Sections 3.7-3.8

Derivatives of Inverse Trigonometric Functions:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\tan^{-x} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Exercise: Find the derivatives of the following functions:

(1) $x^2 \tan^{-1} x$

(2) $\frac{\sin^{-1} x}{\cos x}$

(3) $\frac{2x \operatorname{arccot} x}{\log_2 x}$

Chain Rule:

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$$

Alternatively: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Exercises: Find the derivatives of the following functions. You do not have to simplify your answers.

(1) $y = \sqrt{x^2+1}$

(2) $y = \sin\big(x^2\big)$

(3) $y = \sin^2 x$

(4) $h(x) = (x^3-1)^{100}$

(5) $f(x) = e^{\sin x}$

(6) $y = \cos^{-1}(x^2-1)$

(7) $y = \cos\big(\arcsin^{-1} x\big)$

(8) $y = 1 - \cos(2x)$

(9) $f(x) = \log_2 x^2$

(10) $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$

(11) $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

(12) $y = (2x+1)^5(x^3-x+1)^4$

(13) $f(x) = x^2\ln(2x) + x\ln(3x)$

(14) $f(x) = \ln\big((3x+1)^4(x^4+5x+7)\big)$

Note: This can be extended to longer compositions of functions. For example:

$$\frac{d}{dx}(f(g(h(x)))) = f'(g(h(x)))g'(h(x))h'(x)$$

Exercises:

(1) Differentiate $f(x) = \sin(\cos(\tan x))$

(2) Differentiate $y = \sqrt{\sec(x^3)}$

(3) Differentiate $g(x) = \ln\left(\tan^2 x\right)$

(4) Find the first and second derivatives of $\cos(\sin(3\theta))$

Implicit Differentiation: Some functions cannot be expressed explicitly as $y = f(x)$. To find $\frac{dy}{dx}$ for something expressed implicitly, we use implicit differentiation.

Steps for Implicit Diffferentiation:

(1) Differentiate both sides with respect to the independent variable (usually x)

(2) Anytime you differentiate a dependent variable (usually y), use chain rule $\left(\text{i.e. multiply by } \frac{dy}{dx}\right)$

(3) Solve for $\frac{dy}{dx}$ using algebra techniques

Exercises: Find $\frac{dy}{dx}$ for the following:

(1) $x^2 + y^2 = 25$

(2) $x^3 + y^3 = 6xy$

(3) $\sin(x + y) = y^2 \cos x$

Exercises:

(1) Find an equation of the line tangent to $x^2 + y^2 = 25$ at $(3, 4)$

(2) Find an equation of the line tangent to $x^2 - xy - y^2 = 1$ at $(2, 1)$

(3) Find y'' for $x^4 + y^4 = 16$