**Final Exam Practice Questions**

Note: This document is split up based on major mathematical themes covered after the midterm exam. Please use the midterm practice problems to practice the topics covered before the midterm. A topic that is not on this practice exam may still show up on the actual exam if it was covered in class.

## Extreme Values and Applied Optimization

(1) Find the absolute extreme values, and the $x$ coordinates where they occur, for the function $f(x) = \cos(x) + x$ on the interval $[0, \pi]$.

**Solution**

$$f'(x) = -\sin x + 1$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow -\sin x + 1 = 0 \\ &\Leftrightarrow \sin x = 1 \\ &\Rightarrow x = \frac{\pi}{2} \text{ on } [0, \pi] \end{aligned}$$

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) + \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} f(0) &= \cos(0) + 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(\pi) &= \cos(\pi) + \pi \\ &= -1 + \pi \end{aligned}$$

The absolute max is $(\pi, -1 + \pi)$ and the absolute min is $(0, 1)$

□

(2) Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$

**Solution**

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

Thus there are two critical numbers: 0,1

$$\begin{aligned} f(0) &= 3(0)^4 - 4(0)^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(1) &= 3(1)^4 - 4(1)^3 \\ &= 3 - 4 \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(-1) &= 3(-1)^4 - 4(-1)^3 \\ &= 3 + 4 \\ &= 7 \end{aligned}$$

$$\begin{aligned} f(2) &= 3(2)^4 - 4(2)^3 \\ &= 2^3(6 - 4) \\ &= 8(2) \\ &= 16 \end{aligned}$$

Thus the absolute max is $(2, 16)$ and the absolute min is $(1, -1)$.

□

(3) Find the absolute maximum and minimum values of the following functions of the given intervals.

(a) $f(x) = x^2 - 1, \ -1 \le x \le 2$

(b) $f(x) = \sqrt[3]{x}, \ -1 \le x \le 8$

**Solution**

(a)

$$f'(x) = 2x \text{ so } f'(x) = 0 \Leftrightarrow x = 0$$

$$\begin{aligned} f(-1) &= (-1)^2 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 2^2 - 1 \\ &= \boxed{3 \leftarrow \text{absolute max}} \end{aligned}$$

$$\begin{aligned} f(0) &= 0^2 - 1 \\ &= \boxed{-1 \leftarrow \text{absolute min}} \end{aligned}$$

(b)

$$h'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

This there is a critical point at $x = 0$

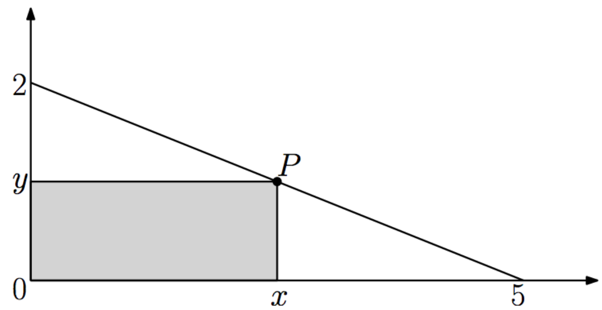
$$h(-1) = \boxed{-1 \leftarrow \text{abs min}}$$

$$h(0) = 0$$

$$\begin{aligned} h(8) &= \sqrt[3]{8} \\ &= \boxed{2 \leftarrow \text{abs max}} \end{aligned}$$

□

(4) Find the coordinates $(x, y)$ of the point P that maximize the area of the shaded rectangle




in the figure below.

**Solution** Since we know the line goes through $(0, 2)$ and $(5, 0)$, we have that the equation of the line is

$$y = 2 - \frac{2}{5}x$$

Thus the area of the rectangle is

$$\begin{aligned} A &= xy \\ &= x\left(2 - \frac{2}{5}x\right) \\ &= 2x - \frac{2x^2}{5} \end{aligned}$$

$$f'(x) = 2 - \frac{4x}{5} = \frac{2(5 - 2x)}{5}$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 5 - 2x = 0 \\ &\Leftrightarrow x = \frac{5}{2} \end{aligned}$$

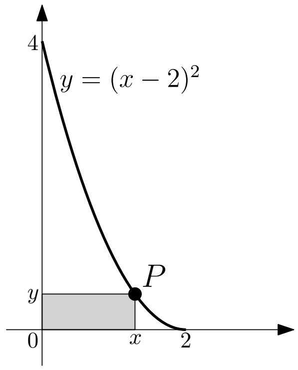
Testing, we get that there is a relative maximum and thus an absolute maximum at $x = \frac{5}{2}$.

$$\begin{aligned} y &= 2 - \frac{2}{5}\left(\frac{5}{2}\right) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

Thus $P = \left(\frac{5}{2}, 1\right)$

□

(5) Consider the parabola $y = (x-2)^2$. Find the coordinates $(x, y)$ of the point $P$ on lying on this parabola between $x = 0$ and $x = 2$ such that the *perimeter* of the rectangle shown below is the smallest.




**Solution**

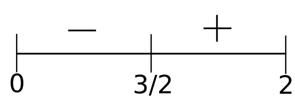
Perimeter $= f(x, y) = 2x + 2y$

Since the point $P$ lies on the parabola $y = (x-2)^2 \Rightarrow$ Perimeter $= f(x) = 2x + 2(x-2)^2$. We need to find the absolute minimum of this function for $x$ in $[0, 2]$

$$f'(x) = 2 + 4(x-2) = 4x - 6$$

$$f'(x) = 0 \Leftrightarrow 4x - 6 = 0 \Leftrightarrow x = \frac{3}{2}$$

Hence, there is only one critical point. Testing the intervals we have



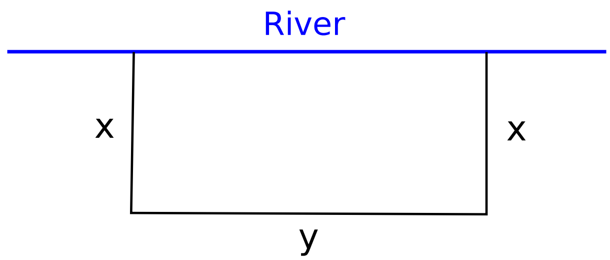

Therefore, there is a relative minimum (and thus an absolute minimum) at $x = \dfrac{3}{2}$. Thus the $x$-coordinate of $P$ is $\frac{3}{2}$ and the $y$-coordinate is:

$$y = \left(\frac{3}{2} - 2\right)^2 = \frac{1}{4}$$

□

(6) A rectangular plot of land will be bounded on one side by a river and on the other three sides by some sort of fence. With 800 m of fencing at your disposal, what is the largest area you can enclose, and what are its dimensions?

