ACMAT161 Summer 2024
Professor Manguba-Glover
Classwork 13 & 14 Name: ______________________________

Complete as many of the following problems as you can with your table in the allotted time. You do not have to go in order.

Classwork 13

1. Suppose that x and y are two functions of t such that the following equation holds for all t:

$$\sin(x+y) + (1+x)^2 + (1+y)^2 = 5$$

and suppose that when $x = 0$ and $y = 0$ we have $dx/dt = -10$. What is the value of dy/dt?

Solution Differentiating both sides with respect to t we have

$$\cos(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) + 2(1+x)\frac{dx}{dt} + 2(1+y)\frac{dy}{dt} = 0$$

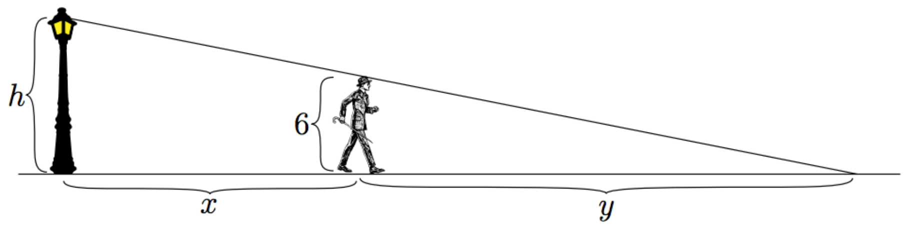
Plugging in the known information we have

$$\cos(0)\left(-10 + \frac{dy}{dt}\right) + 2(1)(-10) + 2(1)\frac{dy}{dt} = 0 \Leftrightarrow \frac{dy}{dt} = 10$$

□

2. A man 6 feet tall is walking away from a lamp post at the rate of 50 feet per minute. When the man is 8 feet away from the post his shadow is 10 feet long. Find the rate at which the length of the shadow is increasing when he is 25 feet away from the post.

Solution Here is our picture:



From geometry we know that

$$\frac{h}{x+y} = \frac{6}{y}$$

We also know that when $x = 8$, then $y = 10$. Thus

$$\frac{h}{18} = \frac{6}{10} \Leftrightarrow h = \frac{54}{5}$$

This gives

$$54y = 30(x+y) \Leftrightarrow 24y = 30x$$

We are given that $dx/dt = 50$. Differentiating with respect to t we have

$$24\frac{dy}{dt} = 3 - \frac{dx}{dt}$$

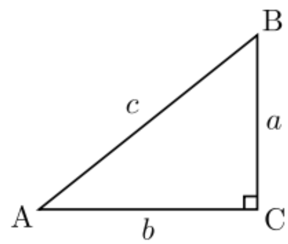
Plugging in we have

$$24\frac{dy}{dt} = 1500 \Leftrightarrow \frac{dy}{dt} = 62.5 \text{ ft/min}$$

□

3. One car leaves a given point and travels north at 30 mph. Another car leaves the same point at the same time and travels west at 40 mph. At what rate is the distance between the two cars changing at the instant when the cars have traveled 2 hours?

Solution Our picture looks like



Using the pythagorean theorem we have

$$a^2 + b^2 = c^2 \Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

We know that $da/dt = 30$ and $db/dt = 40$. We want to know what dc/dt is after 2 hours.

After 2 hours the first car traveling north will have travelled $a = 60$ miles and the second will have travelled $b = 80$. Thus the distance c between them will be

$$c = \sqrt{a^2 + b^2} = \sqrt{60^2 + 80^2} = 100$$

Plugging this in we have:

$$\begin{aligned}
2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt} &\Rightarrow 2(60)(30) + 2(80)(40) = 2(100)\frac{dc}{dt} \\
&\Leftrightarrow 1800 + 3200 = 100\frac{dc}{dt} \\
&\Leftrightarrow 18 + 32 = \frac{dc}{dt} \\
&\Leftrightarrow \frac{dc}{dt} = \boxed{50 \text{ mph}}
\end{aligned}$$

□

4. Consider a bacterial culture growing in a petri dish and suppose that the bacteria reproduce in such a way that they are always forming a disk of growing radius. Suppose that we know the area of the disk is increasing at the rate of 3 cm^2/day. Can we find the rate of change of the radius at the time when that radius is 4 cm?

