ACMAT161 Summer 2024
Professor Manguba-Glover
Classwork 17 & 18 Name: ______________________________

Complete as many of the following problems as you can with your table in the allotted time. You do not have to go in order.

**Classwork 17**

(1) Graph the function $f(x) = 3x^4 - 4x^3$ using curve sketching steps. Indicate inflection points and relative extrema.

(a) Find the open intervals where $f$ is increasing and the intervals where $f$ is decreasing.

(b) Find both coordinates of any local extrema of the graph of $f$.

(c) Find the intervals where $f$ is concave up, and the intervals where $f$ is concave down.

(d) Find both coordinates of the inflection points of $f$.

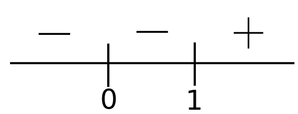
(e) Using the above information, sketch the graph of $y = f(x)$ on the coordinate axes below. You must label both coordinates of any local extrema and inflection points on your graph. (The graph does not need to be to scale.)

**Solution**

(a)

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

Critical numbers: $0, 1$ Plotting this on a number line we have




Thus $f$ is decreasing on $(-\infty, 0)$, $(0, 1)$ and increasing on $(1, \infty)$.

(b) There is a local minimum at $x = 1$.

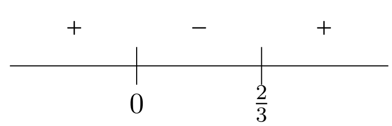
$$f(1) = 3 - 4 = -1$$

So the point is $(1, -1)$

(c)

$$f''(x) = 36x^2 - 24x = 12x(3x-2)$$

So the two points we have to plot are $x = 0$ and $x = 2/3$




So $f$ is concave up on $(-\infty, 0)$, $(2/3, \infty)$ and concave down on $(0, 2/3)$

(d) There are inflection points at $x = 0$ and $x = {}^2\!/_3$

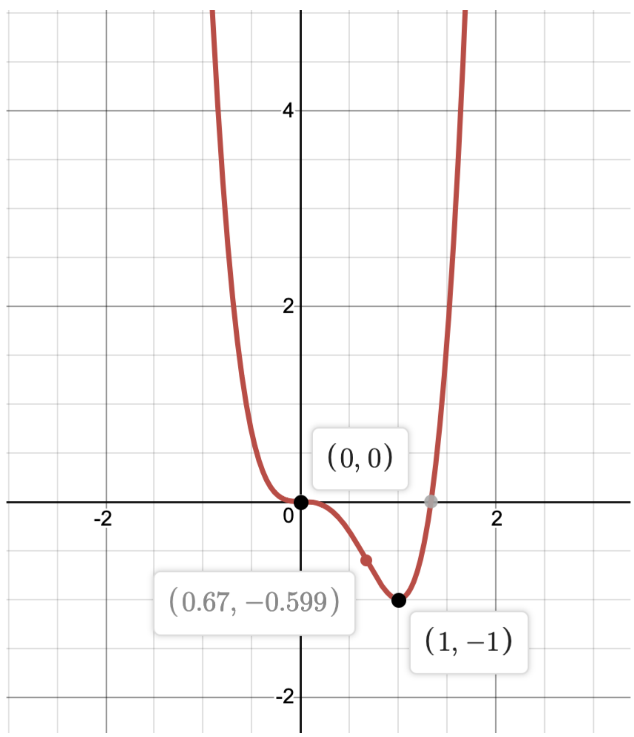
$$f(x) = x^3(3x - 4)$$

$$\begin{aligned} f(0) &= 0 - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f\left(\frac{2}{3}\right) &= \left(\frac{2}{3}\right)^3 \left(3\left(\frac{2}{3}\right) - 4\right) \\ &= \frac{8}{27}(-2) \\ &= -\frac{16}{27} \end{aligned}$$

So the two inflection points are $(0, 0)$ and $({}^2\!/_3, -{}^{16}\!/_{27})$

(e)




□

(2) Graph the function $f(x) = \frac{10x^3}{x^2-1}$. Indicate asymptotes, inflection points, and relative extrema.

**Solution**

Domain: We cannot divide by 0, we we know $x \neq \pm 1$, which gives us a domain of $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$. This also gives us vertical asymptotes of $x = -1$ and $x = 1$

Symmetry:

$$\begin{aligned} f(-x) &= \frac{10(-x)^3}{(-x)^2-1} \\ &= \frac{-10x^3}{x^2-1} \\ &= -f(x) \end{aligned}$$

So the function has origin symmetry.