**Solution** The picture is




We are given

$$2x + y = 800 \Rightarrow y = 800 - 2x$$

Thus

$$A = xy = x(800 - 2x) = 800x - 2x^2$$

Differentiating with respect to $x$ we have

$$\frac{dA}{dx} = 800 - 4x$$

This gives one critical point of $x = 200$. Testing the intervals we have that there is a relative and thus absolute max at $x = 200 \Rightarrow y = 800 - 400 = 400$. So the dimensions are $\boxed{200 \text{ m by } 400 \text{ m}}$

$$A = 200(400) = \boxed{80000\text{m}^2}$$

□

(7) An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with an additional fence across its width. Find the maximum area she can fence in using 1200m of fencing. The area of a rectangle is length times width, the amount of fencing used is the sum of the length of all sides added together. Write your final answer in the form of a complete sentence and use appropriate units.

**Solution** From the problem we know:

$$1200 = 3x + 2y \Leftrightarrow y = \frac{1200 - 3x}{2}$$

$$\begin{aligned} A &= xy \\ &= x \cdot \frac{1200 - 3x}{2} \\ &= \frac{1200x - 3x^2}{2} \end{aligned}$$

$$\begin{aligned} A'(x) &= \frac{1}{2}(1200 - 6x) \\ &= 600 - 3x \end{aligned}$$

$$\begin{aligned} A'(x) = 0 &\Leftrightarrow 600 - 3x = 0 \\ &\Leftrightarrow 600 = 3x \\ &\Leftrightarrow x = 200 \end{aligned}$$

Testing we have that there is a relative max and thus an absolute max at $x = 200$. Thus the maximum area is:

$$A(200) = \frac{3x(400 - x)}{2} = \frac{3(200)(400 - 200)}{2} = \frac{600(200)}{2} = 60,000m^2$$

□

(8) An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with two additional fences across its width, as shown in the sketch. Find the **dimensions** of the pen that has the maximum area she can enclose with 3600 m of fencing.

**Solution** Labeling the width as $x$ and the length as $y$ the equations we have are

$$3600 = 4x + 2y \text{ and } A = xy$$

Solving the first equation for $y$ gives

$$2y = 3600 - 4x \Rightarrow y = 1800 - 2x$$

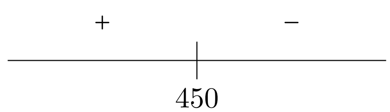
Plugging this into the equation for area gives

$$A(x) = 1800x - 2x^2$$

Taking the derivative we have

$$A'(x) = 1800 - 4x = 4(450 - x)$$

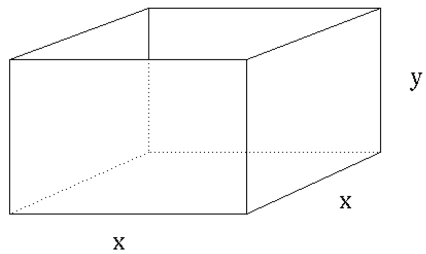
Thus the critical number is $x = 450$. Testing that this is a max we have




Thus $\boxed{\text{the max occurs when } x = 450m \text{ and } y = 1800 - 2(450) = 900m}$

□

(9) A box with a square base must have a volume of 8 in$^3$. What are the dimensions of the box that will minimize the amount of material needed to build it (i.e. minimize surface area).




**Solution**

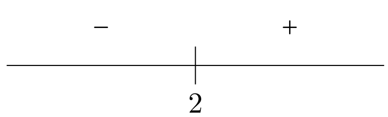
$$x^2 y = 8 \Rightarrow y = \frac{8}{x^2}$$

$$\begin{aligned} A &= 2x^2 + 4xy \\ &= 2x^2 + 4x\left(\frac{8}{x^2}\right) \\ &= 2x^2 + \frac{32}{x} \end{aligned}$$

Taking the derivative and solving for the critical point we have

$$\begin{aligned} A'(x) = 0 &\Leftrightarrow 4x - \frac{32}{x^2} = 0 \\ &\Leftrightarrow \frac{32}{x^2} = 4x \\ &\Leftrightarrow 32 = 4x^3 \\ &\Leftrightarrow x^3 = 8 \\ &\Leftrightarrow x = 2 \end{aligned}$$

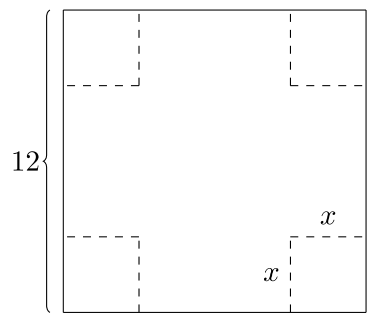
Checking that this is a minimum we have

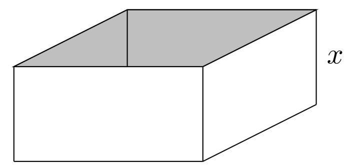



Thus the min occurs when $x = 2$ and $y = \frac{8}{2^2} = 2$. The dimensions are $\boxed{\text{2 in by 2 in by 2in}}$

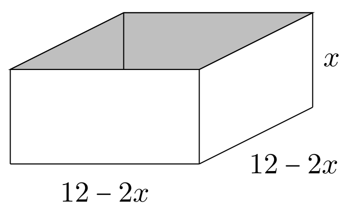
□

(10) A box with no top is constructed by cutting equal-sized squares from the corners of a 12 cm by 12 cm sheet of metal and bending up the sides. What is the largest possible volume of such a box? See the pictures below. (Note: The domain of $x$ is $(0,6)$.)




**Solution** Using the information provided, the picture can be drawn as:



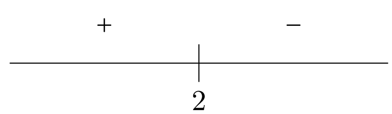

This says that the volume is given by

$$V = (12-2x)(12-2x)(x) = 4x^3 - 48x^2 + 144x$$

Taking the derivative we have

$$V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x-6)(x-2)$$

This gives critical numbers of $x = 6$ and $x = 2$ but since our domain is $(0,6)$, the only critical number we care about is $x = 2$. Testing the intervals we have



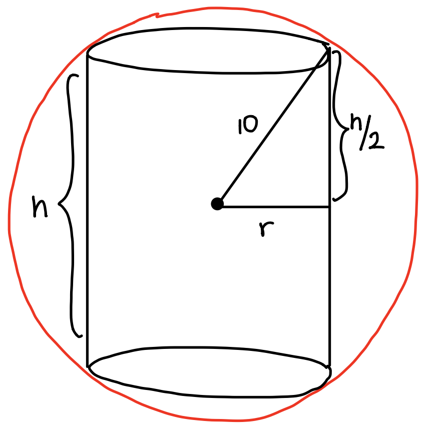

Thus there is a max at $x = 2$ which gives a volume of

$$V = (12-2(2))(12-2(2))(2) = 8\cdot 8\cdot 2 = \boxed{128 \text{ cm}^3}$$

□

(11) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radios 10 cm. What is the maximum volume?

