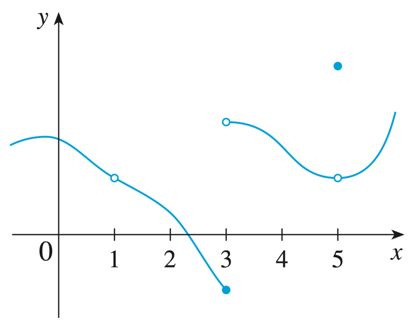
ACMAT161 Summer 2024
Professor Manguba-Glover
Classwork 7 & 8 Name: ______________________________

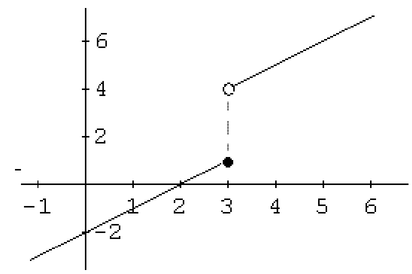
Complete as many of the following problems as you can with your table in the allotted time. You do not have to go in order.

Classwork 7

1. Where is the following function discontinuous? Explain which requirement it fails:



(a)



(b)

Solution The function is discontinuous anywhere there is a break in the graph, so at $x = 1$, $x = 3$, and $x = 5$.

- At $x = 1$, $f(1)$ is undefined
- At $x = 3$, $\lim_{x \to 3} f(x)$ DNE.
- At $x = 5$, $\lim_{x \to 5} f(x) \neq f(5)$

□

2. Where is the function discontinuous? Explain why. $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Solution The top piece $\frac{1}{x^2}$ is continuous for all $x \neq 0$ and the bottom piece 1 is continuous everywhere, so we just have to check when $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} \to \frac{1}{0^+} \to \infty$$

Since the limit as $x \to 0$ does not exist, we know that the function is not continuous at $x = 0$

□

3. Is the $f(x)$ continuous at $x = 3$? At $x = -3$? Where $f(x) = \begin{cases} \dfrac{x^3 - 27}{x^2 - 9} & x \neq 3 \\ \dfrac{9}{2} & x = 3 \end{cases}$

Solution

$$\begin{aligned}
\lim_{x \to 3} f(x) &= \lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9} \\
&= \lim_{x \to 3} \frac{\cancel{(x-3)}(x^2 + 3x + 9)}{\cancel{(x-3)}(x+3)} \\
&= \lim_{x \to 3} \frac{x^2 + 3x + 9}{x + 3} \\
&= \frac{3^2 + 3(3) + 9}{3 + 3} \\
&= \frac{27}{6} \\
&= \frac{9}{2} \\
&= f(3)
\end{aligned}$$

Thus the function is continuous at $x = 3$. However $f(-3)$ is undefined so it is not continuous at $x = -3$.

$\square$

4. Determine the interval(s) where $f(x)$ is continuous if $f(x) = \begin{cases} 0 & x < 0 \\ x^2 - 5x & 0 \le x \le 5 \\ 5 & x > 5 \end{cases}$

Solution Notice that on the interiors of all the intervals present, each piece is continuous. Thus we know our function is continuous for $x \neq 0, 5$. We need to check what happens at $x = 0$ and $x = 5$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x^2 - 5x) \\ &= (0)^2 - 5(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^2 - 5(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 5^-} f(x) &= \lim_{x \to 5^-} (x^2 - 5x) \\ &= (5)^2 - 5(5) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 5^+} f(x) &= \lim_{x \to 5^+} 5 \\ &= 5 \end{aligned}$$

$$f(5) = 0$$

This gives us that it is continuous at $x = 0$ but not at $x = 5$. It also tells us that the function is left-continuous at $x = 5$. Thus we have that $f(x)$ is continuous for $\boxed{(-\infty, 5],\ (5, \infty)}$

□

5. For what value of c is $f(x) = \begin{cases} \dfrac{x^3-1}{x-1} & x < 1 \\ cx - 2 & x \geq 1 \end{cases}$ continuous?

Solution We want the function to match up for $x = 1$.

