

$$\begin{aligned} \text{Area} &= \int_1^2 ((x^2 - 2) - (1 - 3x))\ dx \\ &= \int_1^2 (x^2 - 3 + 3x)\ dx \\ &= \left(\frac{x^3}{3} - 3x + \frac{3x^2}{2}\right)\Bigg|_1^2 \\ &= \left(\frac{2^3}{3} - 6 + \frac{12}{2}\right) - \left(\frac{1}{3} - 3 + \frac{3}{2}\right) \\ &= \frac{8}{3} - 6 + 6 - \frac{1}{3} + 3 - \frac{3}{2} \\ &= \frac{23}{6} \end{aligned}$$

□

3. Find the area between the curves $y = x^4 + \ln(x+10)$ and $y = x^3 + \ln(x+10)$

Solution

(i)

$$\begin{aligned} x^4 + \ln(x+10) = x^3 + \ln(x+10) &\Leftrightarrow x^4 = x^3 \\ &\Leftrightarrow x^4 - x^3 = 0 \\ &\Leftrightarrow x^3(x-1) = 0 \\ &\Leftrightarrow x = 0, 1 \end{aligned}$$

(ii) Thus the area we want to find is from $x = 0$ to $x = 1$.

(iii) Testing we see that $x^3 + \ln(x+10) > x^4 + \ln(x+10)$ on $[0,1]$.

(iv)

$$\begin{aligned}
\text{Area} &= \int_0^1 \left((x^3 + \ln(x+10)) - (x^4 + \ln(x+10))\right)\, dx \\
&= \int_0^1 (x^3 - x^4)\, dx \\
&= \left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 \\
&= \frac{1}{4} - \frac{1}{5} \\
&= \frac{1}{20}
\end{aligned}$$

Alternative Method:

$$\begin{aligned}
\text{Area} &= \left|\int_0^1 \left((x^3 + \ln(x+10)) - (x^4 + \ln(x+10))\right)\, dx\right| \\
\text{or } &\left|\int_0^1 \left((x^4 + \ln(x+10)) - (x^3 + \ln(x+10))\right)\, dx\right| \\
&= \int_0^1 (x^3 - x^4)\, dx \\
&= \left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 \\
&= \frac{1}{4} - \frac{1}{5} \\
&= \frac{1}{20}
\end{aligned}$$

□

4. Find the area of the region bounded by the lines $x = 2$, $x = 4$, and the curves $y = \dfrac{x-1}{4}$, $y = \dfrac{1}{x-1}$

Solution

(i)

$$\begin{aligned}
\frac{x-1}{4} = \frac{1}{x-1} &\Leftrightarrow (x-1)^2 = 4 \\
&\Leftrightarrow x^2 - 2x + 1 = 4 \\
&\Leftrightarrow x^2 - 2x - 3 = 0 \\
&\Leftrightarrow (x-3)(x+1) = 0 \\
&\Leftrightarrow x = 3, -1
\end{aligned}$$

(ii) -1 isn't in the interval so we just have to split to $[2,3]$ and $[2,4]$.

(iii) Testing the intervals we have $\frac{1}{x-1} > \frac{x-1}{4}$ on $[2,3]$ and $\frac{x-1}{4} > \frac{1}{x-1}$ on $[3,4]$

(iv)

$$\begin{aligned}\text{Area} &= \int_2^3 \left(\frac{1}{x-1} - \frac{x-1}{4}\right) dx + \int_3^4 \left(\frac{x-1}{4} - \frac{1}{x-1}\right) dx \\ &= \left(\ln|x-1| - \frac{x^2}{8} + \frac{x}{4}\right)\Bigg|_2^3 + \left(\frac{x^2}{8} - \frac{x}{4} - \ln|x-1|\right)\Bigg|_3^4 \\ &= 2\ln 2 - \ln 3 + \frac{1}{4}\end{aligned}$$

□

5. Find the area between $y = x^4$ and $y = 2x - x^2$

Solution

(i) $x^4 = 2x - x^2 \Leftrightarrow x^4 + x^2 - 2x = 0 \Leftrightarrow x(x^3 + x - 2) = 0 \Leftrightarrow x = 0, 1$. Therefore we have $a = 0$ and $b = 1$.

(ii) There are no intersection points between a and b so we do not need to split the interval.

(iii) Graphing the functions gives

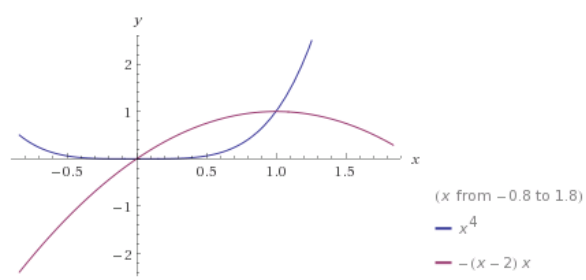


Figure 33

From the graph we see that $y = 2x - x^2$ is above $y = x^4$. Alternatively, you can plug in $\frac{1}{2}$ and see that $2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ which means $2x - x^2 \geq x^4$ on $[0,1]$ so $y = 2x - x^2$ is above. This gives

$$\text{Area} = \int_0^1 (2x - x^2 - x^4) dx$$

(iv)

$$\begin{aligned}
\text{Area} &= \int_0^1 (2x - x^2 - x^4)dx \\
&= \left(\frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 \\
&= \left(x - \frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 \\
&= \left(1 - \frac{1}{3} - \frac{1}{5}\right) - (0 - 0 - 0) \\
&= 1 - \frac{1}{3} - \frac{1}{5} \\
&= \frac{15}{15} - \frac{5}{15} - \frac{3}{15} \\
&= \frac{7}{15}
\end{aligned}$$

