Final Exam Practice Questions

Note: This document is split up based on major mathematical themes covered after the midterm exam. Please use the midterm practice problems to practice the topics covered before the midterm. A topic that is not on this practice exam may still show up on the actual exam if it was covered in class.

Extreme Values and Applied Optimization

(1) Find the absolute extreme values, and the x coordinates where they occur, for the function $f(x) = \cos(x) + x$ on the interval $[0, \pi]$.

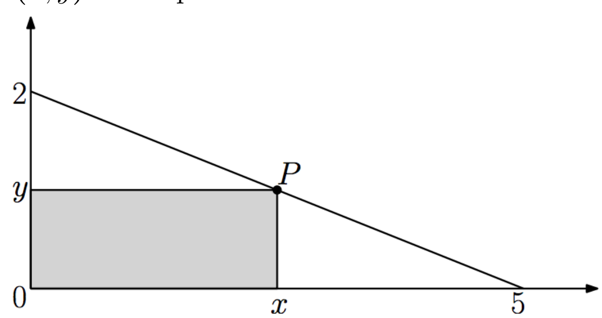
(2) Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$

(3) Find the absolute maximum and minimum values of the following functions of the given intervals.

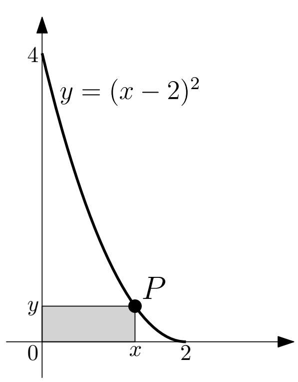
(a) $f(x) = x^2 - 1, \ -1 \le x \le 2$

(b) $f(x) = \sqrt[3]{x}, \ -1 \le x \le 8$

(4) Find the coordinates (x, y) of the point P that maximize the area of the shaded rectangle in the figure below.



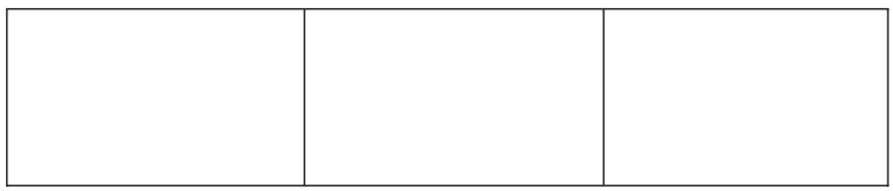
(5) Consider the parabola $y = (x - 2)^2$. Find the coordinates (x, y) of the point P on lying on this parabola between $x = 0$ and $x = 2$ such that the *perimeter* of the rectangle shown below is the smallest.



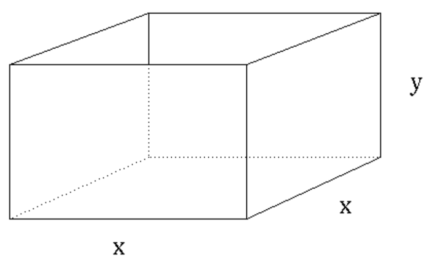
(6) A rectangular plot of land will be bounded on one side by a river and on the other three sides by some sort of fence. With 800 m of fencing at your disposal, what is the largest area you can enclose, and what are its dimensions?

(7) An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with an additional fence across its width. Find the maximum area she can fence in using 1200m of fencing. The area of a rectangle is length times width, the amount of fencing used is the sum of the length of all sides added together. Write your final answer in the form of a complete sentence and use appropriate units.

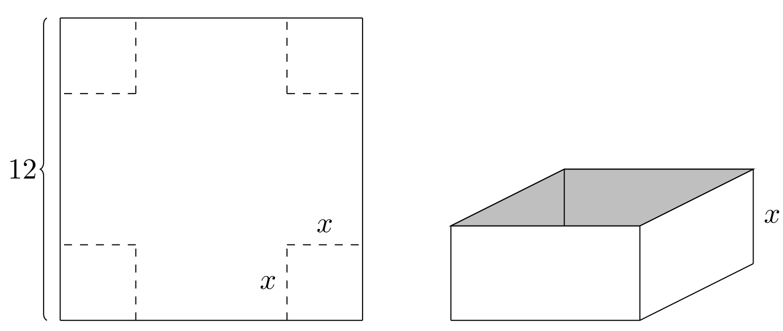
(8) An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with two additional fences across its width, as shown in the sketch. Find the **dimensions** of the pen that has the maximum area she can enclose with 3600 m of fencing.



(9) A box with a square base must have a volume of 8 in^3. What are the dimensions of the box that will minimize the amount of material needed to build it (i.e. minimize surface area).



(10) A box with no top is constructed by cutting equal-sized squares from the corners of a 12 cm by 12 cm sheet of metal and bending up the sides. What is the largest possible volume of such a box? See the pictures below. (Note: The domain of x is $(0, 6)$.)



