Sections 4.7-4.9

L'Hospital's Rule (or L'Hôpital's Rule): Suppose f and g are differentiable functions with $g'(x) \neq 0$ when $x \neq a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$). This also applies if $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$

Note: Informally, L'Hospital's Rule states that if your limit results in an indeterminate form ($\frac{\infty}{\infty}$ or $\frac{0}{0}$), you can take the derivative of the numerator and denominator.

Exercises: Evaluate the following limits:

(1) $\lim_{x\to 1} \frac{x^3 + x^2 - 2x}{x - 1}$

(2) $\lim_{x\to 0} \frac{\sqrt{9+3x} - 3}{x}$

(3) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

(4) $\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

(5) $\lim_{x\to 0} \frac{\sin x}{x}$

(6) $\lim_{x\to \infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$

Other Indeterminate Forms: If you have a limit of on of the following forms:

- $\lim_{x\to a} f(x)(g)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$
- $\lim_{x\to a}(f(x) - g(x))$ where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

then you can rewrite the limit to be of indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to apply L'Hospital's Rule.

Exercises: Evaluate the following limits:

(1) $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right)$

(2) $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

Even More Indeterminate Forms: If you have a limit $\lim_{x\to a} f(x)^{g(x)}$ with indeterminate form 1^∞, 0^0, or ∞^0, then you can complete the following steps to evaluate the limit:

(1) Find the limit $L = \lim_{x\to a} \ln\big(f(x)^{g(x)}\big) = \lim_{x\to a} g(x) \ln f(x)$ using L'Hospital's Rule

(2) Evaluate e^L

Exercises: Evaluate the following limits:

(1) $\displaystyle\lim_{x\to 0^+} x^x$

(2) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

Note: In the past, you've been able to solve equations that look like $f(x) = 0$ for x. In reality, this is not possible for the majority of cases. In these other cases, one helpful took for estimating the roots of a function f (i.e. where $f(x) = 0$) is called Newton's Method.

Newtons Method:

(1) Choose an initial approximation x_0 as close to the root as possible.

(2) Evaluate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 0, 1, 2, ...$

(3) End the calculations when your estimation criterion is met

Brainstorm: Draw a picture to visualize Newton's Method.

Exercises:

(1) Approximate the roots of $f(x) = x^3 - 5x + 1$ using $x_0 = -3$, $x_0 = 1$, and $x_0 = 4$ as initial approximations.

(2) Find the point(s) at which the curves $y = \cos x$ and $y = x$ intersect.

Defintion: A function F is called an antiderivative of f if $F'(x) = f(x)$

Example: If $f(x) = x^2$, then $F(x) = \frac{x^3}{3}$ is an antiderivative of f. $F(x) = \frac{1}{3}x^3 - 3000$ is also an antiderivative of f.

Theorem: If F is an antiderivative of f, then the most general antiderivative can be written as $F(x) + C$, where C is an arbitrary constant.

Antiderivative Formulas: Use your knowledge about derivatives to complete the following table:

$f(x)$	$F(x)$
$cf(x)$	
$f(x) \pm g(x)$	
$x^n (n \neq -1)$	
x^{-1}	
e^x	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$\sec x \tan x$	

Notation: We use an indefinite integral sign $\int$ to indicate general antiderivatives. In other words:

$$\int f(x)\ dx = F(x) + C$$

Exercises: Find the general antiderivative of the following functions:

(1) $4x^3$

(2) 2

(3) $\frac{3}{x}$

(4) $3x^2 - 2x + 4\sin x$

(5) $(x^2 + 1)(2x - 5)$

(6) $\frac{4x^{19} - 5x^{-8}}{x^2}$

Exercises (Initial/Boundary Value Problems):

(1) If $f'(x) = x\sqrt{x}$ and $f(1) = 2$, find $f(x)$

(2) If $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$ and $f(1) = 1$, find $f(x)$

(3) A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff that is 432 feet above ground. Find the equation of the ball's height with respect to time t. When does the ball reach maximum height? When does the ball hit the ground?