□

6. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$, $x = 1$

Solution

(i) $x^2 + 1 = x \Leftrightarrow x^2 - x + 1 = 0$ This is not solvable so there are no intersection points. (Try to do the quadratic formula and you'll end up with a negative inside the square root).

(ii) There are no intersection points so we do not need to split the interval.

(iii) The graph looks like

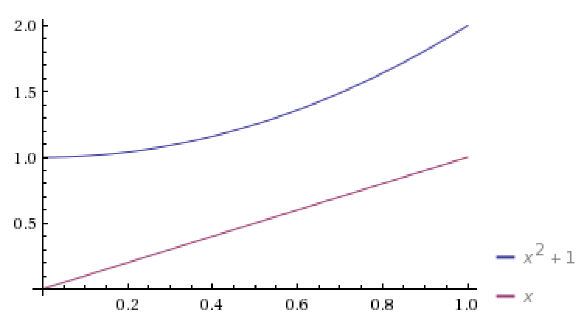


Figure 34

so x^2+1 is on top. Alternatively, you can plug in a value such as $\frac{1}{2}$ to see which is bigger. This tells us that

$$\text{Area} = \int_0^1 (x^2 - x + 1)dx$$

(iv)

$$\begin{aligned}\text{Area} &= \int_0^1 (x^2 - x + 1)dx \\ &= \left(\frac{x^3}{3} - \frac{x^2}{2} + x\right)\Big|_0^1 \\ &= \left(\frac{1}{3} - \frac{1}{2} + 1\right) - (0 - 0 + 0) \\ &= \frac{2}{6} - \frac{3}{6} + \frac{6}{6} \\ &= \frac{5}{6}\end{aligned}$$

□

7. Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, $x = 0$

Solution

(i) $\sin x = \cos x \Leftrightarrow \tan x = 1$ Thus $x = \frac{\pi}{4}$ since we are on $[0, \frac{\pi}{2}]$

(ii) Since an intersection occurs between a and b, we split the interval into $[0, \frac{\pi}{4}]$ and $[\frac{\pi}{4}, \frac{\pi}{2}]$

(iii) The graph is

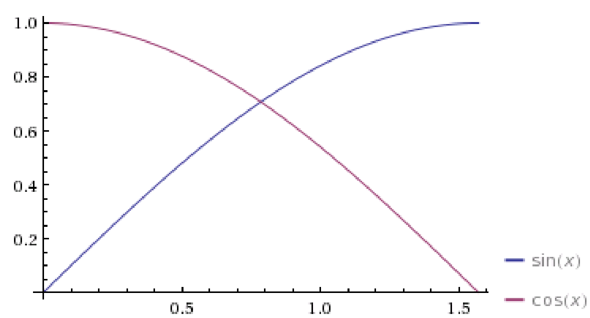


Figure 35

Therefore we know that $\cos x$ is on top during the interval $[0, \frac{\pi}{4})$ and $\sin x$ is on top during the interval $(\frac{\pi}{4}, \frac{\pi}{2}]$. Thus our area is given by

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\frac{\pi}{2}} (\sin x - \cos x)dx$$

(iv)

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx \\
&= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2} \\
&= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] \\
&\quad + \left[\left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right] \\
&= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right] + \left[0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{2}{2} - \frac{2}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\
&= \frac{4\sqrt{2}}{2} - \frac{4}{2} \\
&= 2\sqrt{2} - 2
\end{aligned}$$

□

Lecture 31: Integration By Parts

Recall:

$$\frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + u'(x)v(x)$$

Thus we have

$$\int \frac{d}{dx}(u(x)v(x))\ dx = \int u(x)v'(x)\ dx + \int u'(x)v(x) \Rightarrow u(x)v(x) = \int u(x)v'(x)\ dx + \int u'(x)v(x)\ dx$$
$$\Rightarrow \int u(x)v'(x)\ dx = u(x)v(x) - \int (v(x)u'(x))\ dx$$

Integration By Parts:

The following are equivalent formulas used to integrate by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$
$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx$$
$$\int udv = uv - \int vdu$$
$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

Note: To find du, differentiate u and to find v, find the simplest antiderivative of dv (i.e. integrate and leave off the constant)

Examples:

1. Find $\int (x+1)e^x\ dx$

   Solution Let $u = x+1, dv = e^x dx \Rightarrow du = dx, v = e^x$

$$\int (x+1)e^x\ dx = (x+1)e^x - \int e^x\ dx$$
$$= (x+1)e^x - e^x + C$$
$$= xe^x + C$$

