ACMAT161 Summer 2024
Professor Manguba-Glover
Classwork 11 & 12 Name: ________________________________

Complete as many of the following problems as you can with your table in the allotted time. You do not have to go in order.

Classwork 11

1. Differentiate $\sqrt{13x^2 - 5x + 8}$

   Solution $f(x) = (13x^2 - 5x + 8)^{1/2}$ Thus

$$
\begin{aligned}
f'(x) &= \frac{1}{2}(13x^2 - 5x + 8)^{-1/2}(d/dx(13x^2 - 5x + 8)) \\
&= \frac{26x - 5}{2\sqrt{13x^2 - 5x + 8}}
\end{aligned}
$$

□

2. Differentiate $\frac{(x^3+4)^5}{(1-2x^2)^3}$

   Solution This will be a combination of quotient rule along with the chain rule.

$$
\begin{aligned}
f'(x) &= \frac{(1-2x^2)^3[d/dx(x^3+4)^5] - (x^3+4)^5[d/dx((1-2x^2)^3)]}{[(1-2x^2)^3]^2} \\
&= \frac{(1-2x^2)^3(5(x^3+4)^4(3x^2) - (x^3+4)^5(3(1-2x^2)^2(-4x)}{(1-2x^2)^6} \\
&= \frac{(1-2x^2)^3(15x^2)(x^3+4)^4 - (x^3+4)^5(-12x(1-2x^2)^2)}{(1-2x^2)^6} \\
&= \frac{\cancel{(1-2x^2)^2}[(1-2x^2)(15x^2)(x^3+4)^4 - (x^3+4)^5(-12x)}{\cancel{(1-2x^2)^2}(1-2x^2)^4} \\
&= \frac{(x^3+4)^4[(1-2x^2)(15x^2) - (x^3+4)(-12x)]}{(1-2x^2)^4} \\
&= \frac{(x^3+4)^4(3x)[(1-2x^2)(5x) - (x^3+4)(-4)]}{(1-2x^2)^4} \\
&= \frac{(x^3+4)^4(3x)(5x - 10x^3 + 4x^3 + 16)}{(1-2x^2)^4} \\
&= \frac{(3x)(x^3+4)^4(-6x^3 + 5x + 16)}{(1-2x^2)^4}
\end{aligned}
$$

□

3. Differentiate $\frac{x^2+4x}{(3x^3+2)^4}$

Solution First notice that

$$f(x) = \frac{x^2+4x}{v(u(x))} \text{ where } u(x) = 3x^3+2 \text{ and } v(x) = x^4$$

$$\begin{aligned} f'(x) &= \frac{(3x^3+2)^4(d/dx(x^2+4x)) - (x^2+4x)(d/dx((3x^3+2)^4)}{((3x^3+2)^4)^2} \\ &= \frac{(3x^3+2)^4(2x+4) - (x^2+4x)(v'(u(x))(u'(x)))}{(3x^3+2)^8} \\ &= \frac{(3x^3+2)^4(2x+4) - (x^2+4x)(4(3x^3+2)^3(9x^2)}{(3x^3+2)^8} \\ &= \frac{(3x^3+2)^3((3x^3+2)(2x+4) - 36x^2(x^2+4x))}{(3x^3+2)^8} \\ &= \frac{(3x^3+2)(2x+4) - 36x^3(x^2+4x)}{(3x^3+2)^5} \end{aligned}$$

□

4. Differentiate x^2e^{-x}

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2)e^{-x} + x^2\frac{d}{dx}(e^{-x}) \\ &= 2xe^{-x} - e^{-x}x^2 \end{aligned}$$

□

5. Differentiate $x\tan^3(2x)$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x)\tan^3(2x) + x\frac{d}{dx}(\tan^3(2x)) \\ &= \tan^3(2x) + x(3\tan^2(2x))(\sec^2(2x))(2) \\ &= \tan^3(2x) + 6x\tan^2(2x)\sec^2(2x) \end{aligned}$$