**Solution** The picture looks like




Using the picture we have that

$$\left(\frac{h}{2}\right)^2 + r^2 = 10^2 \Rightarrow r^2 = 100 - \frac{h^2}{4}$$

Plugging this in we have

$$V = \pi\left(100 - \frac{h^2}{4}\right)h = 100\pi h - \frac{\pi h^3}{4}$$

Differentiating with respect to $h$ gives

$$\frac{dV}{dh} = 100\pi - \frac{3\pi h^2}{4} = \frac{400\pi - 3\pi h^2}{4}$$

Solving for the critical point we have

$$400\pi - 3\pi h^2 = 0 \Leftrightarrow 3\pi h^2 = 400\pi \Leftrightarrow h^2 = \frac{400}{3} \Leftrightarrow h = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}}$$

Since the domain for h is $[0, 20]$ we have

$$\begin{aligned}
V(0) &= 0 \\
V(20) &= 0 \\
V\left(\frac{20}{\sqrt{3}}\right) &= 100\pi\left(\frac{20}{\sqrt{3}}\right) - \frac{\pi(20/\sqrt{3})^3}{4} \\
&= \frac{2000\pi}{\sqrt{3}} - \frac{20^3\pi}{4\cdot(\sqrt{3})^3} \\
&= \frac{2000\pi}{\sqrt{3}} - \frac{8000\pi}{4\cdot 3\sqrt{3}} \\
&= \frac{2000\pi}{\sqrt{3}} - \frac{2000\pi}{3\sqrt{3}} \\
&= \frac{6000\pi - 2000\pi}{3\sqrt{3}} \\
&= \frac{4000\pi}{3\sqrt{3}}
\end{aligned}$$

Thus the maximum volume is $\dfrac{4000\pi}{3\sqrt{3}}\text{cm}^3$

□

(12) You fall asleep in class and dream that you are in charge making huge square-based, open-top, rectangular boxes of steel for literally no reason. Dream-you realizes it is absolutely essential that you build one that is 500 $\text{ft}^3$, and this box must be made by welding steel plates together along their edges. Find the dimensions for the base and height that will make the steal box weigh as little as possible. (Note: though you are dreaming, all mathematical rules and laws of physics apply, because you're really psyched for the upcoming midterm.)

**Solution** We have

$$V = 500 = x^2 y \Rightarrow y = \frac{500}{x^2}$$

We want to minimize the material so we want to minimize surface area which is given by

$$S = x^2 + 4xy = x^2 + \frac{2000}{x}$$

Differentiating we have

$$\frac{dS}{dx} = 2x - \frac{2000}{x^2} = \frac{2x^3 - 2000}{x^2}$$

Since $x$ must be positive this gives one critical point of $x = 10$. Testing we have that there is a relative and thus absolute minimum at $x = 10$.

(a)

$$x = 10 \Rightarrow y = \frac{500}{100} = 5$$

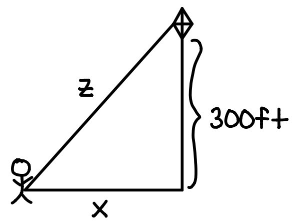
Thus the dimensions are $\boxed{\text{10 ft by 10 ft by 5 ft}}$

(b) By minimizing the amount of material used, we minimize the weight used since the weight depends on the material.

□

(13) A child flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from them at a rate of 25 ft/sec. How fast must they let out the string when the kite is 500 ft away from them?

**Solution** The picture is the following:




Using the pythagorean theorem we have

$$x^2 + 300^2 = z^2 \Rightarrow 2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

When $z = 500$ we have $x^2 + 300^2 = 500^2 \Rightarrow x = 400$. We are also given $\frac{dx}{dt} = 25$ and we want to find $\frac{dz}{dt}$. Plugging in our known information we have

$$2(400)(25) = 2(500)\frac{dz}{dt} \Leftrightarrow \frac{dz}{dt} = \boxed{20 \text{ ft/sec}}$$

□

(14) Explain why $g(t) = \sqrt{t} + \sqrt{1+t} - 4$ has exactly one solution in the interval $(0, \infty)$. State any theorems used.

**Solution**

$$g(1) = 1 + \sqrt{2} - 4 < 0 \text{ and } g(5) = \sqrt{5} + \sqrt{6} - 4 > 0$$

Thus by the Intermediate Value Theorem we have that there is at least on zero on $(0, \infty)$. Assume there are two zeroes. Then by Rolle's Theorem there is a $c$ in $(0, \infty)$ such that $g'(c) = 0$

$$\begin{aligned} g'(t) &= \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{1+t}} \\ &= \frac{\sqrt{1+t} + \sqrt{t}}{\sqrt{t+t^2}} \cdot \frac{\sqrt{1+t} - \sqrt{t}}{\sqrt{1+t} - \sqrt{t}} \\ &= \frac{1}{\sqrt{t+t^2}(\sqrt{1+t} - \sqrt{t})} \end{aligned}$$

$$f'(c) = 0 \Leftrightarrow \frac{1}{\sqrt{c+c^2}(\sqrt{1+c} - \sqrt{c})} = 0 \Leftrightarrow 1 = 0$$

which is impossible thus there cannot be two solutions.

$\square$

(15) Show that $1 + x = x^3$ has exactly one solution in the interval $[1, 2]$

**Solution**

$$1 + x = x^3 \Rightarrow x^3 - x - 1 = 0$$

Let $f(x) = x^3 - x - 1$

$$\begin{aligned} f(1) &= 1 - 1 - 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(2) &= 8 - 2 - 1 \\ &= 5 \end{aligned}$$

Since $f(x)$ is continuous on the interval $[1, 2]$ and since $f(1) < 0$ and $f(2) > 0$ then by the Intermediate Value Theorem, $f(x) = 0$ has at least one solution.

Suppose $f(x) = 0$ has another solution. Since $f(x)$ is continuous and differentiable, by the Mean Value Theorem, $f'(x) = 0$ at some point in $[1, 2]$.

$$\begin{aligned} f'(x) = 0 &\Rightarrow 3x^2 - 1 = 0 \\ &\Rightarrow 3x^2 = 1 \\ &\Rightarrow x^2 = \frac{1}{3} \\ &\Rightarrow x = \pm\sqrt{\frac{1}{3}} \end{aligned}$$

Both of these solutions are outside of $[1, 2]$. Thus by MVT there cannot be two solutions. (i.e. there is exactly one)

$\square$

(16) Show that $x^4 - 4x = 1$ has exactly one solution on $[-1, 0]$. Please state explicitly any theorems and how you are using them.

