ACMAT161 Summer 2024
Professor Manguba-Glover
Classwork 9 & 10 Name: ______________________

Complete as many of the following problems as you can with your table in the allotted time. You do not have to go in order.

Classwork 9

1. Find the first and second derivative of the following functions:

   (a) $f(x) = -x^2 + 3$

   (b) $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

   (c) $r(\theta) = \frac{2}{\theta} - \frac{3}{\theta^3} + \frac{1}{\theta^4}$

   Solution

   (a)

$$\begin{aligned} f'(x) &= -2x^{2-1} + 0 \\ &= \boxed{-2x} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{d}{dx}(-2x) \\ &= \boxed{-2} \end{aligned}$$

   (b)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x\right) \\ &= \frac{1}{3} \cdot 3x^{3-1} + \frac{1}{2} \cdot 2x^{2-1} + \frac{1}{4} \\ &= \boxed{x^2 + x + \frac{1}{4}} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(x^2 + x + \frac{1}{4}\right) \\ &= 2x^{2-1} + 1 \\ &= \boxed{2x + 1} \end{aligned}$$

(c)

$$\begin{aligned} r'(\theta) &= \frac{d}{d\theta}(2\theta^{-1} - 3\theta^{-3} + \theta^{-4}) \\ &= 2(-1)\theta^{-1-1} - 3(-3)\theta^{-3-1} + (-4)\theta^{-4-1} \\ &= -2\theta^{-2} + 9\theta^{-4} - 4\theta^{-5} \end{aligned}$$

$$\begin{aligned} r''(\theta) &= \frac{d}{d\theta}(-2\theta^{-2} + 9\theta^{-4} - 4\theta^{-5}) \\ &= -2(-2)\theta^{-3} + 9(-4)\theta^{-5} - 4(-5)\theta^{-6} \\ &= \boxed{4\theta^{-3} - 36\theta^{-5} + 20\theta^{-6}} \end{aligned}$$

$\square$

2. Find the derivatives of the following:

(a) $f(x) = \frac{4x-2}{2x^2}$

(b) $f(x) = (x^2+1)\left(x+5+\frac{1}{x}\right)$

(c) $y = \frac{2x+5}{3x-2}$

(d) $y = \frac{5x+1}{2\sqrt{x}}$

(e) $y = (x-1)(x^2+x+1)$

Solution

(a) Quotient Rule:

$$f'(x) = \frac{(2x^2)\frac{d}{dx}(4x-2) - (4x-2)\frac{d}{dx}(2x^2)}{(2x^2)^2}$$
$$= \frac{2x^2(4) - (4x-2)(4x)}{4x^4}$$
$$= \frac{8x^2 - 16x^2 + 8x}{4x^4}$$
$$= \frac{-8x^2 + 8x}{4x^4}$$
$$= \frac{4x(-2x+2)}{4x^4}$$
$$= \boxed{\frac{-2x+2}{x^3}}$$

Power Rule:

$$f'(x) = \frac{d}{dx}\left(\frac{4x}{2x^2} - \frac{2}{2x^2}\right)$$
$$= \frac{d}{dx}(2x^{-1} - x^{-2})$$
$$= \boxed{-2x^{-2} + 2x^{-3}}$$

(b) Product Rule:

$$\frac{dy}{dx} = \left(\frac{d}{dx}(x^2+1)\right)\left(x+5+\frac{1}{x}\right) + (x^2+1)\left(\frac{d}{dx}\left(x+5+\frac{1}{x}\right)\right)$$
$$= (2x)\left(x+5+\frac{1}{x}\right) + (x^2+1)\left(1-\frac{1}{x^2}\right) \leftarrow \text{acceptable}$$
$$= 2x^2 + 10x + 2 + x^2 - 1 + 1 - \frac{1}{x^2}$$
$$= \boxed{3x^2 + 10x - \frac{1}{x^2} + 2}$$

Distribution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 5x^2 + x + x + 5 + \frac{1}{x}\right)$$
$$= 3x^2 + 10x + 1 + 1 - \frac{1}{x^2}$$
$$= \boxed{3x^2 + 10x - \frac{1}{x^2} + 2}$$

(c)

$$\frac{dy}{dx} = \frac{(3x-2)(d/dx(2x+5)) - (2x+5)(d/dx(3x-2))}{(3x-2)^2}$$
$$= \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2}$$
$$= \frac{\cancel{6x} - 4 - \cancel{6x} - 15}{(3x-2)^2}$$
$$= \boxed{-\frac{19}{(3x-2)^2}}$$

(d) Quotient Rule:

$$\frac{du}{dx} = \frac{2\sqrt{x}(d/dx(5x+1)) - (5x+1)(d/dx(2\sqrt{x}))}{(2\sqrt{x})^2}$$
$$= \frac{2\sqrt{x}(5) - (5x+1)(2(1/2)x^{-1/2})}{4x}$$
$$= \frac{10\sqrt{x} - (5x+1)x^{-1/2}}{4x}$$
$$= \frac{10\sqrt{x} - 5\sqrt{x} - x^{-1/2}}{4x}$$
$$= \frac{5\sqrt{x} - x^{-1/2}}{4x}$$
$$= \frac{5x-1}{4x^{3/2}}$$

Power Rule:

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{5x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}\right)$$
$$= \frac{d}{dx}\left(\frac{5}{2}x^{1/2}\right) + \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right)$$
$$= \frac{5}{2}\cdot\frac{1}{2}x^{-1/2} + \frac{1}{2}\cdot\frac{-1}{2}x^{-3/2}$$
$$= \frac{5}{4}x^{-1/2} - \frac{1}{4}x^{-3/2}$$