□

2. Find $\ln(5x)\ dx$

   Solution Let $u = \ln(5x), dv = dx \Rightarrow u = \frac{1}{x}, v = x$

$$\ln(5x)\ dx = x\ln(5x) - \int x\frac{1}{x}\ dx$$
$$= x\ln(5x) - x + C$$

$\square$

3. Find $\int x\sin(2x)\ dx$

Solution $u = x, dv = \sin(2x) \Rightarrow du = dx, v = -\frac{1}{2}\cos(2x)$

$$\begin{aligned}\int x\sin(2x)\ dx &= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\ dx \\ &= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C\end{aligned}$$

$\square$

4. Find $\int_1^2 (x^2-1)e^{2x}\ dx$

Solution $u = x^2 - 1, dv = e^{2x}dx \Rightarrow du = 2xdx, v = \frac{1}{2}e^{2x}$

$$\begin{aligned}\int_1^2 (x^2-1)e^{2x}\ dx &= \left(\frac{1}{2}(x^2-1)e^{2x}\right)\Big|_1^2 - \left(-\frac{1}{2}\int_1^2 e^{2x}(2x)\ dx\right) \\ &= \frac{3}{2}e^4 - \int_1^2 xe^{2x}\ dx \\ &\text{Let } u = x, dv = e^{2x} \Rightarrow du = dx, v = \frac{1}{2}e^{2x} \\ &= \frac{3}{2}e^4 - \left(\left(\frac{1}{2}xe^{2x}\right)\Big|_1^2 - \frac{1}{2}\int_1^2 e^{2x}\ dx\right) \\ &= \frac{3}{2}e^4 - \left(\left(e^4 - \frac{1}{2}e^2\right) - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right)\Big|_1^2\right) \\ &= \frac{3}{2}e^4 - \left(e^4 - \frac{1}{2}e^2 - \frac{1}{4}(e^4 - e^2)\right) \\ &= \frac{3}{4}e^4 + \frac{1}{4}e^2\end{aligned}$$

$\square$

5. The area covered by a patch of moss is growing at the rate of

$$A'(t) = \sqrt{t}\ln t \text{cm}^2/\text{day}$$

for $t \geq 1$. Find the area consumed by the moss between days 4 and 9.

Solution We want $A(9) - A(4)$ i.e. $\displaystyle\int_4^9 \sqrt{t} \ln t \; dt$

$$\begin{aligned}
\int_4^9 \sqrt{t} \ln t \; dx &= \left(\frac{2}{3} t^{3/2} \ln t\right)\Bigg|_4^9 - \frac{2}{3}\int_4^9 t^{3/2}\frac{1}{t} \; dt \\
&= \left(18 \ln 9 - \frac{16}{3}\ln 4\right) - \frac{2}{3}\int_4^9 t^{1/2} \; dt \\
&= \left(36 \ln 3 - \frac{32}{3}\ln 2\right) \frac{4}{9} t^{3/2}\Bigg|_4^9 \\
&= \left(36 \frac{32}{3}\ln 2\right) - \frac{4}{9}(3^3 - 2^3) \\
&= 36 \ln 3 - \frac{32}{3}\ln 2 - \frac{76}{9}
\end{aligned}$$

$\square$

6. Find $\displaystyle\int xe^x dx$

Solution Let $u = x$ and $dv = e^x dx$ so $du = dx$ and $v = e^x$. This gives

$$\begin{aligned}
\int xe^x dx &= xe^x - \int e^x dx \\
&= xe^x - e^x + C
\end{aligned}$$

$\square$

7. Find $\displaystyle\int x \sin x dx$

Solution Let $u = x$ and $dv = \sin x dx$ so $du = dx$ and $v = -\cos x$. So we have

$$\begin{aligned}
\int x \sin x dx &= x(-\cos x) - \int (-\cos x) dx \\
&= -x \cos x + \sin x + C
\end{aligned}$$

$\square$

8. Find $\displaystyle\int \ln x dx$

Solution Let $u = \ln x$ and $dv = dx$, then $du = \dfrac{1}{x} dx$ and $v = x$. We get

$$\begin{aligned}
\int \ln x dx &= x \ln x - \int \cancel{x}\frac{1}{\cancel{x}} dx \\
&= x \ln x - \int dx = x \ln x - x + C
\end{aligned}$$

□

9. Find $\int \frac{\ln x}{x^5} dx$

Solution Let $u = \ln x$ and $dv = \frac{1}{x^5} dx = x^{-5} dx$, then $du = \frac{1}{x} dx$ and $v = -\frac{1}{4x^4}$. We have

$$\begin{aligned}\int \frac{\ln x}{x^5} dx &= \ln x \left(-\frac{1}{4x^4}\right) - \int -\frac{1}{4x^4}\left(\frac{1}{x}\right) dx \\ &= -\frac{\ln x}{4x^4} + \frac{1}{4}\int x^{-5} dx \\ &= -\frac{\ln x}{4x^4} - \frac{1}{16x^4} + C\end{aligned}$$

□

10. Find $\int x^2 e^{3x} dx$

Solution We'll have to do integration by parts twice to solve this one.