Slant Asymptote: Since the degree of the top is one more than the bottom, there is a slant asymptote. Doing long division, we have

$$\begin{array}{r|l} & \quad\quad 10x \\ \hline x^2-1 & 10x^3 \\ & -10x^3+10x \\ \hline & \quad\quad 10x \end{array}$$

This gives a slant asymptote of $y = 10x$.

First Derivative:

$$f'(x) = \frac{10x^2(x^2-3)}{(x^2-1)^2}$$

Plotting our critical points and testing the intervals, we have that $f$ is increasing on $(-\infty,-\sqrt{3}),(\sqrt{3},\infty)$ and decreasing on $(-\sqrt{3},-1),(-1,0),(0,1),(1,\sqrt{3})$.

This means that there is a local max at $x = -\sqrt{3}$ and a local min at $x = \sqrt{3}$. Plugging into our original function we have:

$$f(-\sqrt{3}) = -15\sqrt{3} \text{ and } f(\sqrt{3}) = 15\sqrt{3}$$

Second Derivative:

$$f''(x) = \frac{20x(x^2+3)}{(x^2-1)^3}$$

Plotting our critical points for $f''$ and testing the intervals, we have that $f$ is concave down on $(-\infty,-1),(0,1)$ and concave up on $(-1,0),(1,\infty)$. We also have an inflection point at $x = 0$, which corresponds to a $y$-value of $f(0) = 0$.

□

(3) Sketch the graph of $\dfrac{x-1}{x+2}$. Indicate asymptotes, inflection points, and relative extrema.

**Solution**

Domain: $x$ cannot be $-2$ so the domain is $(-\infty,-2)\bigcup(-2,\infty)$

$x$-intercept:

$$\begin{aligned}\frac{x-1}{x+2}=0 &\Leftrightarrow x-1=0\\ &\Leftrightarrow x=1\end{aligned}$$

Thus the $x$-intercept is 1

$y$-intercept:

Plugging in $x=0$ we have

$$\frac{0-1}{0+2}=-\frac{1}{2}$$

Thus the $y$-intercept is 0.

Horizontal Asymptote:

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{x-1}{x+2} &= \lim_{x\to\pm\infty}\frac{x/x-1/x}{x/x+2/x}\\ &= \lim_{x\to\pm\infty}\frac{1-1/x}{1+2/x}\\ &= \frac{1-0}{1+0}\\ &= 1\end{aligned}$$

Vertical Asymptote:

$$\frac{x-1}{x+2}\text{ is undefined}\Leftrightarrow x+2=0\Leftrightarrow x=-2$$

Thus the vertical asymptote is $x=-2$

Symmetry:

$$f(-x)=\frac{-x-1}{-x+2}$$

so there is no symmetry.

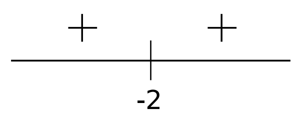
First Derivative Test:

$$\frac{d}{dx}\left(\frac{x-1}{x+2}\right) = \frac{(x+2)d/dx(x-1)-(x-1)d/dx(x+2)}{(x+2)^2}$$

$$= \frac{\cancel{x}+2-\cancel{x}+1}{(x+2)^2}$$

$$= \frac{3}{(x+2)^2}$$

So our key point is $x = -2$. Testing the intervals we have




Thus the function is always increasing and there is no relative extrema.

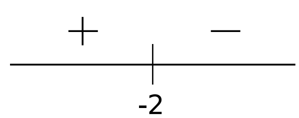
Second Derivative Test:

$$\frac{d}{dx}\left(\frac{3}{(x+2)^2}\right) = \frac{d}{dx}(3(x+2)^{-2})$$

$$= 3(-2)(x+2)^{-3}$$

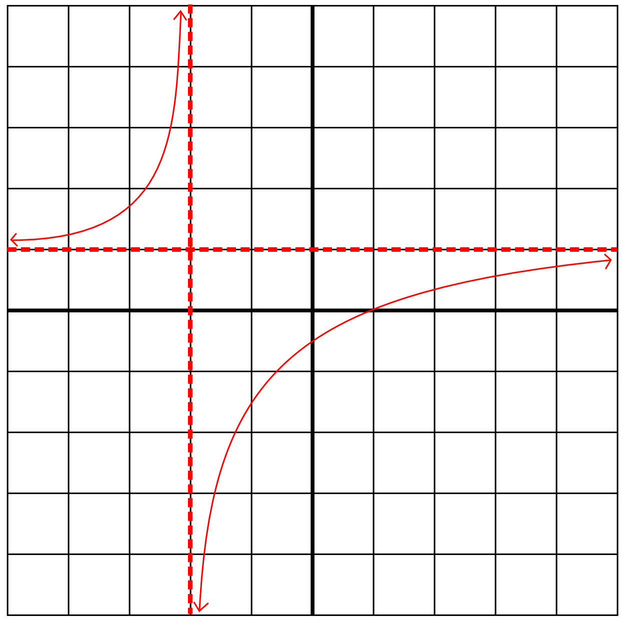
$$= -\frac{6}{(x+2)^2}$$

Thus the important point is $x = -2$. Checking the intervals we have




So the function is concave up on $(-\infty, -2)$ and concave down on $(-2, \infty)$. However, there is no inflection point since the function is undefined at $x = -2$