**Solution** Let $f(x) = x^4 - 4x - 1$. $(x^4 - 4x = 1 \Leftrightarrow f(x) = 0)$

$$\begin{aligned} f(-1) &= (-1)^4 - 4(-1) - 1 \\ &= 1 + 4 - 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^4 - 4(0) - 1 \\ &= -1 \end{aligned}$$

Since $f(x)$ is a continuous function on $[-1.0]$ with $f(-1) > 0$ and $f(0) < 0$, by the intermediate value theorem there is at least one solution to $f(x) = 0$. Assume there is more than one solution. Since $f(x)$ is differentiable on $(-1, 0)$, by the mean value theorem (or by Rolle's theorem), $f'(x) = 0$ in $(-1, 0)$.

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 4x^3 - 4 = 0 \\ &\Leftrightarrow 4x^3 = 4 \\ &\Leftrightarrow x^3 = 1 \\ &\Leftrightarrow x = 1 (\text{not in } (-1, 0)) \end{aligned}$$

Thus there can't be more than one solution. I.e. there is exactly one.

□

(17) Show that $f(x) = 2x^3 + 3x^2 + 6x + 1$ has exactly one real root in $[-1, 0]$. Be sure to state and explain any theorems that you use.

**Solution**

$$\begin{aligned} f(-1) &= 2(-1)^3 + 3(-1)^2 + 6(-1) + 1 \\ &= 2(-1) + 3(1) - 6 + 1 \\ &= -2 + 3 - 6 + 1 \\ &= -4 \end{aligned}$$

$$\begin{aligned} f(0) &= 2(0)^3 + 3(0)^2 + 6(0) + 1 \\ &= 1 \end{aligned}$$

Since $f$ is continuous, by IVT there is at least one solution. If there were more than one then since $f$ is differentiable, by MVT $f'(x) = 0$ at some point in $[-1, 0]$

$$f'(x) = 6x^2 + 6x + 6 = 6(x^2 + x + 1)$$

$$\begin{aligned} x^2 + x + 1 = 0 \Rightarrow x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\ \Rightarrow x &= \frac{-1 \pm \sqrt{-3}}{2} \end{aligned}$$

Which is impossible since you cant square root a negative number. Thus there is exactly one solution.

□

**First Derivative Test, Second Derivative Test, Graphing**

(1) Let $f(x) = x^3 + 3x^2$

(a) Find the (open) intervals where $f$ is increasing and where $f$ is decreasing.

(b) Find all relative extrema (both $x$ and $y$ coordinates). Indicate whether it is a relative maximum or relative minimum.

(c) Find the (open) intervals where $f$ is concave up and where $f$ is concave down

(d) Find all inflection point(s) (both $x$ and $y$ coordinates)

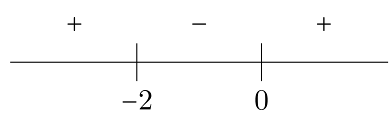
(e) Using the information from parts (a)-(d), graph the function. Label all relative extrema and inflection point(s).

**Solution**

(a)

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

So the critical numbers are $x = 0, -2$. Plotting these and testing the intervals we have




Thus $f$ is $\boxed{\text{increasing on } (-\infty, -2),\ (0, \infty) \text{ and decreasing on } (-2, 0)}$

(b) There is a relative max at $x = -2$ and a relative min at $x = 0$. Plugging these into the original function we have'

$$\begin{aligned} f(-2) &= (-2)^3 + 3(-2)^2 \\ &= -8 + 12 \\ &= 4 \end{aligned}$$

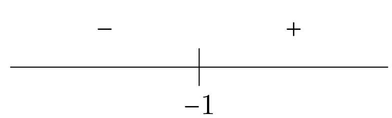
$$\begin{aligned} f(0) &= (0)^3 + 3(0) \\ &= 0 \end{aligned}$$

So the $\boxed{\text{relative max is } (-2, 4) \text{ and the relative min is } (0, 0)}$

(c)

$$f''(x) = 6x + 6 = 6(x+1)$$

So the point we need to plot is $x = -1$. Testing the intervals we have



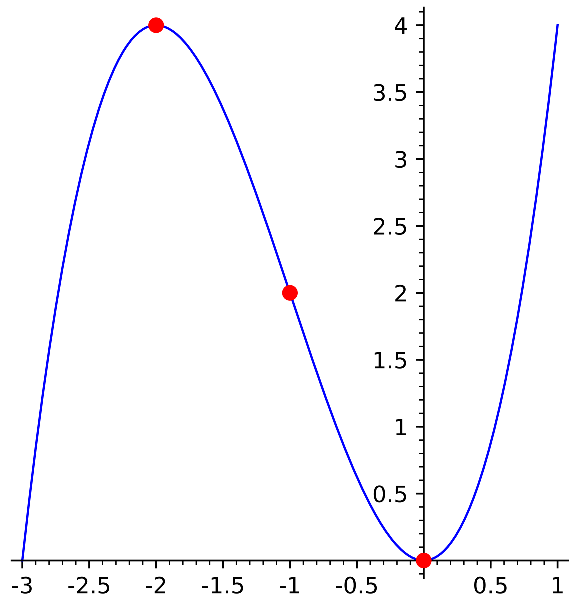

So $f$ is $\boxed{\text{concave down on } (-\infty, -1) \text{ and concave up on } (-1, \infty)}$

(d) There is an inflection point at $x = -1$. Plugging this in gives

$$\begin{aligned} f(-1) &= (-1)^3 + 3(-1)^2 \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

So the inflection point is $\boxed{(-1, 2)}$

(e)




$\square$

(2) Consider the function $f(x) = x^3 - 6x^2 + 9x$

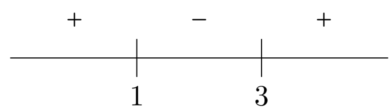
(a) Find the open intervals where $f$ is increasing and the intervals where $f$ is decreasing.
(b) Find both coordinates of any local extrema of the graph of $f$.
(c) Find the intervals where $f$ is concave up, and the intervals where $f$ is concave down.
(d) Find the both coordinates of any inflection point(s) of $f$.

