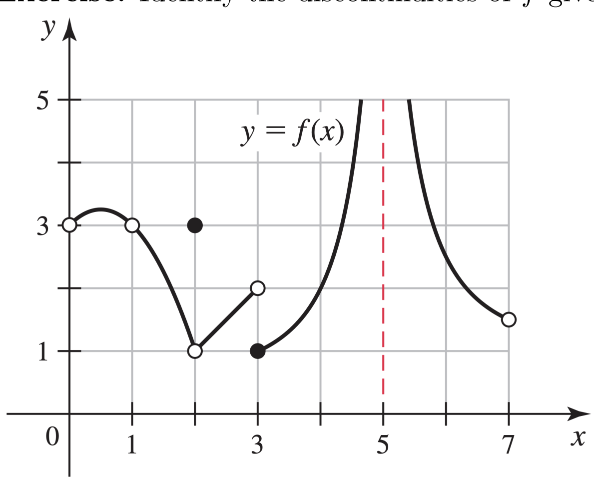
Definition: A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$. If the function is not continuous, we say it is discontinuous at a or that f has a discontinuity at a.

- Informally, we say a function f is continuous at a if the graph of f does not have a hole or a break at a.

Checklist: In order for the function to be continuous at a, the following three things must be satisfied:

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

Exercise: Identify the discontinuities of f given the graph below.



Exercise: Determine whether the following functions are continuous at the given a value. Justify your answers:

(1) $f(x) = \frac{3x^2+2x+1}{x-1}$; $a = 1$

(2) $g(x) = \frac{3x^2+2x+1}{x-1}$; $a = 2$

(3) $h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Theorem: If f and g are continuous at a, then the following functions are also continuous at a.

- $f + g$
- $f - g$
- cf, c a constant
- fg
- $\frac{f}{g}$, $g(a) \neq 0$
- $(f(x))^n$, $n > 0$

Theorem: The following functions are continuous at every point in their domain:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Exponential functions
- Logarithmic functions

Exercise: On what interval(s) are the following functions continuous:

(1) $f(x) = x^{100} - 2x^{37} + 75$

(2) $g(x) = \frac{x^2+2x+17}{x^2-1}$

(3) $h(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1}$

Theorem:

- If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a
- If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$
  - In both of these cases: $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

Exercise: Evaluate $\lim_{x \to 2} \cos\left(\frac{x^2 - 4}{x - 2}\right)$

Defintions:

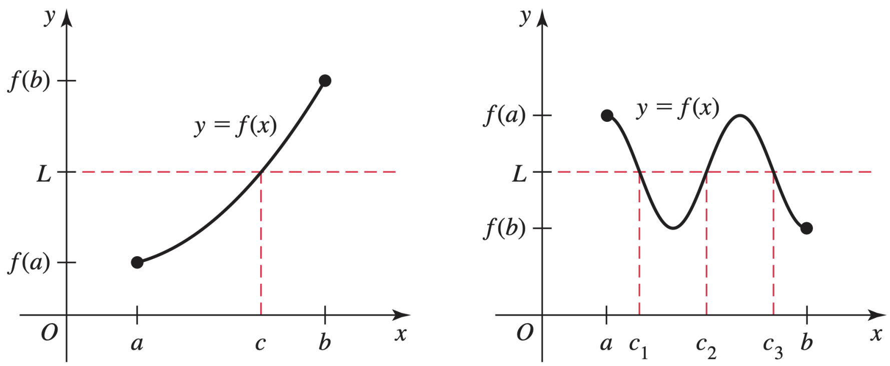
- A function f is continuous from the right if $\lim_{x \to a^+} f(x) = f(a)$
- A function is continuous from the left if $\lim_{x \to a^-} f(x) = f(a)$
- A function is continuous on an interval if it is continuous at every point in that interval. If the interval contains its endpoints, that means it's continuous from the right or left at the respective endpoints.

Example: Determine the intervals of continuity for $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ 3x + 5 & \text{if } x > 0 \end{cases}$

Note: People often classify discontinuities depending on the behavior of the graph at that point. The three types are: removable discontinuities, infinite discontinuities, and jump discontinuities.