Graph:



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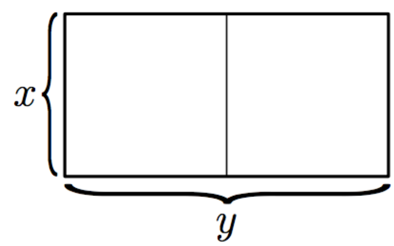
**Classwork 18**

(1) A 216 $m^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to the one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

**Solution** The picture is as follows




We are given that $216 = xy \Leftrightarrow y = \dfrac{216}{x}$ and we want to minimize

$$P = 3x + 2y = 3x + \frac{432}{x}$$

Taking the derivative we have

$$\frac{dP}{dx} = 3 - \frac{432}{x^2} = \frac{3x^2 - 432}{x^2}$$

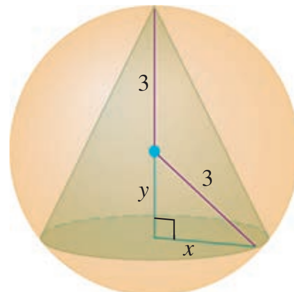
$\dfrac{dP}{dx}$ is undefined at $x = 0$ and

$$\frac{dP}{dx} = 0 \Leftrightarrow x^2 = 144 \Rightarrow x = \pm 12$$

Since $x$ must be positive we have one critical point of $x = 12$. Testing the intervals we have that this critical point is a minimum and thus the dimensions needed are $x = 12$m and $y = \dfrac{216}{12} = 18$m

$\square$

(2) Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.




**Solution** We have the volume of the cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2(y+3)$$

We also have that

$$x^2 + y^2 = 3^2 \Rightarrow x^2 = 9 - y^2$$

Using this substitution we have

$$V = \frac{1}{3}\pi(9-y^2)(y+3) = \frac{\pi}{3}(9y + 27 - y^3 - 3y^2)$$

Thus we have

$$\frac{dV}{dy} = \frac{\pi}{3}(9 - 3y^2 - 6y) = -\pi(y^2 + 2y - 3) = -\pi(y+3)(y-1)$$

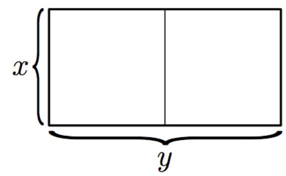
Thus there are critical points at $y = -3$ and $y = 1$. Since $y$ must be positive we have $y = 1$ is the only valid critical point. Testing we have that $dV/dt$ switches from positive to negative at $y = 1$ so there is a relative and thus absolute max at $y = 1$. Thus gives

$$V = \frac{1}{3}\pi(9-1)(1+3) = \frac{32\pi}{3}$$

□

(3) A farmer is constructing a rectangular pen with one additional fence across its width. Find the maximum area that can be enclosed with 2400m of fencing.

**Solution** The picture looks like:




We have that

$$2400 = 3x + 2y \text{ and } A = xy$$

$$2400 = 3x + 2y \Rightarrow 2y = 2400 - 3x \Rightarrow y = 1200 - \frac{3}{2}x$$

Plugging this in to the area equation we have

$$A(x) = x\left(1200 - \frac{3}{2}x\right) = 1200x - \frac{3}{2}x^2$$

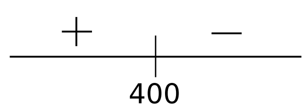
Taking the derivative we get

$$A'(x) = 1200 - 3x$$

Solving for the critical point:

$$A'(x) = 0 \Leftrightarrow 1200 - 3x = 0 \Leftrightarrow 3x = 1200 \Leftrightarrow x = 400$$

Testing this is a max we have



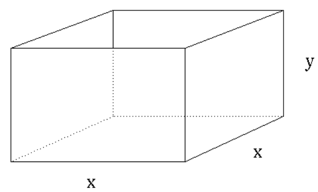

So the max occurs at $x = 400$.

$$\begin{aligned} A(400) &= 400\left(1200 - \frac{3}{2}(400)\right) \\ &= 400(1200 - 600) \\ &= 400(600) \\ &= \boxed{240000 \text{ ft}^2} \end{aligned}$$

□

(4) A rectangular box has a base that is a square. The perimeter of the base plus the height of the box is equal to 3 feet. What is the largest possible volume for such a box, and what are its dimensions?