**Solution**

(a)

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 3)(x - 1)$$

This gives critical points of $x = 1, 3$. Testing the intervals we have




Thus $f$ is increasing on $(-\infty, 1)$, $(3, \infty)$ and decreasing on $(1, 3)$

(b) There is a relative max at $x = 1$ and a relative min at $x = 3$. Plugging these into the original function we have

$$\begin{aligned} f(1) &= 1 - 6 + 9 \\ &= 4 \end{aligned}$$

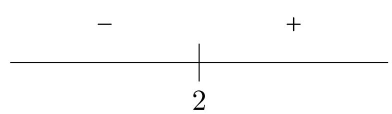
$$\begin{aligned} f(3) &= 27 - 6(9) + 27 \\ &= 27 - 54 + 27 \\ &= 0 \end{aligned}$$

Thus the relative max is $(1, 4)$ and the relative min is $(3, 0)$

(c)

$$f''(x) = 6x - 12 = 6(x - 2)$$

Plotting $x = 2$ and testing we have




Thus $f$ is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$

(d) The inflection point occurs at $x = 2$. Plugging it into the original we have

$$\begin{aligned} f(2) &= 8 - 6(4) + 18 \\ &= 2 \end{aligned}$$

Thus the inflection point is (2,2)

□

(3) For the following functions, **a)** find the critical points, **b)** classify them as local maxima, local minima, or neither, **c)** find where the function is increasing, **d)** find where the function is concave up, and **e)** sketch the graph.

(a) $y = x^4 - 2x^2$

(b) $y = x^5 - 5x^4$

**Solution**

(a) Let $y = f(x)$

(a)

$$f'(x) = 4x^3 - 4x$$

$$f'(x) = 0 \Leftrightarrow 4x(x^2 - 1) = 0 \Leftrightarrow x = 0, \pm 1$$

So the critical points are $x = -1$, $x = 0$, and $x = 1$

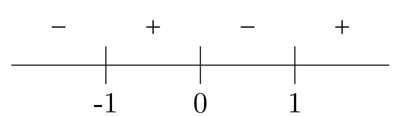
(b)

$$f''(-1) = 8 > 0,\ f''(0) = -4 < 0, \text{ and } f''(1) = 8 > 0$$

$$f(-1) = -1,\ f(0) = 0, \text{ and } f(1) = -1$$

Thus there are relative mins of -1 at $x = \pm 1$ and a relative max of 0 at $x = 0$.

(c) The intervals for $f'$ are given by



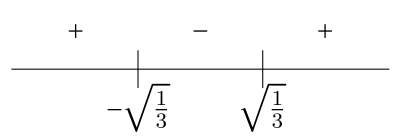

Thus $f$ is increasing on $(-1, 0)$, $(1, \infty)$ and decreasing on $(-\infty, -1)$, $(0, 1)$

(d)

$$f''(x) = 12x^2 - 4$$

$$f''(x) = 0 \Leftrightarrow 4(3x^2 - 1) = 0 \Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$$

Plotting and testing the intervals we have



Thus $f$ is concave up on $(-\infty, -\sqrt{1/3}, (\sqrt{1/3}, \infty)$ and concave down on $(-\sqrt{1/3}, \sqrt{1/3})$.

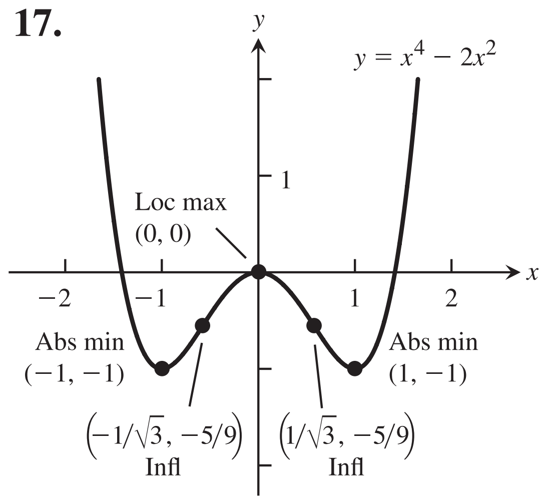
$$f\left(-\sqrt{\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1}{3} - 2\right)$$
$$= \frac{1}{3} \cdot -\frac{5}{3}$$
$$= -\frac{5}{9}$$

$$f\left(\sqrt{\frac{1}{3}}\right) = \frac{1}{3} \cdot -\frac{5}{3}$$
$$= -\frac{5}{9}$$

Therefore there are two inflection points of $\left(-\sqrt{\frac{1}{3}}, -\frac{5}{9}\right)$ and $\left(\sqrt{\frac{1}{3}}, -\frac{5}{9}\right)$

(e) Using the above information we have:

**17.**



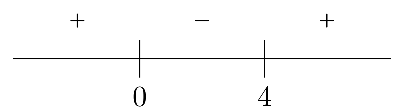
(b) Let $y = f(x) = x^4(x - 5)$

(a)

$$f'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)$$

So the critical points are $x = 0$ and $x = 4$.

(b) Testing the intervals we have




$$f(0) = 0 \text{ and } f(4) = -256$$

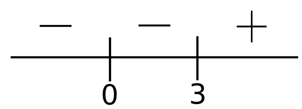
So there is a relative max of 0 at $x = 0$ and a relative min of $-256$ at $x = 4$

(c) From the above number line we have that $f$ is increasing on $(-\infty, 0)$, $(4, \infty)$ and decreasing on $(0, 4)$

(d)

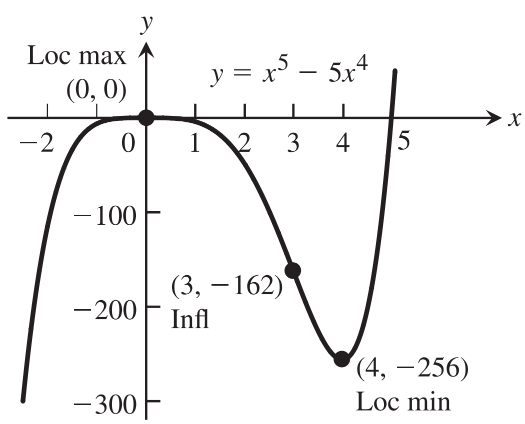
$$f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3)$$

This gives important points of $x = 0$ and $x = 3$. Testing the intervals we have




So $f$ is concave down on $(-\infty, 0)$, $(0, 3)$ and concave up on $(3, \infty)$.

(e) Putting the previous steps together we get




□

(4) For each of the following functions, determine the critical points, inflection points, relative and absolute extrema, intervals where the function is increasing, decreasing, concave up and down, vertical and horizontal asymptotes, and sketch the graph.