(11) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radios 10 cm. What is the maximum volume?

(12) You fall asleep in class and dream that you are in charge making huge square-based, open-top, rectangular boxes of steel for literally no reason. Dream-you realizes it is absolutely essential that you build one that is 500 ft^3, and this box must be made by welding steel plates together along their edges. Find the dimensions for the base and height that will make the steal box weigh as little as possible. (Note: though you are dreaming, all mathematical rules and laws of physics apply, because you're really psyched for the upcoming midterm.)

(13) Explain why $g(t) = \sqrt{t} + \sqrt{1+t} - 4$ has exactly one solution in the interval $(0, \infty)$. State any theorems used.

(14) Show that $1 + x = x^3$ has exactly one solution in the interval $[1, 2]$

(15) Show that $x^4 - 4x = 1$ has exactly one solution on $[-1, 0]$. Please state explicitly any theorems and how you are using them.

(16) Show that $f(x) = 2x^3 + 3x^2 + 6x + 1$ has exactly one real root in $[-1, 0]$. Be sure to state and explain any theorems that you use.

First Derivative Test, Second Derivative Test, Graphing

(1) Let $f(x) = x^3 + 3x^2$

(a) Find the (open) intervals where f is increasing and where f is decreasing.

(b) Find all relative extrema (both x and y coordinates). Indicate whether it is a relative maximum or relative minimum.

(c) Find the (open) intervals where f is concave up and where f is concave down

(d) Find all inflection point(s) (both x and y coordinates)

(e) Using the information from parts (a)-(d), graph the function. Label all relative extrema and inflection point(s).

(2) Consider the function $f(x) = x^3 - 6x^2 + 9x$

(a) Find the open intervals where f is increasing and the intervals where f is decreasing.

(b) Find both coordinates of any local extrema of the graph of f.

(c) Find the intervals where f is concave up, and the intervals where f is concave down.

(d) Find the both coordinates of any inflection point(s) of f.

(3) For the following functions, **a)** find the critical points, **b)** classify them as local maxima, local minima, or neither, **c)** find where the function is increasing, **d)** find where the function is concave up, and **e)** sketch the graph.

(a) $y = x^4 - 2x^2$

(b) $y = x^5 - 5x^4$

(4) For each of the following functions, determine the critical points, inflection points, relative and absolute extrema, intervals where the function is increasing, decreasing, concave up and down, vertical and horizontal asymptotes, and sketch the graph.

(a) $f(x) = \dfrac{2+x}{x-1}$

(b) $f(x) = x^4 - 4x^3 + 7$ on $[-1, 4]$

(c) $f(x) = 2 + 2x - 3x^{2/3}$ on $[-1, 2]$

(5) Consider the function $f(x) = \dfrac{1}{3}x^3 - 9x$

(a) Find the intervals where f is increasing and the intervals where f is decreasing.

(b) Find both coordinates of the local max and local min of the graph of f.

(c) Find the intervals where f is concave up, and the intervals where f is concave down.

(d) Find both coordinates of the inflection point of f.

(6) Consider the function $f(x) = x^4 - 4x^3$

(a) Find the open intervals where f is increasing and the intervals where f is decreasing.

(b) Find both coordinates of any local extrema of the graph of f.

(c) Find the intervals where f is concave up, and the intervals where f is concave down.

(d) Find both coordinates of the inflection points of f.

(e) Using the above information, sketch the graph of $y = f(x)$ on the coordinate axes below. You must label both coordinates of any local extrema and inflection points on your graph. (The graph does not need to be to scale.)

(7) Given

$$f(x) = \frac{(x+1)(x+3)}{x^2+3} \qquad f'(x) = \frac{4(3-x^2)}{(x^2+3)^2} \qquad f''(x) = \frac{8x(x^2-9)}{(x^2+3)^3}$$

(a) List all x and y intercepts

(b) Find the intervals of increase and decrease

(c) Find the intervals of concavity and any inflection points.

(d) Find any asymptoptes

(e) Sketch the graph

Riemann Sums

(1) Use mid-points to approximate the area above the x-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

(2) Approximate the integral of $f(x) = x^2$ on the interval $[0, 4]$ by using 4 equal subintervals and evaluating the function at midpoints.

(3) Use mid-points to approximate the area above the x-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

(4) For each of the following functions over the given intervals, calculate the Riemann sum using **a)** left-hand endpoint, **b)** right-hand endpoint, and **c)** midpoint of the subinterval, with four subintervals of equal length.

(a) $f(x) = x^2 - 1,\ [0, 2]$

(b) $f(x) = \sin x,\ [-\pi, \pi]$

(5) Using 4 rectangles of equal length and the following rules find Riemann sums estimates for $f(x) = -x^2 + 16$ from $x = -2$ to $x = 2$ (i.e. to estimate $\int_{-2}^{2}(-x^2 + 16)\ dx$).