$$\begin{aligned}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (cx - 2) \\
&= c(1) - 2 \\
&= c - 2
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} \frac{x^3 - 1}{x - 1} \\
&= \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}} \\
&= \lim_{x \to 1^-} x^2 + x + 1 \\
&= (1)^2 + 1 + 1 \\
&= 3
\end{aligned}$$

We want them equal so we want $c - 2 = 3 \Leftrightarrow \boxed{c = 5}$

$\square$

6. Find $\lim_{x \to \infty} e^{\frac{2x^2+7x-3}{x^2-5x+1}}$

Solution

$$\begin{aligned}
\lim_{x \to \infty} e^{\frac{2x^2+7x-3}{x^2-5x+1}} &= e^{\lim_{x \to \infty} \frac{2x^2+7x-3}{x^2-5x+1}} \\
&= e^{\lim_{x \to \infty} \frac{\frac{2x^2}{x^2} + \frac{7x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}} \\
&= e^{\lim_{x \to \infty} \frac{2 + \frac{7}{x} - \frac{3}{x^2}}{1 - \frac{5}{x} + \frac{1}{x^2}}} \\
&= e^{\frac{2+0-0}{1-0+0}} \\
&= \boxed{e^2}
\end{aligned}$$

$\square$

7. Explain why the equation $\cos x = x$ has at least one solution. (Hint: get everything to one side first)

Solution Let $f(x) = \cos x - x$

$$\begin{aligned} f(0) &= \cos 0 - 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \cos \frac{\pi}{2} - \frac{\pi}{2} \\ &= 0 - \frac{\pi}{2} \\ &= -\frac{\pi}{2} \end{aligned}$$

Thus by the intermediate value theorem, there must be a value of x between 0 and $\frac{\pi}{2}$ such that $f(x) = 0$ i.e. such that $\cos x = x$.

$\square$

Classwork 8

1. Find an equation of the line tangent to the graph of $f(x) = \frac{3}{x}$ at $\left(2, \frac{3}{2}\right)$

   Solution First, we can find the slope using the limit definition of a derivative.

$$
\begin{aligned}
m &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{\frac{3}{2+h} - \frac{3}{2}}{h} \\
&= \lim_{h \to 0} \frac{\frac{3}{2+h} - \frac{3}{2}}{h} \cdot \frac{2(2+h)}{2(2+h)} \\
&= \lim_{h \to 0} \frac{3(2) - 3(2+h)}{2h(2+h)} \\
&= \lim_{h \to 0} \frac{6 - 6 - 3h}{2h(2+h)} \\
&= \lim_{h \to 0} \frac{-3\cancel{h}}{2\cancel{h}(2+h)} \\
&= \lim_{h \to 0} \frac{-3}{2(2+h)} \\
&= \frac{-3}{2(2+0)} \\
&= -\frac{3}{4}
\end{aligned}
$$

   Using point-slope form we have

$$\boxed{y - \frac{3}{2} = -\frac{3}{4}(x - 2)}$$

$\square$

2. Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1,5)$

Solution First, we can find the slope using the limit definition of a derivative.

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)^3 + 4(1-h) - 5}{h} \\
&= \lim_{h\to 0} \frac{(1+h)(1+h)(1+h) + 4 - 4h - 5}{h} \\
&= \lim_{h\to 0} \frac{(1+2h+h^2)(1+h) + 4 - 4h - 5}{h} \\
&= \lim_{h\to 0} \frac{h^3 + 3h^2 + 3h + 1 + 4 - 4h - 5}{h} \\
&= \lim_{h\to 0} \frac{h^3 + 3h^2 + 7h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(h^2 + 3h + 7)}{\cancel{h}} \\
&= \lim_{h\to 0} (h^2 + 3h + 7) \\
&= 0 + 0 + 7 \\
&= 7
\end{aligned}$$