□

6. Differentiate $x\tan^{-1}\left(\frac{x}{2}\right)$

Solution

$$\begin{aligned}
\frac{d}{dx}\left(x\tan^{-1}\left(\frac{x}{2}\right)\right) &= \frac{d}{dx}(x)\tan^{-1}\left(\frac{x}{2}\right) + x\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right) \\
&= (1)\tan^{-1}\left(\frac{x}{2}\right) + x\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{d}{dx}\left(\frac{1}{2}x\right)\right) \\
&= \tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{1+\left(\frac{x^2}{4}\right)}\cdot\frac{1}{2} \\
&= \tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{2(1+\frac{x^2}{4})} \\
&= \tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{2+\frac{x^2}{2}}\cdot\frac{2}{2} \\
&= \boxed{\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{4+x^2}}
\end{aligned}$$

□

7. Differentiate $\log_5(2x+1)$

Solution

$$\begin{aligned}
\frac{d}{dx}(\log_5(2x+1)) &= \frac{1}{\ln 5(2x+1)}\cdot\frac{d}{dx}(2x+1) \\
&= \frac{1}{\ln 5(2x+1)}\cdot 2 \\
&= \frac{2}{\ln 5(2x+1)}
\end{aligned}$$

□

8. Differentiate $\frac{\ln x^2}{x^2}$

Solution

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\ln x^2}{x^2}\right) &= \frac{x^2 \frac{d}{dx}(\ln x^2) - \ln x^2 \frac{d}{dx}(x^2)}{(x^2)^2} \\
&= \frac{x^2\left(\frac{1}{x^2}\cdot\frac{d}{dx}(x^2)\right) - \ln x^2(2x)}{x^4} \\
&= \frac{x^2\left(\frac{1}{x^2}\cdot 2x\right) - 2x\ln x^2}{x^4} \\
&= \frac{x^2\left(\frac{2x}{x^2}\right) - 2x\ln x^2}{x^4} \\
&= \frac{2x - 2x\ln x^2}{x^4} \\
&= \frac{2x(1-\ln x^2)}{x^4} \\
&= \frac{2(1-\ln x^2)}{x^3}
\end{aligned}$$

□

Classwork 12

1. Find $\dfrac{dy}{dx}$ for $x + \ln y = x^2 y^3$

   Solution

$$\begin{aligned} 1 + \frac{1}{x}\frac{dy}{dx} = 2xy^3 + 3x^2y^2\frac{dy}{dx} &\Leftrightarrow y + \frac{dy}{dx} - 3x^2y^3\frac{dy}{dx} = 2xy^4 \\ &\Leftrightarrow \frac{dy}{dx}(1 - 3x^2y^3) = 2xy^4 - y \\ &\Leftrightarrow \frac{dy}{dx} = \frac{2xy^4 - y}{1 - 3x^2y^3} \end{aligned}$$

$\square$

2. Find $\dfrac{dy}{dx}$ of $e^{xy} = e^{4x} - e^{5y}$

   Solution Differentiating everything with respect to x we get

$$\begin{aligned} e^{xy}(y + x\frac{dy}{dx}) = e^{4x}(4) - e^{5y}(5\frac{dy}{dx}) &\Leftrightarrow e^{xy}y + xe^{xy}\frac{dy}{dx} = 4e^{4x} - 5e^{5y}\frac{dy}{dx} \\ &\Leftrightarrow \frac{dy}{dx}(xe^{xy} + 5e^{5y}) = 4x^{4x} - e^{xy}y \\ &\Leftrightarrow \frac{dy}{dx} = \frac{4e^{4x} - e^{xy}y}{xe^{xy} + 5e^{5y}} \end{aligned}$$

$\square$

3. Find y' for $\sin xy = x^2 + y$

Solution

$$\begin{aligned}
\sin xy = x^2 + y \Rightarrow \cos(xy)\frac{d}{dx}(xy) &= 2x + (1)\frac{dy}{dx} \\
\Rightarrow \cos(xy)\left(\frac{d}{dx}(x)y + x\frac{d}{dx}(y)\right) &= 2x + \frac{dy}{dx} \\
\Rightarrow \cos(xy)\left((1)y + x(1)\frac{dy}{dx}\right) &= 2x + \frac{dy}{dx} \\
\Leftrightarrow y\cos(xy) + x\cos(xy)\frac{dy}{dx} &= 2x + \frac{dy}{dx} \\
\Leftrightarrow x\cos(xy)\frac{dy}{dx} - \frac{dy}{dx} &= 2x - y\cos(xy) \\
\Leftrightarrow \frac{dy}{dx}(x\cos(xy) - 1) &= 2x - y\cos(xy) \\
\Leftrightarrow \boxed{\frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) - 1}}
\end{aligned}$$