Solution We know that the area of a disk is given by $A = \pi r^2$ where the area and the radius of the disk depend on time. From the problem we know that $dA/dt = 3$ and we need to find dr/dt when $r = 4$. Differentiating both sides of the equation with respect to t we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Plugging in our known info we have

$$3 = 2\pi(4)\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt} = \frac{3}{8\pi}\text{cm/day}$$

□

Classwork 14

1. Find the slope of the tangent line to $r = 1 + \sin\theta$ when $\theta = \frac{\pi}{3}$

   Solution

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\
&= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta-\sin\theta} \\
&= \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}
\end{aligned}
$$

$$
\begin{aligned}
m &= \frac{dy}{dx} \text{ at } \theta = \frac{\pi}{3} \\
&= \frac{\cos\frac{\pi}{3}\left(1+2\sin\frac{\pi}{3}\right)}{\left(1+\sin\frac{\pi}{3}\right)\left(1-2\sin\frac{\pi}{3}\right)} \\
&= \frac{\frac{1}{2}(1+\sqrt{3})}{\left(1+\frac{\sqrt{3}}{2}\right)(1-\sqrt{3})} \\
&= \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} \\
&= \frac{1+\sqrt{3}}{-1-\sqrt{3}} \\
&= \boxed{-1}
\end{aligned}
$$

□

2. Find the derivative of $r = 3 + 2\cos\theta$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{(-2\sin\theta)\sin\theta + (3+2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (3+2\cos\theta)\sin\theta} \leftarrow \text{acceptable} \\
&= \frac{-2\sin^2\theta + 3\cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - 3\sin\theta - 2\sin\theta\cos\theta} \\
&= \frac{2(\cos^2\theta - \sin^2\theta) + 3\cos\theta}{-2(2\sin\theta\cos\theta) - 3\sin\theta} \\
&= \frac{2\cos 2\theta + 3\cos\theta}{-2\sin 2\theta - 3\sin\theta} \\
&= \boxed{-\frac{2\cos(2\theta) + 3\cos\theta}{2\sin(2\theta) + 3\sin\theta}}
\end{aligned}$$

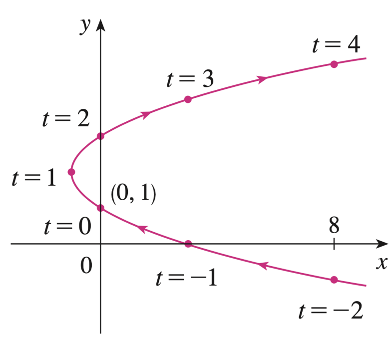
□

3. Sketch the curve defined by $x = t^2 - 2t$, $y = t + 1$

Solution Plugging in t-values, we can get the following table

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5

Graphing the points (x, y) gives us:



□

4. Find an equation of the tangent line to the curve $x = \sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ when $t = \frac{\pi}{4}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dy}} \\ &= \frac{\sec^2 t}{\sec t \tan t} \\ &= \frac{\sec t}{\tan t} \\ &= \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} \\ &= \frac{1}{\sin t}\end{aligned}$$

$$\begin{aligned}m &= \frac{dy}{dx} \text{ at } t = \frac{pi}{4} \\ &= \frac{1}{\sin \frac{\pi}{4}} \\ &= \frac{1}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\end{aligned}$$

Plugging in $t = \frac{\pi}{4}$, we have:

$$x = \sec \frac{\pi}{4} = \sqrt{2} \text{ and } y = \tan \frac{\pi}{4} = 1$$

Using point-slope form we have

$$\boxed{y - 1 = \sqrt{2}(x - \sqrt{2})}$$

□

5. A curve is defined by $x = t^2$, $y = t^3 - 3t$. Find $\frac{dy}{dx}$ and determine where the tangent is horizontal or vertical.

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \boxed{\frac{3t^2 - 3}{2t}}$$

The tangent line is horizontal when the numerator is equal to 0

$$3t^2 - 3 = 0 \Leftrightarrow 3t^2 = 3 \Leftrightarrow t^2 = 1 \Leftrightarrow t = \pm 1$$

Plugging in $t = 1$, we have $x = (1)^2 = 1$ and $y = (1)^3 - 3(1) = -2$.
Plugging in $t = -1$, we have $x = (-1)^2 = 1$ and $y = (-1)^3 - 3(-1) = 2$

So the tangent line is horizontal at $\boxed{(1, -2), (-1, 2)}$

The tangent line is vertical when the denominator is equal to 1

$$2t = 0 \Leftrightarrow t = 0$$

Plugging in $t = 0$, we have $x = (0)^2 = 0$ and $y = (0)^3 - 3(0) = 0$.

So the tangent line is vertical at $\boxed{(0, 0)}$

□

6. Find the equation of the slope of the line tangent to $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{r\sin\theta}{r(1 - \cos\theta)}$$

$$= \boxed{\frac{\sin\theta}{1 - \cos\theta}}$$

□