Using point-slope form we have

$$\boxed{y - 5 = 7(x - 1)}$$

□

3. Let $f(x) = \sqrt{2x} + 1$ and compute $f'(2)$

Solution

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(2+h)} + 1 - (\sqrt{2(2)} + 1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+2h} + \cancel{1} - \sqrt{4} - \cancel{1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+2h} - 2}{h} \cdot \frac{\sqrt{4+2h} + 2}{\sqrt{4+2h} + 2} \\
&= \lim_{h\to 0} \frac{4 + 2h - 4}{h(\sqrt{4+2h} + 2} \\
&= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{4+2h} + 2)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{4+2h} + 2} \\
&= \frac{2}{\sqrt{4} + 2} \\
&= \frac{2}{4} \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

□

4. Find the derivative of $f(x) = -x^2 + 6x$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{-(x+h)^2 + 6(x+h) - (-x^2+6x)}{h} \\
&= \lim_{h\to 0} \frac{-(x^2+2xh+h^2) + 6x + 6h + x^2 - 6x}{h} \\
&= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 6x + 6h + x^2 - 6x}{h} \\
&= \lim_{h\to 0} \frac{-2xh - h^2 + 6h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x - h + 6)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x - h + 6) \\
&= \boxed{2x + 6}
\end{aligned}$$

□

5. Find the derivative of $g(t) = \frac{1}{t^2}$

Solution

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{t^2+2th+h^2} - \frac{1}{t^2}}{h} \cdot \frac{(t^2+2th+h^2)(t^2)}{(t^2+2th+h^2)(t^2)} \\
&= \lim_{h\to 0} \frac{t^2 - (t^2+2th+h^2)}{h(t^2+2th+h^2)(t^2)} \\
&= \lim_{h\to 0} \frac{\cancel{t^2} - \cancel{t^2} - 2th - h^2}{h(t^2+2th+h^2)(t^2)} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2t-h)}{\cancel{h}(t^2+2th+h^2)(t^2)} \\
&= \lim_{h\to 0} \frac{-2t-h}{(t^2+2th+h^2)(t^2)} \\
&= \frac{-2t+0}{(t^2+0+0)(t^2)} \\
&= \frac{-2t}{t^4} \\
&= \boxed{\frac{-2}{t^3}}
\end{aligned}
$$

□

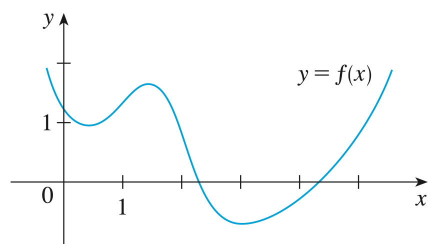
6. Find f' for $f(x) = \frac{1-x}{2+x}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1-x-h}{2+x+h} - \frac{1-x}{2+x}}{h} \cdot \frac{(2+x+h)(2+x)}{(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{2+x-2x-x^2-2h-xh-(2+x+h-2x-x^2-xh)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{\cancel{2}+\cancel{x}-\cancel{2x}-\cancel{x^2}-2h-\cancel{xh}-\cancel{2}-\cancel{x}-h+\cancel{2x}+\cancel{x^2}+\cancel{xh}}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} \\
&= \boxed{\frac{-3}{(2+x)^2}}
\end{aligned}$$

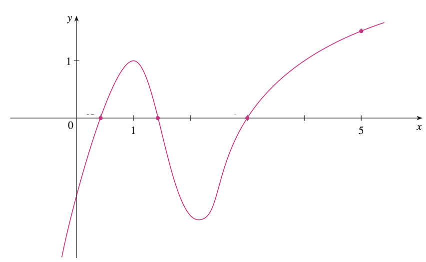
□

7. Sketch the derivative of $f(x)$:



Solution The derivative is the slope of the tangent line, so we need to see where the graph's tangent slope is positive (graph is moving upward) vs where the tangent slope is negative (graph is going downward).

Negative slope means the derivative is negative (i.e. the graph of f' will be below the x-axis). Positive slope means the derivative is positive (i.e. the graph of f' will be above the x-axis).



$\square$