Let $u = x^2, dv = e^{3x} dx \Rightarrow du = 2x dx, v = \frac{1}{3} e^{3x}$.

$$\begin{aligned}\int x^2 e^{3x} dx &= x^2 \left(\frac{1}{3}\right) - \int \frac{1}{3} e^{3x} (2x) dx \\ &= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx\end{aligned}$$

Now we do integration by parts to solve $\int x e^{3x} dx$

Let $u_1 = x, dv_1 = e^{3x} dx \Rightarrow du_1 = dx v_1 = \frac{1}{3} e^{3x}$.

$$\begin{aligned}\int x e^{3x} dx &= x \left(\frac{1}{3}\right) e^{3x} - \int \frac{1}{3} e^{3x} dx \\ &= \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx \\ &= \frac{1}{3} x e^{3x} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right) + C\end{aligned}$$

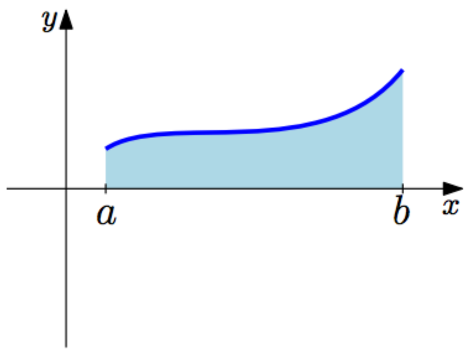
Putting this into our original equation we have

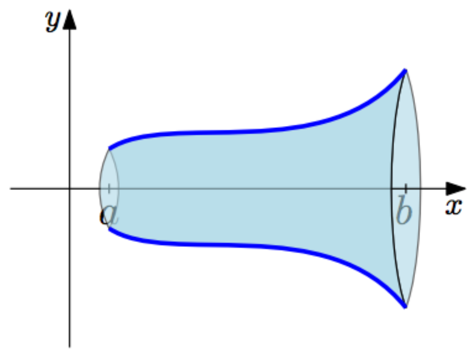
$$\begin{aligned}\int x^2e^{3x}dx &= x^2\left(\frac{1}{3}\right) - \int \frac{1}{3}e^{3x}(2x)dx\\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx\\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right) + C\right)\\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C\end{aligned}$$

□

Lecture 32: Volume and Average Value

Consider a continuous function over the interval $[a, b]$ and imagine we revolve the region bounded by the graph of f and the lines $y = 0$, $x = a$, and $x = b$ around the x-axis to create a solid called a *solid of revolution*. Our question is, how do we compute the volume of this solid?





<u>Volume of a Solid</u>

If f is a continuous function on $[a, b]$, then the volume found by rotating the region bounded by f, $y = 0$, $x = a$, and $x = b$ about the x-axis is given by:

$$\text{Volume} = \int_a^b \pi(f(x))^2 dx$$

Note: This equation is essentially finding the volume of a cylinder of radius $f(x)$

<u>Examples:</u>

1. Compute the volume of the solid obtained by rotating the region bounded by $f(x) = \sqrt{2x+1}$ and the lines $y = 0$, $x = 1$, and $x = 4$ around the x-axis.

   Solution

$$\begin{aligned}\text{Volume} &= \int_1^4 \pi(\sqrt{2x+1})^2\ dx \\ &= \pi \int_1^4 (2x+1)\ dx \\ &= \pi(x^2 + x)|_1^4 \\ &= \pi((4^2+4) - (1^2+1)) \\ &= 18\pi\end{aligned}$$

□

2. Compute the volume of the solid obtained by rotating the region bounded by $f(x) = \sec x$ and the lines $y = 0$, $x = 0$, and $x = \pi/4$ around the x-axis.

Solution

$$\begin{aligned}\text{Volume} &= \int_0^{\pi/4} \pi \sec^2 x \ dx \\ &= \pi \tan x\Big|_0^{\pi/4} \\ &= \pi\end{aligned}$$

□

3. Find the volume of the solid bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and $y = 0$ rotated about the x-axis.

Solution Using our formula we have:

$$\begin{aligned}\text{Volume} &= \int_1^4 \pi(x^2 - 4x + 5)^2 dx \\ &= \pi \int (x^4 - 8x^3 + 26x^2 - 40x + 25)dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\ &= \pi\left[\left(\frac{1}{5}(4)^5 - 2(4)^4 + \frac{26}{3}(4)^3 - 20(4)^2 + 25(4)\right) - \left(\frac{1}{5}(1)^5 - 2(1)^4 + \frac{26}{3}(1)^3 - 20(1)^2 + 25(1)\right)\right] \\ &= \pi\left[\frac{1024}{5} - 512 + \frac{1664}{3} - 320 + 100 - \frac{1}{5} + 2 - \frac{26}{3} + 20 - 25\right] \\ &= \frac{78\pi}{5}\end{aligned}$$

□

4. Find the volume of the solid bounded by $f(x) = \sqrt{x}$, $x = 1$, $x = 4$, and $y = 0$ rotated about the x-axis.

Solution Using the formula gives:

$$\begin{aligned}\text{Volume} &= \int_1^4 \pi(\sqrt{x})^2 dx \\ &= \pi \int_1^4 x dx \\ &= \pi\left(\frac{x^2}{2}\right)\Bigg|_1^4 \\ &= \pi\left(\frac{(4)^2}{2} - \frac{(1)^2}{2}\right) \\ &= \frac{15\pi}{2}\end{aligned}$$