Exercise: Draw graphical examples of the three types of discontinuities.

The Intermediate Value Theorem: Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(x) = L$.



Exercise: Show that there is a solution to the equation $4x^3 - 6x^2 + 3x - 2 = 0$

Definition: A secant line is a line through two points on a curve.

Brainstorm: Draw a generic secant line and determine the slope.

Definition: The tangent line to f at a is the limiting position of the secant line as $x \to a$. This means that the slope can be determine by:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ OR } \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Examples: Find an equation for the tangent line for:

(1) $y = x^2$ at $(1, 1)$

(2) $y = \frac{3}{x}$ at $x = 3$

Definition: The derivative of a function f at $x = a$, denoted $f'(a)$, is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ OR } \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists.

- If $f'(a)$ exists, we say that f is differentiable at $x = a$
- A function is differentiable on an open interval if it is differentiable at every point in that interval.

Examples: Find the derivative of:

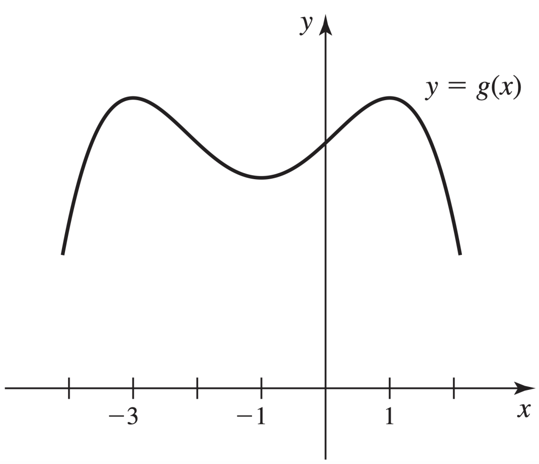
(1) $f(x) = x^2 - 8x + 9$ (2) $f(x) = \sqrt{x}$

Interpretations:

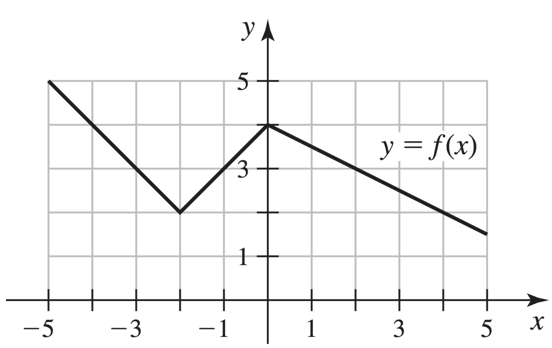
- $f'(x)$ is the slope of the tangent line to f at x
- The slope of a secant line between points (x_1, y_1) and (x_2, y_2) is the average rate of change of y with respect to x over the interval $[x_1, x_2]$
- $f'(a)$ is the instantaneous rate of change of $y = f(x)$ at $x = a$
- If $s(t)$ is the position of an object at time t, $v(t)$ is the velocity of the object, and $a(t)$ is the acceleration of the object, then the following is true: $s'(t) = v(t)$, $v'(t) = a(t)$

Brainstorm: Given the graph of f, sketch f':

(1)



(2)



Notation: The following notations are different ways of writing the derivative of a function $y = f(x)$

$$f'(x) = f' = y' = \frac{dy}{dx} = \frac{d}{dx}y = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Theorem: If f is differentiable at a, then f is continuous at a

Brainstorm: When might a function not be differentiable at $x = a$?

Higher Derivatives:

- The second derivative of f is the derivative of $f'(x)$. It is written with the following notations:
$$f''(x) = \frac{d^2y}{dx^2}$$
- The third derivative of f is the derivative of $f''(x)$. It is written with the following notations:
$$f'''(x) = \frac{d^3y}{dx^3}$$
- The nth derivative of f is the derivative of $f^{(n-1)}(x)$. It is written with the following notations:
$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Example: With physics applications, this gives us that if $s(t)$ is position, then $s'(t)$ is velocity, and $s''(t)$ is acceleration.