(a) Left-hand endpoints

(b) Right-hand endpoints

(c) Midpoints

Linear Approximations, Differentials, L'Hospital's Rule, and Newton's Method

(1) Use linear approximation to estimate the following numbers (you do not need to simplify your answers):

(a) $(.95)^{10}$

(b) $\sqrt{10}$

(c) $\frac{1}{101}$ (using that $1/100 = 0.01$)

(d) $29^{1/3}$

(2) Use linear approximation to find the following approximations.

(a) $\frac{1}{0.9}$ given that $\frac{1}{1} = 1$

(b) $\sqrt[3]{8.5}$ given that $\sqrt[3]{8} = 2$

(c) $\frac{1.3}{1+1.3}$ given that $\frac{1}{1+1} = \frac{1}{2}$.

(3) Approximate $\sqrt{37}$ using linear approximation. (Recall the formula for linear approximation is $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$)

(4) Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

(5) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$

(b) $\lim_{x\to 0} \frac{8x^2}{\cos x - 1}$

(c) $\lim_{x\to\infty} \ln(2x) - \ln(x+1)$

(6) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{x^2 + 8}{6x^2 - x}$

(b) $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$

(c) $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

Position, Velocity, Acceleration Problems

(1) A particle's acceleration is given by $a(t) = 6t + 2$. Its velocity at 1 sec is -1 m/s. Its initial position is given by $s(0) = 5$. Find the position function $s(t)$.

(2) An object moves along the x-axis with velocity $v(t) = 2t - 2$. Its initial position was $x(0) = 5$. Find the position of the object at time $t = 3$.

(3) If a particle's motion is given by the equation $s(t) = 4t^3 - 10t^2 + 5$, find its velocity and acceleration as functions of t. What is its speed at $t = 1$

(4) The acceleration of an object is given by $\frac{3t}{8}$ find the position given that $v(4) = 3$ and $s(4) = 4$.

(5) A ball is thrown from a cliff that is 6 feet from the ground ($s(0) = 6$) with initial velocity 100ft/sec ($v(0) = 100$). If the acceleration due to gravity is -32 ft/sec^2 ($a(t) = -32$), find the equation $s(t)$ for the position of the ball at time t.

Antiderivatives and Integrals

(1) Find the following (definite and indefinite) integrals.

(a) $\int_1^4 \frac{x+4}{\sqrt{x}}\, dx$

(b) $\int e^{5x}\, dx$

(c) $\int x\sqrt{4+x^2}\, dx$

(d) $\int_1^e \left(5x^4 - \frac{1}{x}\right) dx$

(e) $\int_0^3 \sqrt{9-x^2}\, dx$

(2) Find the following (definite and indefinite) integrals.

(a) $\int_1^4 \frac{2-5x^2-3x}{\sqrt{x}} dx$

(b) $\int xe^{x^2+2} dx$

(c) $\int \frac{\sin(\ln x)}{x} dx$

(3) Find the following integrals:

(a) $\int \frac{1+2t^3}{4t}\, dt$

(b) $\int \tan^4 x \sec^2 x\, dx$

(c) $\int_{1/2}^{e/2} \frac{\ln(2x)}{x}\, dx$

(4) Find the most general antiderivative for the following. Check your answer by differentiation.

(a) $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

(b) $\int 2x(1-x^{-3})\, dx$

(5) Evaluate the following integrals

(a) $\int_1^4 \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

(b) $\int_1^2 x^{-3}(x+1)\, dx$

(c) $\int_0^{\pi/3} 2\sec^2 x\, dx$

(6) Calculate the following integrals.

(a) $\int \left(\frac{x^2+7x^5+5}{x^2}\right) dx$

(b) $\int_1^2 (x^2+3x-1)\, dx$

(c) $\int_0^4 \frac{x}{\sqrt{x^2+9}}\, dx$

(7) Find the following integrals:

(a) $\int_0^4 2(\sqrt{t}-t)\, dt$

(b) $\int \frac{1+2t^3}{t^3}\, dt$

(c) $\int_0^\pi 2\sin x \cos^2 x$

(d) $\int \frac{x}{(x^2+2)^3}$

(8) Solve the initial value problem

(a) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $y(4) = 0$

(b) $\frac{ds}{dt} = 12t(3t^2 - 1)^2$, $s(1) = 3$

(9) Solve the initial value problem $\frac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$

(10) Solve the following initial value problems.

(a) $\frac{dr}{d\theta} = -\pi \sin \pi\theta$, $r(0) = 0$

(b) $\frac{d^3y}{dx^3} = 6$; $y''(0) = -8$, $y'(0) = 0$, $y(0) = 5$