□

Average Value

The *average value* of a continuous function f over an interval $[a, b]$ is:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Examples:

1. Compute the average value of $f(x) = x\ln x$ on the interval $[1, e]$

   Solution

$$\begin{aligned}
\text{Average Value} &= \frac{1}{e-1}\int_1^e x\ln x\ dx \\
&\text{Let } u = \ln x, dv = xdx \Rightarrow du = \frac{1}{x}dx, v = \frac{x^2}{2} \\
&= \frac{1}{e-1}\left(\left(\frac{x^2}{2}\ln x\right)\Big|_1^e - \frac{1}{2}\int x^2\frac{1}{x}\ dx\right) \\
&= \frac{1}{e-1}\left(\frac{e^2}{2} - \frac{1}{2}\int_1^e x\ dx\right) \\
&= \frac{1}{e-1}\left(\frac{e^2}{2} - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^e\right) \\
&= \frac{1}{e-1}\left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right) \\
&= \frac{1}{e-1}\cdot\frac{e^2+1}{4} \\
&= \frac{e^1+1}{4(e-1)} \\
&\approx 1.2206
\end{aligned}$$

□

2. Find the average value of $f(x) = x^2 - 2x$ on $[1, 4]$.

   Solution

$$\frac{1}{4-1}\int_1^4 (x^2-2x)dx = \frac{1}{3}\left(\frac{x^3}{3}-x^2\right)\Bigg|_1^4$$
$$= \frac{1}{3}\left[\left(\frac{(4)^3}{3}-(4)^2\right)-\left(\frac{(1)^3}{3}-(1)^2\right)\right]$$
$$= \frac{1}{3}\left[\frac{64}{3}-16-\frac{1}{3}+1\right]$$
$$= 2$$

□

3. Find the average value of $y = x^2 + 1$ from $x = 0$ to $x = 2$.

Solution

$$\frac{1}{2-0}\int_0^2 (x^2+1)dx = \frac{1}{2}\left(\frac{x^3}{3}+x\right)\Bigg|_0^2$$
$$= \frac{1}{2}\left[\left(\frac{(2)^3}{3}+2\right)-\left(\frac{(0)^3}{3}+0\right)\right]$$
$$= \frac{1}{2}\left[\frac{8}{3}+2\right]$$
$$= \frac{7}{3}$$

□

Lecture 33: Improper Integrals

Definition: Let f be a continuous function. An *improper integral* of f is an integral of one of the following forms:

- $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$
- $\int_{-\infty}^a f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$
- $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ for any constant c
- $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$ if $f(a)$ is undefined
- $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$ if $f(b)$ is undefined

If the limit for the integral exists then we say that the integral *converges*. Otherwise it *diverges*.

Examples:

1. Determine if the following improper integral converges. If so, find its value: $\int_2^\infty \frac{1}{x\ln x}\, dx$

   Solution

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\, dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\, dx \\ &= \lim_{b\to\infty} \ln|u|\Big|_{\ln 2}^{\ln b} \\ &= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) \\ &= \infty\end{aligned}$$

   Thus the integral diverges.

□

2. Find $\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}\, dx$

   Solution

$$\begin{aligned}\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}\, dx &= \int_{-\infty}^0 \frac{x}{(1+x^2)^2}\, dx + \int_0^\infty \frac{x}{(1+x^2)^2}\, dx \\ &= 0\end{aligned}$$

$\square$

3. Find $\int_0^\infty xe^{-x}\ dx$

   Solution

$$\begin{aligned}
\int_0^\infty xe^{-x}\ dx &= \lim_{b\to\infty} \int_0^b xe^{-x}\ dx \\
&= \lim_{b\to\infty} ((-xe^{-x})|_0^b + \int_0^b e^{-x}\ dx) \\
&= \lim_{b\to\infty} (-be^{-x} + (-e^{-x})|_0^b) \\
&= \lim_{b\to\infty} -be^{-b} - e^{-b} + 1 \\
&= 1
\end{aligned}$$

$\square$

4. Find $\int_1^\infty \frac{1}{x^2} dx$

   Solution

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2} dx &= \lim_{b\to\infty} \frac{1}{x^2} dx \\
&= \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b \\
&= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

$\square$

5. Find $\int_1^\infty \frac{1}{x} dx$

   Solution

$$\begin{aligned}
\int_1^\infty \frac{1}{x} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x} dx \\
&= \lim_{b\to\infty} (\ln x)|_1^b \\
&= \lim_{b\to\infty} \ln b \\
&= \infty
\end{aligned}$$

So the integral diverges. If you don't understand why the limit goes to infinity, look at the graph of $\ln x$

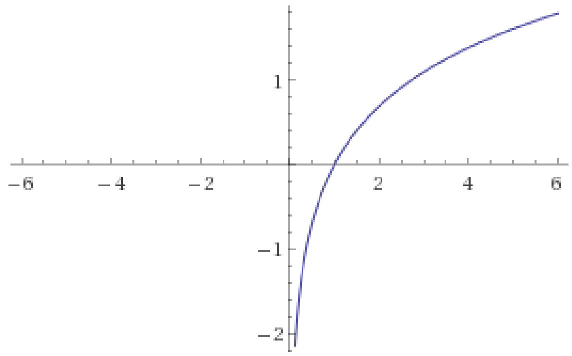


Figure 36

From the graph you can see as x increases, $\ln x$ goes towards infinity.