(a) $f(x) = \dfrac{2+x}{x-1}$

(b) $f(x) = x^4 - 4x^3 + 7$ on $[-1, 4]$

(c) $f(x) = 2 + 2x - 3x^{2/3}$ on $[-1, 2]$

**Solution**

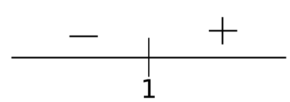
(a)

$$\begin{aligned} f'(x) &= \frac{(x-1)(1) - (2+x)(1)}{(x-1)^2} \\ &= \frac{x-1-2-x}{(x-1)^2} \\ &= -\frac{3}{(x-1)^2} \end{aligned}$$

$f'(x)$ is never zero and is undefined at $x = 1$. Since $f'(x) < 0$ everywhere else we have that $f$ is decreasing on $(-\infty, 1)$, $(1, \infty)$ and there are no extrema.

$$\begin{aligned} f''(x) &= \frac{d}{dx}(-3(x-1)^{-2}) \\ &= 6(x-1)^{-3} \\ &= \frac{6}{(x-1)^3} \end{aligned}$$

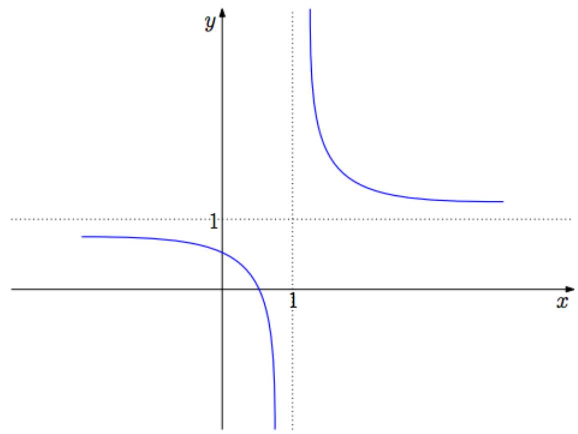
Thus there is one important number at $x = 1$. Testing the intervals we have




So the function is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$. Since the function is undefined at $x = 1$, there are no inflection points.

$$\begin{aligned} \lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{2+x}{x-1} \\ &= \lim_{x\to\pm\infty} \frac{2/x+1}{1-1/x} \\ &= 1 \end{aligned}$$

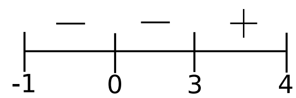
So there is a horizontal asymptote of $y = 1$. Since $f$ is undefined at $x = 1$, that is the vertical asymptote.. Putting this all together we have the following graph:



(b)

$$f'(x) = 4x^2 - 12x^2 = 4x^2(x-3)$$

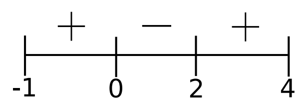
Thus there are critical numbers of $x = 0, 3$. Testing the intervals we have




Thus $f$ is decreasing on $[-1, 0)$, $(0, 3)$ and increasing on $(3, 4]$. Further, there is a relative minimum at $x = 3$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus there is important numbers of $x = 0$ and $x = 2$. Testing the intervals we have




Thus $f$ is concave up on $[-1, 0)$, $(2, 4]$ and concave down on $(0, 2)$. Further, there is an inflection point at $x = 0$ and $x = 2$.

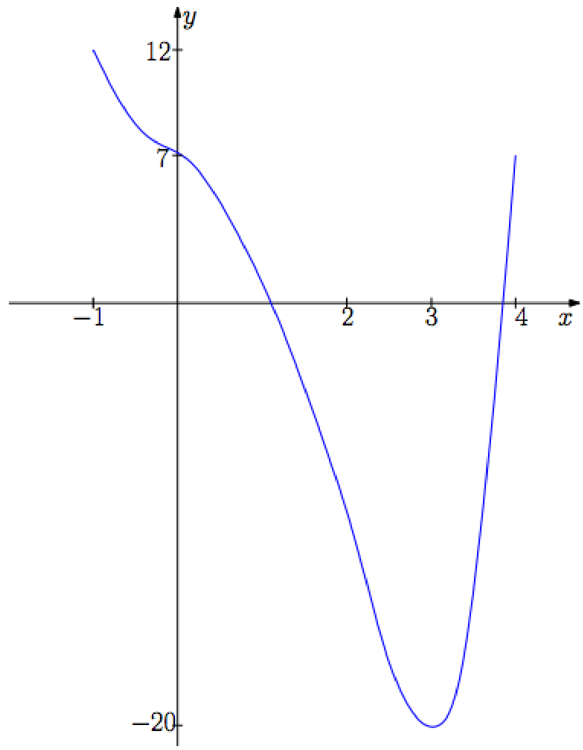
$$\begin{aligned} f(-1) &= (-1)^4 - 4(-1)^3 + 7 \\ &= 1 + 4 + 7 \\ &= 12 \end{aligned}$$

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^3 + 7 \\ &= 7 \end{aligned}$$

$$\begin{aligned} f(3) &= (3)^4 - 4(3)^3 + 7 \\ &= 81 - 108 + 7 \\ &= -20 \end{aligned}$$

$$\begin{aligned} f(4) &= (4)^4 - 4(4)^4 + 7 \\ &= 7 \end{aligned}$$

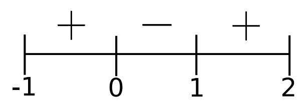
There are no asymptotes. Putting this together we get:



(c)

$$
\begin{aligned}
f'(x) &= 2 - 2x^{-1/3} \\
&= 2 - \frac{2}{x^{1/3}} \\
&= \frac{2x^{1/3} - 2}{x^{1/3}} \\
&= \frac{2(x^{1/3} - 1)}{x^{1/3}}
\end{aligned}
$$

Thus there are two critical numbers: $x = 0, 1$. Testing the intervals we have




Thus $f$ is increasing on $[-1, 0)$, $(1, 2]$ and decreasing on $(0, 1)$. Further, there is a relative maximum at $x = 0$ and a relative minimum at $x = 1$.

$$
f''(x) = \frac{2}{3}x^{-4/3}
$$

Thus there is one important number $x = 0$ and everywhere else $f''(x) < 0$, thus $f$ is concave up on $[-1, 0)$, $(0, 2]$. There are no inflection points.

