Sections 4.1-4.4

Definition: Let c be a number in the domain D of a function f.

- $f(c)$ is called an absolute (or global) maximum value if $f(c) \geq f(x)$ for all x in D
- $f(c)$ is called an absolute (or global) minimum value if $f(c) \leq f(x)$ for all x in D
- $f(c)$ is called a relative (or local) maximum value if $f(c) \geq f(x)$ for x near c
- $f(c)$ is called a relative (or local) minimum value if $f(c) \leq f(x)$ for x near c

Note:

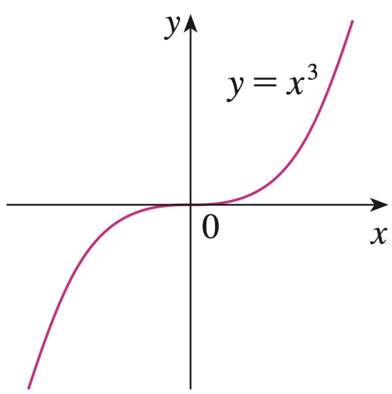
- Maxima and minima are also referred to as the extreme values of f
- Some functions do not have extreme values

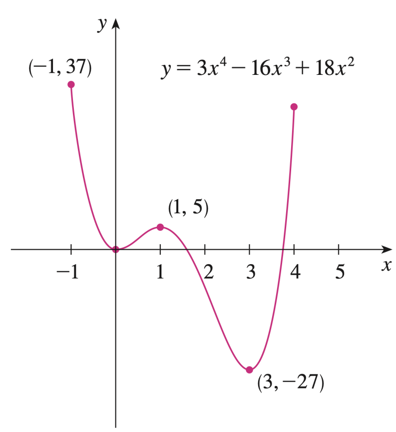
Brainstorm: Draw a picture to visualize the above definitions.

Exercises:

(1) Find the relative and absolute extrema of $f(x) = x^3$

(2) Find the relative and absolute extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$





The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and absolute minimum in $[a, b]$.

Brainstorm: Draw some pictures to visualize the Extreme Value Theorem.

Definition: A critical number of f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

- Note: the converse is NOT true.
- With the above definition, Fermat's theorem states that if f has a local extrema at $x = c$, then c is a critical number.

Exercise: Find the critical numbers of $f(x) = 4x^{3/5} - x^{8/5}$

Steps for finding absolute extrema on $[a, b]$:

(1) Find all critical numbers in (a, b) and evaluate the function at those values.

(2) Find $f(a)$ and $f(b)$

(3) Compare

Exercises:

(1) Find the absolute maximum and minimum values of $f(x) = x^3 - 3x^2 + 1$ for $-\frac{1}{2} \leq x \leq 4$

(2) Find the absolute maximum and minimum values of $f(x) = x - 2\sin x$ on $[0, 2\pi]$

The Mean Value Theorem: Let f be a function that satisfies the following: f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ or equivalently } f'(c)(b - a) = f(b) - f(a)$$

Rolle's Theorem: Let f be a function that satisfies the following: f is continuous on $[a, b]$, f is differentiable on (a, b), and $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$

Exercises:

(1) Prove that $x^3 + x - 1 = 0$ has exactly one solution

(2) Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

More Theorems/Corollaries:

- If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b) (i.e. $f(x) = c$, where c is a constant)
- If $f'(x) = g'(x)$ for all x in (a, b), then $f - g$ is constant on (a, b) (i.e. $f(x) = g(x) + c$, where c is a constant)

Definitions:

- f is increasing if $f(x_1) \le f(x_2)$ for $x_1 < x_2$
- f is decreasing if $f(x_1) \ge f(x_2)$ for $x_1 < x_2$
- f is strictly increasing if $f(x_1) < f(x_2)$ for $x_1 < x_2$
- f is strictly decreasing if $f(x_1) > f(x_2)$ for $x_1 < x_2$

Increasing/Decreasing Test:

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Exercise: Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

The First Derivative Test: Suppose that c is a critical number of f.

- If f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$
- If f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$
- If f' does not change signs at $x = c$, then there is neither a local max nor a local min at $x = c$

Brainstorm: Draw a picture to make sense of the First Derivative Test

Examples:

(1) Find any local maxima/minima of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

(2) Find any local maxima/minima of $f(x) = x + 2\sin x$ for $0 \leq x \leq 2\pi$

Note: There are different ways to increase/decrease. We refer to this shape of the graph as its concavity.

Definitions:

- If the graph of f lies above all of its tangent lines on an interval, then f is concave upward on that interval
- If the graph of f lies below all of its tangent lines on an interval, then f is concave downward on that interval

Brainstorm: Draw a picture to visualize the above definitions

Concavity Test:

- If $f''(x) > 0$ on an interval, then f is concave upward on that interval
- If $f''(x) < 0$ on an interval, then f is concave downward on that interval

Definition: A point on a curve $y = f(x)$ is called an inflection point if f changes concavity at that point.

The Second Derivative Test:

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$
- if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

(1) Sketch a graph satisfying all of the following

(a) $f(0) = 0,\ f(2) = 3,\ f(4) = 6,\ f'(0) = f'(4) = 0$

(b) $f'(x) > 0$ for $0 < x < 4$, $f'(x) < 0$ for $x < 0$ and $x > 4$

(c) $f''(x) > 0$ for $x < 2$, $f''(x)$¡0 for $x > 0$

(2) Discuss the shape of $y = x^4 - 4x^3$ using derivatives.