**Solution** The picture looks like:




We have

$$4x + y = 3 \text{ and } V = x^2 y$$

$$4x + y = 3 \Rightarrow y = 3 - 4x$$

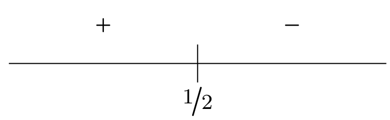
Plugging this into the equation for volume gives

$$V(x) = x^2(3 - 4x) = 3x^2 - 4x^3$$

Taking the derivative gives

$$V'(x) = 6x - 12x^2 = 6x(1 - 2x)$$

We have two critical numbers of $x = 0, 1/2$.
Since $x$ is the length of the side we know $x > 0$ so we just need to check that $x = 1/2$ is a max.




Therefore the maximum volume occurs when $x = 1/2$

$$y = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

So the $\boxed{\text{dimensions are 1 ft by 1/2 ft by 1/2 ft}}$.

$$\begin{aligned} V\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^2\left(3 - 4\left(\frac{1}{2}\right)\right) \\ &= \frac{1}{4}(3 - 4) \\ &= \boxed{\frac{1}{4}\text{ ft}^3} \end{aligned}$$

□

(5) Find the linearization $L(x)$ of $f(x)$ at $x = a$: $f(x) = x^3 - 2x + 3$, $a = 2$

**Solution**

$$f(2) = 8 - 4 + 3 = 7$$

$$f'(x) = 3x^2 - 2 \Rightarrow f'(2) = 10$$

Using the linearization formula we have

$$L(x) \approx 7 + 10(x - 2) = \boxed{10x - 13}$$

□

(6) Find the linearization $L(x)$ of $f(x)$ at $x = a$: $f(x) = \tan x$, $a = \pi$

**Solution**

$$f(\pi) = \tan \pi = 0$$

$$f'(x) = \sec^2 x \Rightarrow f'(\pi) = \sec^2 \pi = 1$$

Using the linearization formula we have

$$L(x) \approx 0 + (x - \pi) = \boxed{x - \pi}$$

□

(7) Find $dy$ of $y = x^3 - 3\sqrt{x}$

**Solution**

$$\frac{dy}{dx} = 3x^2 - \frac{3}{2\sqrt{x}}$$

Thus

$$dy = \boxed{\left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx}$$

□

(8) Use linear approximation to approximate $\sqrt[3]{29}$. Is this an over or underestimate?

**Solution** We choose $a = 27$ and $f(x) = \sqrt[3]{x}$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt[3]{27} + \frac{1}{3(\sqrt[3]{27})^2}(x-27) \\ &= 3 + \frac{1}{3(3)^2}(x-27) \\ &= 3 + \frac{1}{27}(x-27) \end{aligned}$$

$$\begin{aligned} \sqrt[3]{29} &\approx L(29) \\ &= 3 + \frac{1}{27}(29-27) \\ &= 3 + \frac{1}{27}(2) \\ &= \boxed{3 + \frac{2}{27} \text{ or } \frac{83}{27}} \end{aligned}$$

□

(9) Use linear approximation to approximate $\sqrt[3]{65}$

**Solution** Let $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \dfrac{1}{3}x^{-2/3}$
Choose $x_0 = 64$,

$$\begin{aligned} f(x_0) &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

$$\begin{aligned} f'(x_0) &= \frac{1}{3}(64)^{-2/3} \\ &= \frac{1}{3}((64)^{-1/3})^2 \\ &= \frac{1}{3}\left(\frac{1}{4}\right)^2 \\ &= \frac{1}{3}\left(\frac{1}{16}\right) \\ &= \frac{1}{48} \end{aligned}$$

This gives

$$\begin{aligned} f(65) &= \sqrt[3]{65} \\ &\approx f(64) + f'(64)(65 - 64) \\ &= 4 + \frac{1}{48} \end{aligned}$$

□

(10) Find the differential $dy$ for the following:

(a) $y = x\sqrt{1 - x^2}$
(b) $y = \cos(x^2)$

**Solution**

(a)

$$\begin{aligned} dy &= f'(x)dx \\ &= \left(\sqrt{1 - x^2} + x\left(\frac{1}{\cancel{2}}(1 - x^2)^{-1/2}(-\cancel{2}x)\right)\right)dx \\ &= (\sqrt{1 - x^2} - x^2(1 - x^2)^{-1/2})dx \end{aligned}$$

(b)

$$\begin{aligned} dy &= f'(x)dx \\ &= -\sin(x^2)(2x)dx \\ &= -2x\sin(x^2)dx \end{aligned}$$

□