Sections 5.1-5.3

Note: The captial Greek letter sigma (Σ) is used to denote sums, where the beginning index and ending index are indicated on the bottom and top. For instance

$$\sum_{i=m}^{n} f(x_i) = f(x_m) + f(x_{m+1}) + f(x_{m+2}) + \ldots + f(x_n)$$

Exercises: Expand the following sigma notations:

(1) $\sum_{i=0}^{5} x^i$

(2) $\sum_{n=1}^{4} (n+1)^2$

Riemann Sums: Sometimes it is useful to find the area between a curve and the x-axis (often called the area under the curve). To get a general idea of the size of the area, we can approximate it using rectangles (or any other shape that we know the area of).

There are three standard ways of approximating area using rectangles. These are referred to as left endpoint approximations, right endpoint approximations, and midpoint approximations.

Brainstorm: Draw a generic function curve and sketch how you might use rectangles of equal length to estimate the area under the curve.

Brainstorm: Can we determine if an estimation is an overestimate or underestimate?

Definition: We call a partition regular if all segments of the partition are of equal length.

Steps for Estimating Area under $f(x)$ on $[a,b]$ using n regular rectangles:

(1) Find the width of the rectangles: $\Delta x = \frac{b-a}{n}$

(2) Divide $[a,b]$ into n intervals of equal width, label each partition as $a = x_0$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x, ..., x_n = b$

(3) If using left endpoints:

$$\text{Area } \approx \sum_{i=0}^{n-1} f(x_i)\Delta x = L_n$$

If you're using right endpoints then,

$$\text{Area } \approx \sum_{i=1}^{n} f(x_i)\Delta x = R_n$$

Lastly, if you're using midpoints then,

$$\text{Area } \approx \sum_{i=1}^{n} f(\frac{x_i + x_{i-1}}{2})\Delta x = M_n$$

These types of sums are called *Riemann Sums*.

Exercise: Approximate the area under the curve $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ using left endpoints, right endpoints, and midpoints with 4 rectangles.

(space for work)

Definition: The exact area under $f(x)$ is given by the limit as the amount of rectangles $n \to \infty$. In other words,

$$\text{Area} = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{x\to\infty} M_n$$

Note: The above definition can be generalized to other shapes, including shapes given by non-regular partitions.

Helpful Sums:

- $\displaystyle\sum_{i=1}^{n} c = nc$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Exercise: Find the exact area under $f(x) = x^2$ from $x = 0$ to $x = 1$

Definition: A definite integral is defined as follows:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \text{ where } x_i^* \text{ are any sample points}$$

If this limit exists and is the same for all x_i^*, then we say f is integrable on $[a, b]$.

Note: In other words, the definite integral is another way of writing an area under a curve.

Exercises: Use geometry to find the following integrals:

(1) $\int_0^1 \sqrt{1-x^2}\ dx$

(2) $\int_0^3 (x-1)\ dx$

Properties of the Definite Integral:

- $\int_a^a f(x)\ dx = 0$
- $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
- $\int_a^b c\ dx = c(b-a)$
- $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
- $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
- $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

Exercise: If $\int_0^{10} f(x)\ dx = 17$ and $\int_0^8 f(x)\ dx = 12$, find $\int_8^{10} f(x)\ dx$

The Fundamental Theorem of Calculus (Part 1): If f is continuous on $[a, b]$, then the function $g(x) = \int_0^x f(t)\ dt$, $a \leq x \leq b$ is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$

The Fundamental Theorem of Calculus (Part 2): Let f be a continuous function on the interval $[a, b]$ and let F be any antiderivative of f. Then:

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Note: Putting these two theorems together, we know that derivatives and integrals will cancel each other out.

Exercises: Evaluate the following integrals:

(1) $\int_{-1}^{1} x^3\ dx$

(2) $\int_0^{\pi/4} \sin x\ dx$

(3) $\int_0^4 2(\sqrt{t} - t)\ dt$

(4) $\int_1^2 \frac{2x^5 - x + 3}{x^2}\ dx$