□

6. Find $\int_0^3 \frac{1}{\sqrt{3-x}} dx$

Solution The integrand isn't defined for $x = 3$ so we have

$$\begin{aligned}
\int_0^3 \frac{1}{\sqrt{3-x}} dx &= \lim_{c \to 3^-} \int_0^c \frac{1}{\sqrt{3-x}} dx \\
&= \lim_{c \to 3^-} (-2\sqrt{3-x})\Big|_0^c \text{(This is } u\text{-sub with } u = 3 - x) \\
&= \lim_{c \to 3^-} (-2\sqrt{3-c} + 2\sqrt{3}) \\
&= 2\sqrt{3} - 2\sqrt{3-3} \\
&= 2\sqrt{3}
\end{aligned}$$

□

7. Find $\int_{-2}^3 \frac{1}{x^3} dx$

Solution At first glance this doesn't seem like an improper integral. However the integrand is undefined at $x = 0$ which falls between the limits of integration so we must split the integral into $\int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^0 \frac{1}{x^3} dx + \int_0^3 \frac{1}{x^3} dx$. Let's focus on the first integral.

$$\begin{aligned}\int_{-2}^{0} \frac{1}{x^3}dx &= \lim_{c\to 0^-} \int_{-2}^{c} \frac{1}{x^3}dx \\ &= \lim_{c\to 0^-} \left(-\frac{1}{2x^2}\right)\Bigg|_{-2}^{c} \\ &= \lim_{c\to 0^-} \left(-\frac{1}{2c^2} + \frac{1}{2(-2)^2}\right) \\ &= \lim_{c\to 0^-} \left(-\frac{1}{2c^2} + \frac{1}{8}\right) \\ &= -\infty\end{aligned}$$

Thus the integral diverges. Here is the graph of $y = -\dfrac{1}{2x^2}$ to clear up why the limit is $-\infty$

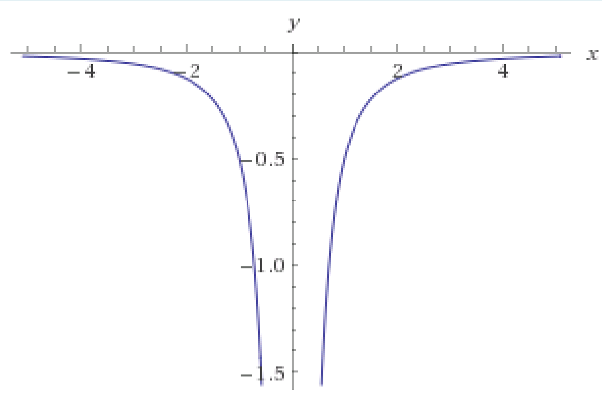


Figure 37

$\square$

Lecture 34: Elementary and Separable Differential Equations

Definition: The family of all solutions to a differential equation $\frac{dy}{dx} = f(x)$ is called the *general solution*. If you are given an initial value, $y(x_0) = y_0$, then you can find the *particular solution* (i.e. a solution without the constant term C)

Examples:

1. Find the general solution of the following differential equation:

$$\frac{dy}{dt} = -4t + 6t^2$$

Solution

$$\begin{aligned} y &= \int -4t + 6t^2 \; dt \\ &= -2t^2 + 2t^3 + C \end{aligned}$$

□

2. Solve $\frac{dy}{dx} = 10 - x$, if $y(0) = -1$

Solution

$$\begin{aligned} y &= \int (10 - x)dx \\ &= 10x - \frac{x^2}{2} + C \end{aligned}$$

Plugging in the initial condition we have

$$-1 = 10(0) - \frac{(0)^2}{2} + C \Leftrightarrow C = -1$$

$$\Rightarrow y = 10x - \frac{x^2}{2} - 1$$

□

3. Solve $\frac{dy}{dx} = \cos x + \sin x$, if $y(\pi) = 1$

Solution

$$\begin{aligned} y &= \int \cos x + \sin x \\ &= \sin x - \cos x + C \end{aligned}$$

Using the initial condition gives us

$$1 = \sin\pi - \cos\pi + C = 0 + 1 + C \Leftrightarrow C = 0$$
$$\Rightarrow y = \sin x - \cos x$$

□

4. Solve

$$2\frac{dy}{dt} = 4te^{-t}, y(0) = 42$$

Solution

$$2\frac{dy}{dt} = 4te^{-t} \Leftrightarrow \frac{dy}{dt} = 2te^{-t}$$
$$\Rightarrow y = 2\int te^{-t}\ dt$$
Let $u = t, dv = e^{-t} \Rightarrow du = dt, v = -e^{-t}$
$$\Rightarrow y = 2(-te^{-t} + \int e^{-t}\ dt)$$
$$\Rightarrow y = 2(-te^{-t} - e^{-t} + C)$$
Plugging in the initial value we have
$$\Rightarrow 42 = 2(0 - 1 + C)$$
$$\Leftrightarrow C = 22$$
$$\Rightarrow y = 2(-te^{-t} - e^{-t} + 22)$$

