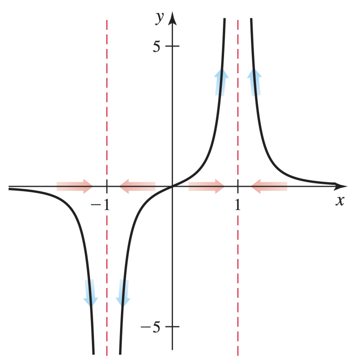
Definition: If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$, the line $x = a$ is called a vertical asymptote of f.

Exercises:

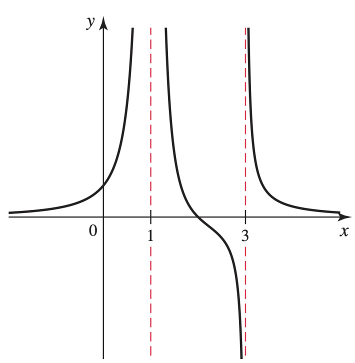
(1) Consider the graph of $\frac{x}{(x^2-1)^2}$ and find the following limits.



(a) $\lim_{x\to 1} \frac{x}{(x^2-1)^2}$

(b) $\lim_{x\to -1} \frac{x}{(x^2-1)^2}$

(2) Consider the graph of $\frac{x-2}{(x-1)^2(x-3)}$ and find the following limits.



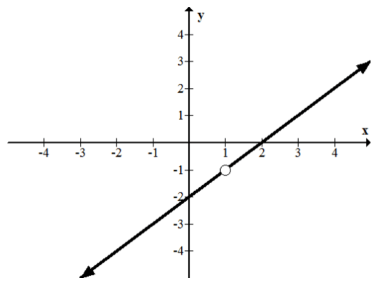
(a) $\lim_{x\to 1} \frac{x-2}{(x-1)^2(x-3)}$

(b) $\lim_{x\to 3} \frac{x-2}{(x-1)^2(x-3)}$

Brainstorm: What relationship do you notice between the function and the equation(s) of its vertical asymptote(s)?

Note: If you can completely cancel the factor, then that corresponds to a hole in the graph, not a vertical asymptote.

Example: The following is a graph of $y = \frac{(x-2)(x-1)}{x-1}$



Exercises: Find the vertical asymptotes of the following functions:

(1) $\frac{-x^3+5x^2-6x}{-x^3-4x^2}$

(2) $\frac{x^2-4x+3}{x^2-1}$

Fact:

- $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$
- $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

Strategy: To figure out whether the limit as x approaches your vertical asymptote is ∞, $-\infty$, or DNE, you need to figure out the one-sided limits by determining whether your function is heading towards an arbitrarily large positive number or an arbitrarily large negative number.

Example: Consider the function $f(x) = \frac{x-1}{(x+2)(x-3)}$. If I want to find $\lim\limits_{x \to 3} f(x)$, I would need to look at the one-sided limits, since there is a vertical asymptote at $x = 3$.

Approaching from the left of 3 means that we're looking at $x < 3$ for x values **close** to $x = 3$. Looking at what sign the function would be at such an x value, we would have:

$$\frac{(\#\text{ close to, but less than } 3) - 1}{((\#\text{ close to, but less than } 3) + 2)((\#\text{ close to, but less than } 3) - 3)} \to \frac{(+)}{(+)(0^-)} \to \text{negative}$$

This gives us. $\lim\limits_{x \to 3^-} \dfrac{x-1}{(x+2)(x-3)} = -\infty$

You can do the same to see that $\lim\limits_{x \to 3^+} \dfrac{x-1}{(x+2)(x-3)} \to \dfrac{+}{(+)(0^+)} \to \infty$

Exercises: Figure out all of the above limits analytically instead of graphically.