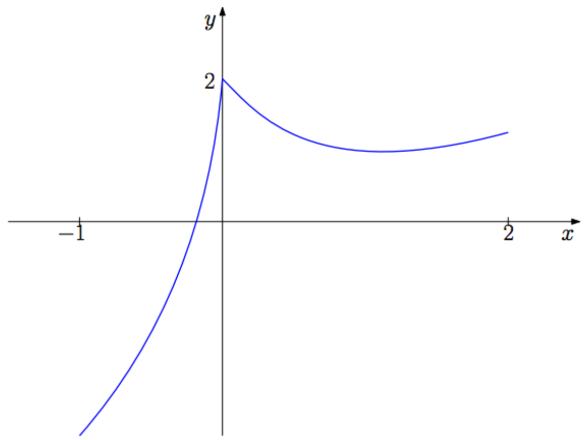
$$
\begin{aligned}
f(0) &= 2 + 2(0) - 3(0)^{2/3} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
f(1) &= 2 + 2(1) - 3(1)^{2/3} \\
&= 2 + 2 - 3 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
f(-1) &= 2 + 2(-1) - 3(-1)^{2/3} \\
&= -3
\end{aligned}
$$

$$
\begin{aligned}
f(2) &= 2 + 2(2) - 3(2)^{2/3} \\
&= 2 + 4 - 3\sqrt[3]{4} \\
&= 6 - 3\sqrt[3]{4}
\end{aligned}
$$

Thus the absolute minimum is $(-1, -3)$ and the absolute maximum is $(0, 2)$. Putting this together we have




□

(5) Consider the function $f(x) = \frac{1}{3}x^3 - 9x$

(a) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.

(b) Find both coordinates of the local max and local min of the graph of $f$.

(c) Find the intervals where $f$ is concave up, and the intervals where $f$ is concave down.

(d) Find both coordinates of the inflection point of $f$.

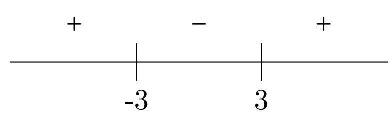
**Solution**

(a)

$$f'(x) = x^2 - 9$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow x^2 - 9 = 0 \\ &\Leftrightarrow (x-3)(x+3) = 0 \\ &\Leftrightarrow x = \pm 3 \end{aligned}$$

Testing the intervals we have




Thus $f$ is increasing on $(-\infty, -3)$, $(3, \infty)$ and decreasing on $(-3, 3)$.

(b)

$$\begin{aligned} f(-3) &= \frac{1}{3}(-3)^3 - 9(-3) \\ &= -\frac{27}{3} + 27 \\ &= -9 + 27 \\ &= 18 \end{aligned}$$

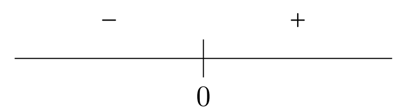
$$\begin{aligned} f(3) &= \frac{1}{3}(3)^3 - 9(3) \\ &= \frac{27}{3} - 27 \\ &= 9 - 27 \\ &= -18 \end{aligned}$$

Thus there is a local max at $(-3, 18)$ and a local min at $(3, -18)$.

(c)

$$f''(x) = 2x$$

Thus there is only one important point: $x = 0$. Testing the intervals we have:




So $f$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

(d)

$$\begin{aligned} f(0) &= \frac{1}{3}(0)^3 - 9(0) \\ &= 0 \end{aligned}$$

So the infection point is $(0, 0)$.

$\square$

(6) Consider the function $f(x) = x^4 - 4x^3$

(a) Find the open intervals where $f$ is increasing and the intervals where $f$ is decreasing.

(b) Find both coordinates of any local extrema of the graph of $f$.

(c) Find the intervals where $f$ is concave up, and the intervals where $f$ is concave down.

(d) Find both coordinates of the inflection points of $f$.

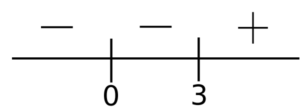
(e) Using the above information, sketch the graph of $y = f(x)$ on the coordinate axes below. You must label both coordinates of any local extrema and inflection points on your graph. (The graph does not need to be to scale.)

**Solution**

(a)

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

Thus there are two critical numbers: 0 and 3. Testing the intervals we have




Thus $f$ is increasing on $(3,\infty)$ and decreasing on $(-\infty,0)$, $(0,3)$.

(b) There is a relative min at $x = 3$

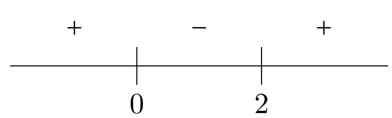
$$\begin{aligned} f(3) &= (3)^4 - 4(3)^3 \\ &= 3^3(3-4) \\ &= 27(-1) \\ &= -27 \end{aligned}$$

So the relative min is $(3,-27)$

(c)

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus there are two important numbers: 0 and 2. Testing the intervals we have




Thus $f$ is concave up on $(-\infty,0)$, $(2,\infty)$ and concave down on $(0,2)$.

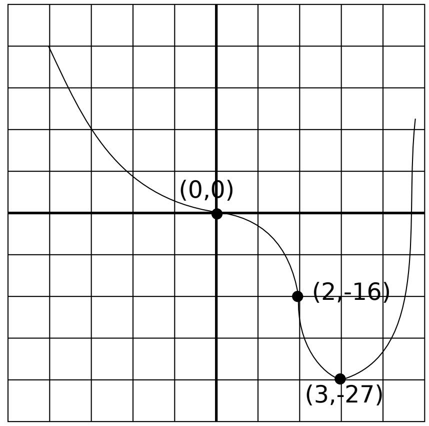
(d) There are inflection points at $x = 0$ and $x = 2$

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^4 - 4(2)^3 \\ &= 2^3(2-4) \\ &= 8(-2) \\ &= -16 \end{aligned}$$

So the inflection points are $(0,0)$ and $(2,-16)$

(e)




□

(7) Given

$$f(x) = \frac{(x+1)(x+3)}{x^2+3} \qquad f'(x) = \frac{4(3-x^2)}{(x^2+3)^2} \qquad f''(x) = \frac{8x(x^2-9)}{(x^2+3)^3}$$

(a) List all $x$ and $y$ intercepts
(b) Find the intervals of increase and decrease
(c) Find the intervals of concavity and any inflection points.
(d) Find any asymptoptes
(e) Sketch the graph

**Solution**

(a)

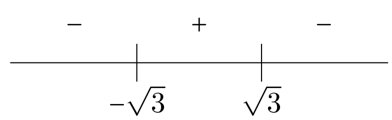
$$f(x) = 0 \Leftrightarrow x = \boxed{-1, -3}$$

$$f(0) = \frac{3}{3} = \boxed{1}$$

(b) $f'(x) = 0$ when $3 - x^2 = 0$

$$3 - x^2 = 0 \Leftrightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

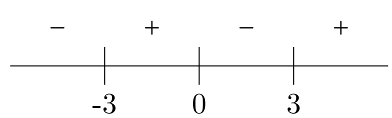
Potting this on a number line gives




$\boxed{f \text{ increases on } (-\sqrt{3}, \sqrt{3}) \text{ and decreases on } (-\infty, -\sqrt{3}), (\sqrt{3}, \infty)}$