□

5. Solve

$$x^2\frac{dy}{dx} - y\sqrt{x} = 0, y(1) = e^{-2}$$

Solution

$$\begin{aligned}
x^2\frac{dy}{dx} - y\sqrt{x} = 0 &\Rightarrow \frac{dy}{dx} - yx^{-3/2} \\
&\Rightarrow \frac{dy}{y} = x^{-3/2}\ dx \\
&\Rightarrow \int \frac{dy}{y} = x^{-3/2}\ dx \\
&\Rightarrow \ln|y| = -2x^{-1/2} + C_1 \\
&\Rightarrow |y| = e^{-2/\sqrt{x}+C_1} \\
&\Rightarrow |y| = e^{C_1}e^{-2/\sqrt{x}} \\
&\Rightarrow y = \pm e^{C_1}e^{-2/\sqrt{x}} \\
&\Rightarrow y = Ce^{-2/\sqrt{x}}, \text{ where } C = \pm e^{C_1} \\
&\text{Plugging in the initial condition we have} \\
&\Rightarrow e^{-2} = Ce^{-2} \\
&\Rightarrow C = 1 \\
&\Rightarrow y = e^{-2}\sqrt{x}
\end{aligned}$$

□

Definition: A differential equation of the form $\frac{dy}{dx} = \frac{p(x)}{q(y)}$ (i.e. a differential equation where you can separate the x terms from the y terms) is called *separable*. To solve a separable equation, use the following argument:

$$\begin{aligned}
\frac{dy}{dx} = \frac{p(x)}{q(y)} &\Leftrightarrow q(y)dy = p(x)dx \\
&\Leftrightarrow \int q(y)dy = \int p(x)dx \\
&\Leftrightarrow Q(y) + C_1 = P(x) + C_2, \text{ where } Q'(y) = q(y) \text{ and } P'(x) = p(x) \\
&\Leftrightarrow Q(y) = P(x) + C, \text{ where } C = C_2 - C_1
\end{aligned}$$

Note that whenever you have an expression that doesn't depend on x in your final answer, you can write it as a constant C.

Examples:

1. Find the general solution of

$$\frac{dy}{dt} = y^2e^{2t}$$

Solution

$$\begin{aligned}\frac{dy}{dt} = y^2 e^{2t} &\Rightarrow \frac{1}{y^2}\ dy = e^{2t}\ dt \\ &\Rightarrow \int \frac{1}{y^2}\ dy = \int e^{2t}\ dt \\ &\Rightarrow -\frac{1}{y} + C_1 = \frac{1}{2}e^{2t} + C_2 \\ &\Rightarrow -\frac{1}{y} = \frac{1}{2}e^{2t} + C,\ C = C_2 - C_1 \\ &\Rightarrow y = -\frac{2}{2C + e^{2t}}\end{aligned}$$

□

2. Solve $\dfrac{dy}{dx} = xy + \dfrac{y}{x}$

Solution

$$\begin{aligned}\frac{dy}{dx} = xy + \frac{y}{x} &\Leftrightarrow \frac{dy}{dx} = y\left(x + \frac{1}{x}\right) \\ &\Leftrightarrow \frac{dy}{y} = \left(x + \frac{1}{x}\right)dx \\ &\Leftrightarrow \int \frac{dy}{y} = \int \left(x + \frac{1}{x}\right)dx \\ &\Leftrightarrow \ln|y| = \frac{x^2}{2} + \ln|x| + C_1 \\ &\Leftrightarrow e^{\ln|y|} = e^{x^2/2 + \ln|x| + C_1} \\ &\Leftrightarrow |y| = e^{x^2/2} e^{\ln|x|} e^{C_1} \\ &\Leftrightarrow y = \pm e^{x^2/2}|x|\, e^{C_1} \\ &\Leftrightarrow y = C_2 e^{x^2/2}|x|,\ C_2 = \pm e^{C_1} \\ &\Leftrightarrow y = Cxe^{x^2/2},\ \text{since } |x| \text{ is either } x \text{ or } -x\end{aligned}$$

□

Lecture 35: Equilibrium Solutions (OPTIONAL)

Definition: Given a differential equation $\frac{dy}{dx} = f(y)$, an *equilibrium solution*, y_e, is a solution to the equation $f(y) = 0$.

1. If f changes from positive to negative at y_e then y_e is called *stable*
2. If f changes from negative to positive at y_e then y_e is called *unstable*
3. If f does not change sign at y_e then y_e is called *semi-stable*

Examples:

1. Find and classify all equilibrium solutions to $\frac{dy}{dx} = y^2 - y - 6$

   Solution $y^2 - y - 6 = 0 \Leftrightarrow (y-3)(y+2) = 0 \Leftrightarrow y = 3, 2$. Plotting these solutions on a number line shows us that $f > 0$ on $(-\infty, -2)$, $(3, \infty)$ and $f < 0$ on $(-2, 3)$ which tells us that $y = -2$ is stable and $y = 3$ is unstable.