□

4. Find y'' for $x^2 + y^2 = 1$

Solution

$$\begin{aligned}
x^2 + y^2 = 1 &\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \\
&\Leftrightarrow 2y\frac{dy}{dx} = -2x \\
&\Leftrightarrow \frac{dy}{dx} = \frac{-2x}{2y} \\
&\Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} = -\frac{x}{y} &\Rightarrow \frac{d^2y}{dx^2} = \frac{y\frac{d}{dx}(-x) - (-x)\frac{d}{dx}(y)}{y^2} \\
&\Leftrightarrow y'' = \frac{y(-1) + x(1)\frac{dy}{dx}}{y^2} \\
&\Leftrightarrow y'' = \frac{-y + x\frac{dy}{dx}}{y^2} \\
&\Leftrightarrow y'' = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} \leftarrow \text{acceptable} \\
&\Leftrightarrow y'' = \frac{-y - \frac{x^2}{y}}{y^2} \cdot \frac{y}{y} \\
&\Leftrightarrow y'' = \frac{-y^2 - x^2}{y^3} \\
&\Leftrightarrow y'' = \frac{-(x^2 + y^2)}{y^3} \\
&\Leftrightarrow \boxed{y'' = -\frac{1}{y^3}} \text{ since } x^2 + y^2 = 1 \text{ in our original equation}
\end{aligned}$$

□

5. Find the equation of the line tangent to $3(x^2+y^2)^2 = 25(x^2-y^2)$ at $(2,1)$

Solution

$$3(x^2+y^2)^2 = 25(x^2-y^2) \Leftrightarrow 6(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 25\left(2x-2y\frac{dy}{dx}\right)$$

We know that $x = 2$ and $y - 1$ so plugging this in we get

$$\begin{aligned} 6(4+1)\left(4+2\frac{dy}{dx}\right) = 25\left(4-2\frac{dy}{dx}\right) &\Leftrightarrow 30\left(4+2\frac{dy}{dx}\right) = 100-50\frac{dy}{dx} \\ &\Leftrightarrow 120+60\frac{dy}{dx} = 100-50\frac{dy}{dx} \\ &\Leftrightarrow 110\frac{dy}{dx} = -20 \\ &\Leftrightarrow \frac{dy}{dx} = -\frac{2}{11} \end{aligned}$$

Using point slope form we get

$$y-1 = -\frac{2}{11}(x-2)$$

□

6. Find the equation of the tangent line to $x^2 + \tan\left(\frac{\pi}{4}xy\right) = 2$ at $(1,1)$

Solution

$$\begin{aligned} x^2+\tan\left(\frac{\pi}{4}xy\right) = 2 &\Leftrightarrow 2x + \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}xy\right)\left(y+x\frac{dy}{dx}\right) = 0 \\ &\Leftrightarrow y + x\frac{dy}{dx} = -\frac{8x}{\pi\sec^2((\pi/4)(xy))} \\ &\Leftrightarrow \frac{dy}{dx} = -\frac{8x}{\pi x\sec^2((\pi/4)(xy))} - \frac{y}{x} \end{aligned}$$

Plugging in $x = 1$ and $y = 1$ we have

$$\frac{dy}{dx} = -\frac{16}{\pi} - 1$$

So our equation is

$$y-1 = -\left(\frac{16}{\pi}+1\right)(x-1)$$

□

7. Find the equation of the tangent line to $x^2 + xy + y^2 = 3$ at $(1, 1)$

Solution Differentiating everything with respect to x gives

$$\begin{aligned} 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Leftrightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -2x - y \\ \Leftrightarrow \frac{dy}{dx}(x + 2y) &= -2x - y \\ \Leftrightarrow \frac{dy}{dx} &= \frac{-2x - y}{x + 2y} \end{aligned}$$

Plugging in our point $(1, 1)$ we get

$$\frac{dy}{dx} = \frac{-3}{3} = -1$$

Now using point-slope form, we get that the equation of the tangent line is

$$y - 1 = -(x - 1) \Leftrightarrow y = -x + 2$$

$\square$