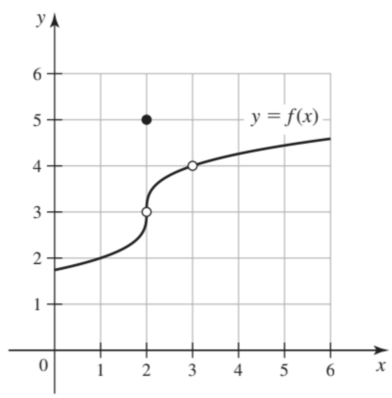
Sections 2.1-2.3

Definition: We write $\lim_{x \to a} f(x) = L$ and say "the limit of $f(x)$ as x approaches a equals L" if $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a.

Note: We do not care what happens with $x = a$, only for the values of x that are close to a. The function doesn't even have to be defined at $x = a$!

Brainstorm: What are some examples of graphs that you can draw for which $\lim_{x \to a} f(x) = L$?

Example: Use the graph below to determine the following values.



(1) $f(1)$ and $\lim_{x \to 1} f(x)$

(2) $f(2)$ and $\lim_{x \to 2} f(x)$

(3) $f(3)$ and $\lim_{x \to 3} f(x)$

Note: The limit defined above is referred to as a two-sided limit.

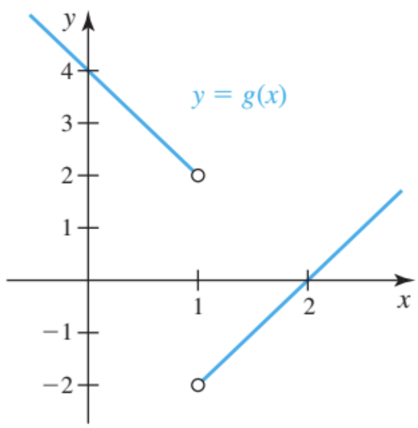
Definitions:

- We write $\lim_{x \to a^-} f(x) = L$ and say "the left-sided limit of $f(x)$ as x approaches a" or "the limit of $f(x)$ as x approaches a from the left" is equal to L if $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a with $x < a$.

- We write $\lim_{x \to a^+} f(x) = L$ and say "the right-sided limit of $f(x)$ as x approaches a" or "the limit of $f(x)$ as x approaches a from the right" is equal to L if $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a with $x > a$.

Note: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$. Otherwise, we say the limit does not exist.

Example: Use the graph below to determine the following values:



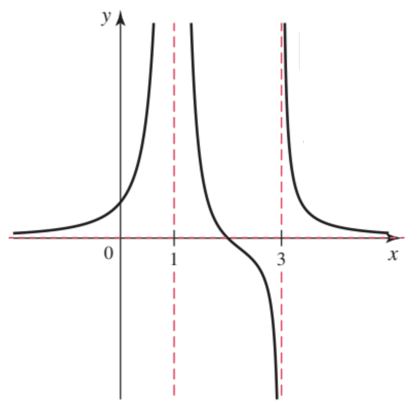
(1) $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$

(2) $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$, and $\lim_{x \to 2} f(x)$

Definitions:

- We say $\lim_{x \to a} f(x) = \infty$ if $f(x)$ gets arbitrarily large for all x sufficiently close (but not equal to) a.
- We say $\lim_{x \to a} f(x) = -\infty$ if $f(x)$ is negative and gets arbitrarily large in magnitude for all x sufficiently close (but not equal to) a.
- We say $\lim_{x \to \infty} f(x) = L$ if $f(x)$ is arbitrarily close to L for all sufficiently large and positive x.
- We say $\lim_{x \to -\infty} f(x) = L$ if $f(x)$ is arbitrarily close to L for all sufficiently large in magnitude and negative x.

Example: Use the graph of $f(x)$ below to determine the following values:



(1) $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

(2) $\lim_{x \to 1} f(x)$

(3) $\lim_{x \to 3} f(x)$

Limit Laws: Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and $n > 0$ is an integer.

- $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$
- $\lim_{x\to a} f(x)^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$, provided $f(x) > 0$ for x near a if n is even

Example: Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$, and $\lim_{x\to 2} h(x) = 8$. Compute the following

(1) $\lim_{x\to 2} \dfrac{f(x) - g(x)}{h(x)}$

(2) $\lim_{x\to 2}(6f(x)g(x) + h(x))$

(3) $\lim_{x\to 2}(g(x))^3$

Theorem: If $p(x)$ and $q(x)$ are polynomials, the following is true:

- $\lim_{x \to a} p(x) = p(a)$
- $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$

Note: In general, as long as a function is "nice", you can just plug in a to evaluate the limit.

Example: Evaluate the following:

(1) $\lim_{x \to 2} \frac{3x^2 - 4x}{5x^3 - 36}$

(2) $\lim_{x \to 2} \frac{\sqrt{2x^3 + 9} + 3x - 1}{4x + 1}$

Note: When evaluating piecewise functions, you need to use the equation from the correct piece.

Example: Consider the following piecewise function and use it to calculate $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$

$$f(x) = \begin{cases} -2x + 4 & \text{if } x \leq 1 \\ \sqrt{x - 1} & \text{if } x > 1 \end{cases}$$

Note: Sometimes simply plugging in $x = a$ does not work. If this happens, you need to use strategies to rewrite the limit. Some strategies include:

- Simplifying
- Factoring and canceling
- Multiplying by the conjugate
- Using theorems and/or identities
- Graphing

Example (Factoring): Evaluate $\lim_{x \to 2} \frac{x^2 - 6x + 8}{x^2 - 4}$

Example (Conjugate): Evaluate $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

Example (Identities): Evaluate $\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ for x near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

Brainstorm: When might the squeeze theorem come in handy?

Example: Evaluate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$