   □

2. Find and classify all equilibrium solutions to $\frac{dy}{dx} = (y^2 - 4)(y+1)^2$

   Solution $(y^2 - 4)(y+1)^2 = 0 \Leftrightarrow (y-2)(y+2)(y+1)^2 = 0 \Leftrightarrow y = -2, 2, -1$. Testing our intervals gives us that $f > 0$ on $(-\infty, -2), (3, \infty)$ and $f < 0$ on $(-2, -1)$, $(-1, 3)$. This tells us that $y = 2$ is unstable, $y = -2$ is stable, and $y = -1$ is semi-stable.

   □

Lecture 36: Linear First Order Differential Equations

Definition: A differential equation of the form $\frac{dy}{dx} + p(x)y = q(x)$ is called a *linear first-order differential equation*. To solve an equation of this type we'll need the *integrating factor* $I(x) = e^{\int p(x)dx}$ (Note: We can generalize this to $I(x) = e^{\text{any antiderivative of } p(x)}$)

Once we find this, we use the following steps:

(i) Find $I(x)$

(ii) Multiply the entire equation by $I(x)$

(iii) This new equation becomes $\frac{d}{dx}[I(x)y] = I(x)q(x)$

Proof. Since $\frac{dI}{dx} = I(x)p(x)$ we have

$$\begin{aligned}\frac{dy}{dx} + p(x)y = q(x) &\Leftrightarrow I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x)\\ &\Leftrightarrow I(x)\frac{d}{dx}(y) + \frac{d}{dx}(I(x))y = I(x)q(x)\\ &\Leftrightarrow \frac{d}{dx}(I(x)y) = I(x)q(x)\end{aligned}$$

□

(iv) Integrate both sides and solve for y

Examples:

1. Find the solution to $x\frac{dy}{dx} + 2y = x^2 - x + 1$, where $y(1) = \frac{1}{2}$

Solution Divide by x to get it in the proper form $\frac{dy}{dx} + \frac{2}{x}y = x - 1 + \frac{1}{x}$ which gives us that $p(x) = \frac{2}{x}$ so $I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{ln|x|^2} = |x|^2 = x^2$. Multiplying by the integration factor gives us:

$$\frac{dy}{dx}[x^2y] = x^3 - x^2 + x \Leftrightarrow x^2y = \int (x^3 - x^2 + x)dx$$
$$\Leftrightarrow x^2y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$
$$\Leftrightarrow y(t) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{c}{x^2}$$

Using the initial condition gives us $\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \Leftrightarrow C = \frac{1}{12}$ which gives us the final solution $y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12x^2}$

□

2. Find the general solution of $2xy + x^2 = x\frac{dy}{dx}$

Solution

$$2xy + x^2 = x\frac{dy}{dx} \Rightarrow x\frac{dy}{dx} - 2xy = x^2$$
$$\Rightarrow \frac{dy}{dx} - 2y = x$$

So $p(x) = -2 \Rightarrow I(x) = e^{-2x}$

$$\Rightarrow \frac{d}{dx}(e^{-2x}y) = xe^{-2x}$$
$$\Rightarrow e^{-2x}y = \int xe^{-2x}\ dx$$

Let $u = x, dv = e^{-2x} \Rightarrow du = dx, v = -\frac{1}{2}e^{-2x}$

$$\Rightarrow e^{-2x}y = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\ dx$$
$$\Rightarrow e^{-2x}y = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$\Rightarrow y = -\frac{1}{2}x - \frac{1}{4} + Ce^{2x}$$

□

3. Solve $x\frac{dy}{dx} - 3y + 2 = 0,\ y(1) = 8$

Solution

$$x\frac{dy}{dx} - 3y + 2 = 0 \Rightarrow x\frac{dy}{dx} - 3y = -2$$
$$\Rightarrow \frac{dy}{dx} - \frac{3}{x}y = -\frac{2}{x}$$

Thus

$$p(x) = -\frac{3}{x} \Rightarrow I(x) = e^{\ln|1|x^3} \Rightarrow I(x) = \left|\frac{1}{x^3}\right|$$

Since we can choose any antiderivative of $p(x)$, let's drop the absolute values to get

$$I(x) = \frac{1}{x^3}$$

This gives us

$$\begin{aligned} x\frac{dy}{dx} - 3y + 2 = 0 \Rightarrow \frac{d}{dx}\left(\frac{y}{x^3}\right) &= -\frac{2}{x^4} \\ \Rightarrow \frac{y}{x^3} &= -\int 2x^{-4}\ dx \\ \Rightarrow \frac{y}{x^3} &= \frac{2}{3x^3} + C \\ \Rightarrow y &= \frac{2}{3} + Cx^3 \end{aligned}$$

Using our initial condition we have

$$8 = \frac{2}{3} + C \Leftrightarrow C = \frac{22}{3} \Rightarrow y = \frac{2}{3}\frac{22}{3}x^3$$

□