(1) $\displaystyle\lim_{x\to 1} \frac{x}{(x^2-1)^2}$

(2) $\displaystyle\lim_{x\to -1} \frac{x}{(x^2-1)^2}$

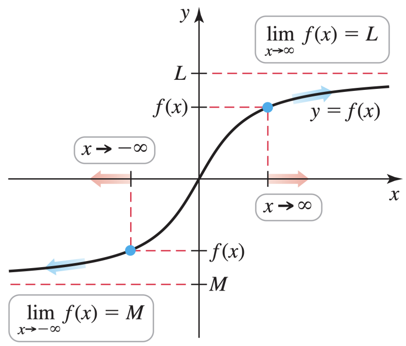
(3) $\displaystyle\lim_{x\to 1} \frac{x-2}{(x-1)^2(x-3)}$

(4) $\displaystyle\lim_{x\to 3} \frac{x-2}{(x-1)^2(x-3)}$

Optional Practice: Figure out the behavior around the vertical asymptotes for $y = \frac{-x^3+5x^2-6x}{-x^3-4x^2}$ and $y = \frac{x^2-4x+3}{x^2-1}$

Definition: The line $y = L$ is a horizontal asymptote of $y = f(x)$ if either of the following is true: $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

Example: In the following graph, both $y = L$ and $y = M$ are horizontal asymptotes.



Fact: $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ for $n > 0$

Exercises:

(1) Find $\lim_{x\to-\infty} \left(2 + \frac{10}{x^2}\right)$

(2) Find $\lim_{x\to\infty} \left(5 + \frac{\sin x}{\sqrt{x}}\right)$

More Facts:

- $\lim_{x\to\pm\infty} x^n = \infty$, if $n > 0$ is even
- $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$, if $n > 0$ is odd
- $\lim_{x\to\pm\infty} (a_n x^n + a_{n-a} x^{n-a} + \ldots + a_1 x + a_0) = \lim_{x\to\pm\infty} a_n x^n$

Exercises: Determine the end behavior (i.e. the limits at $x \to \pm\infty$) of the following:

(1) $3x^4 - 6x^2 + x - 10$

(2) $-2x^3 + 3x^2 - 12$

Rational Functions: When finding the end behavior of rational functions, divide by the largest power of x in the denominator.

Example: Find the end behavior of $f(x) = \frac{3x+2}{x^2-1}$

$$\lim_{x \to \pm\infty} \frac{3x+1}{x^2-1} = \lim_{x \to \pm\infty} \frac{\frac{3x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \frac{0+0}{1-0}$$
$$= 0$$

Exercises: Find the end behavior of the following:

(1) $g(x) = \frac{40x^4+4x^2-1}{10x^4+8x^2+1}$

(2) $h(x) = \frac{x^3-2x+1}{2x+4}$

Brainstorm: What is a pattern we could notice about the end behavior of rational functions in relation to the degrees of the numerator and denominator?

Special Case: If the degree of the denominator is one less than the degree of the numerator, the graph has a slant asymptote. You can find this asymptote by performing polynomial division.

Example (reminder on how to do polynomial division): Divide $x^3 + 1$ by $x - 1$

$$
\begin{array}{rrrrr}
 & & x^2 & +x & +1 \\ \hline
x-1 \,\big) & x^3 & +0x^2 & +0x & +1 \\
 & -x^3 & +x^2 & & \\ \hline
 & & x^2 & +0x & \\
 & & -x^2 & +x & \\ \hline
 & & & x & +1 \\
 & & & -x & +1 \\ \hline
 & & & & 2
\end{array}
$$

We can write the results as the following: $\frac{x^3+1}{x-1} = x^2 + x + 1 + \frac{2}{x-1}$

Note: Some of you may have learned something called synthetic division. Please remember that this only works when dividing by a polynomial that looks like $x - c$. You still need to know long division for every other case.

Exercise: Find the slant asymptote for $f(x) = \frac{2x^2+6x-2}{x+1}$

Even More Facts:

- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to\infty} e^{-x} = 0$
- $\lim_{x\to 0^+} \ln x = -\infty$
- $\lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to-\infty} e^{-x} = \infty$
- $\lim_{x\to\infty} \ln x = \infty$

Exercises: Find the limit as $x \to \infty$ of the following:

(1) $f(x) = e^{5x}$

(2) $f(x) = 4 - \ln 3x$