(e) Product Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \left(\frac{d}{dx}(x-1)\right)(x^2+x+1) + (x-1)\left(\frac{d}{dx}(x^2+x+1)\right) \\
&= (1)(x^2+x+1) + (x-1)(2x+1) \\
&= x^2+x+1+(x-1)(2x+1) \leftarrow \text{ acceptable} \\
&= x^2+x+1+2x^2+x-2x-1 \\
&= 3x^2
\end{aligned}$$

Distribution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^3+x^2+x-x^2-x-1) \\
&= \frac{d}{dx}(x^3-1) \\
&= 3x^2
\end{aligned}$$

□

3. Let $f(x) = 2x^3 - 15x^2 + 24x$. For what values of x does the line tangent to the graph of f have a slope of 6?

Solution

$$\begin{aligned}
f'(x) = 6 &\Leftrightarrow 2(3x^2) - 15(2x) + 24 = 6 \\
&\Leftrightarrow 6x^2 - 30x + 24 = 6 \\
&\Leftrightarrow 6x^2 - 30x - 18 = 0 \\
&\Leftrightarrow x^2 - 5x + 3 = 0 \\
&\Leftrightarrow x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} \\
&\Leftrightarrow x = \frac{5 \pm \sqrt{25-12}}{2} \\
&\Leftrightarrow \boxed{x = \frac{5 \pm \sqrt{13}}{2}}
\end{aligned}$$

□

Classwork 10

1. Find the derivative of the following:

(a) $y = e^{-x}$

(b) $f(x) = \sqrt[3]{x}e^x$

(c) $y = e^x \cos x$

(d) $y = \sin x - x\cos x$

(e) $y = \sec x \csc x$

(f) $y = \frac{1+\sin x}{1-\sin x}$

(g) $\frac{4xe^x}{x^2+1}$

(h) $3e^x + 10x^3 \ln x$

Solution

(a)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{e^x}\right) \\ &= \frac{e^x(0) - (1)e^x}{(e^x)^2} \\ &= \frac{-e^x}{(e^x)^2} \\ &= \frac{-1}{e^x} \\ &= \boxed{-e^{-x}} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{1/3}e^x) \\ &= \frac{d}{dx}(x^{1/3})e^x + x^{1/3}\frac{d}{dx}(e^x) \\ &= \boxed{\frac{1}{3}x^{-2/3}e^x + x^{1/3}e^x} \end{aligned}$$

(c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(e^x)\cos x + e^x\frac{d}{dx}(\cos x) \\ &= \boxed{e^x\cos x - e^x \sin x} \end{aligned}$$

(d)

$$\begin{aligned} \frac{dy}{dx} &= \cos x - \left(\frac{d}{dx}(x)\cos x + x\frac{d}{dx}(\cos x)\right) \\ &= \cos x - (\cos x - x\sin x) \\ &= \cos x - \cos x + x\sin x \\ &= \boxed{x\sin x} \end{aligned}$$

(e)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sec x)\csc x + \sec x \frac{d}{dx}(\csc x)\\ &= (\sec x \tan x)\csc x + \sec x(-\csc x \cot x)\\ &= \boxed{\sec x \tan x \csc x - \sec x \csc x \cot x}\end{aligned}$$

(f)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1-\sin x)\frac{d}{dx}(1+\sin x) - (1+\sin x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}\\ &= \boxed{\frac{(1-\sin x)(\cos x) - (1+\sin x)(-\cos x)}{(1-\sin x)^2}}\end{aligned}$$

(g)

$$\begin{aligned}\frac{d}{dx}\left(\frac{4xe^x}{x^2+1}\right) &= \frac{(x^2+1)\frac{d}{dx}(4xe^x) - 4xe^x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\\ &= \frac{(x^2+1)\left(\frac{d}{dx}(4x)e^x + 4x\frac{d}{dx}(e^x)\right) - 4xe^x(2x)}{(x^2+1)^2}\\ &= \boxed{\frac{(x^2+1)(4e^x+4xe^x) - 8x^2e^x}{(x^2+1)^2}}\end{aligned}$$

(h)

$$\begin{aligned}\frac{d}{dx}(3e^x + 10x^3\ln x) &= 3e^x + \frac{d}{dx}(10x^3)\ln x + 10x^3\frac{d}{dx}(\ln x)\\ &= \boxed{3e^x + 30x^2\ln x + 10x^3\left(\frac{1}{x}\right)}\end{aligned}$$

□

2. Write an equation of the line tangent to the graph of $f(x) = 2x - \frac{e^x}{2}$ at $\left(0, -\frac{1}{2}\right)$

Solution First we can find the slope:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x - \frac{1}{2}e^x\right) \\ &= 2 - \frac{1}{2}e^x \end{aligned}$$

$$\begin{aligned} m &= f'(0) \\ &= 2 - \frac{1}{2}e^0 \\ &= 2 - \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Using point-slope form we have

$$y - \left(-\frac{1}{2}\right) = \frac{3}{2}(x - 0) \Leftrightarrow \boxed{y + \frac{1}{2} = \frac{3}{2}x}$$

□

3. Find the second derivative of $y = \csc x$

Solution

$$y' = -\csc x \cot x$$

$$\begin{aligned} y'' &= \frac{d}{dx}(-\csc x)\cot x + (-\csc x)\frac{d}{dx}(\cot x) \\ &= -(-\csc x \cot x - \csc x(-\csc^2 x) \\ &= \boxed{\csc x \cot x + \csc^3 x} \end{aligned}$$

□

4. Find the second derivative of $y = 3^x$

Solution

$$\begin{aligned} y' &= 3^x \ln 3 \\ &= \ln 3 3^x \end{aligned}$$

$$\begin{aligned} y'' &= \ln 3(3^x \ln 3) \\ &= \boxed{(\ln 3)^2 3^x} \end{aligned}$$

□