(c) $f''(x) = 0$ when $8x(x^2 - 9) = 0$

$$8x(x^2 - 9) = 0 \Rightarrow x = 0, \pm 3$$



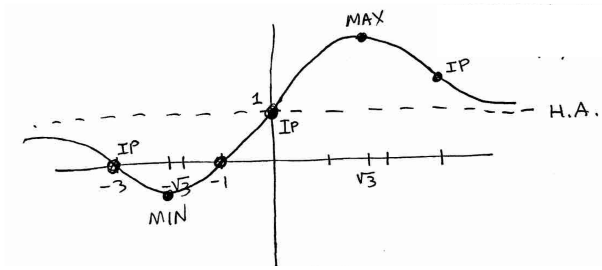

$\boxed{\text{f is concave up on } (-3, 0),\ (3, \infty) \text{ and concave down on } (-\infty, -3),\ (0, 3)}$
$\boxed{\text{There are inflection points at } x = -3, 0, 3}$

(d) $f(x)$ is never undefined so there are no vertical asymptotes.

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{(x+1)(x+3)}{x^2+3} &= \lim_{x\to\pm\infty} \frac{(1 + {}^1/_x)(1 + {}^3/_x)}{1 + {}^3/_{x^2}} \\ &= \frac{(1+0)(1+0)}{1+0} \\ &= 1 \end{aligned}$$

So there is a $\boxed{\text{horizontal asymptote of } y = 1}$

(e)

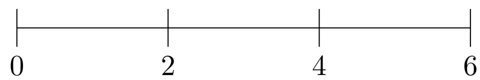



□

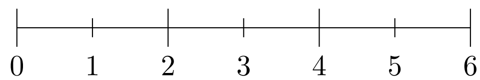
## Riemann Sums

(1) Use mid-points to approximate the area above the $x$-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

**Solution** Splitting the interval into 3 subintervals will give us this




So each rectangle has width 2. Finding the midpoint of each interval will give us




Thus the area is approximately

$$\begin{aligned} M_3 &= 2f(1) + 2f(3) + 2f(5) \\ &= 2(1^2 + 6) + 2(3^3 + 6) + 2(5^2 + 6) \\ &= 2(7) + 2(15) + 2(31) \\ &= 14 + 30 + 62 \\ &= \boxed{106} \end{aligned}$$

□

(2) Approximate the integral of $f(x) = x^2$ on the interval $[0, 4]$ by using 4 equal subintervals and evaluating the function at midpoints.

**Solution**

$$\Delta x = \frac{4-0}{4} = 1$$

Using midpoints we have

$$\begin{aligned} \text{Area} &= f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) \\ &= \frac{1}{4} + \frac{3}{4} + \frac{9}{4} + \frac{25}{4} + \frac{49}{4} \\ &= \frac{84}{4} \\ &= 21 \end{aligned}$$

□

(3) Use mid-points to approximate the area above the $x$-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

**Solution**

$$\Delta x = \frac{6-0}{3} = 2$$

So our midpoints are 1,3,5

$$\begin{aligned} \text{Area} &\approx 2(f(1) + f(3) + f(5)) \\ &= 2(7 + 15 + 31) \\ &= 2(53) \\ &= 106 \end{aligned}$$

□

(4) For each of the following functions over the given intervals, calculate the Riemann sum using **a)** left-hand endpoint, **b)** right-hand endpoint, and **c)** midpoint of the subinterval, with four subintervals of equal length.

(a) $f(x) = x^2 - 1,\ [0, 2]$

(b) $f(x) = \sin x,\ [-\pi, \pi]$

**Solution**

(a)

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$
$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$$

a)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=0}^{3} f(x_i)\Delta x \\
&= \frac{1}{2}\left(f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right) \\
&= \frac{1}{2}\left(\cancel{0} - 1 + \frac{1}{4} - 1 + \cancel{1} - \cancel{1} + \frac{9}{4} - 1\right) \\
&= \frac{1}{2}\left(\frac{-4 + 1 - 4 + 9 - 4}{4}\right) \\
&= \frac{1}{2} \cdot \frac{-1}{2} \\
&= -\frac{1}{4}
\end{aligned}$$

b)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=1}^{4} f(x_i)\Delta x \\
&= \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right) \\
&= \frac{1}{2}\left(\frac{1}{4} - 1 + \cancel{1} - \cancel{1} + \frac{9}{4} - 1 + 4 - 1\right) \\
&= \frac{1}{2}\left(\frac{1 - 4 + 9 - 4 + 16 - 4}{4}\right) \\
&= \frac{1}{2}\left(\frac{14}{4}\right) \\
&= \frac{7}{4}
\end{aligned}$$

c)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x \\
&= \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{16} - 1 + \frac{9}{16} - 1 + \frac{25}{16} - 1 + \frac{49}{16} - 1\right) \\
&= \frac{1}{2}\left(\frac{1 - 16 + 9 - 16 + 25 - 16 + 49 - 16}{16}\right) \\
&= \frac{1}{2}\left(\frac{20}{16}\right) \\
&= \frac{10}{16} \\
&= \frac{5}{8}
\end{aligned}$$

(b)

$$\Delta x = \frac{\pi - (-\pi)}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x_0 = -\pi,\ x_1 = -\frac{\pi}{2},\ x_2 = 0,\ x_3 = \frac{\pi}{2},\ x_4 = \pi$$

(a)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=0}^{3} f(x_i)\Delta x \\
&= \frac{\pi}{2}\left(f(-\pi) + f\left(-\frac{\pi}{2}\right) + f(0) + f\left(\frac{\pi}{2}\right)\right) \\
&= \frac{\pi}{2}(0 - 1 + 0 + 1) \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=1}^{4} f(x_i)\Delta x \\
&= \frac{\pi}{2}\left(f\left(-\frac{\pi}{2}\right) + f(0) + f\left(\frac{\pi}{2}\right) + f(\pi)\right) \\
&= \frac{\pi}{2}(-1 + 0 + 1 + 0) \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\text{Riemann Sum} &= \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x \\
&= \frac{\pi}{2}\left(f\left(-\frac{3\pi}{4}\right) + f\left(-\frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right)\right) \\
&= \frac{\pi}{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \\
&= 0
\end{aligned}$$

□

(5) Using 4 rectangles of equal length and the following rules find Riemann sums estimates for $f(x) = -x^2 + 16$ from $x = -2$ to $x = 2$ (i.e. to estimate $\int_{-2}^{2}(-x^2 + 16)\ dx$).

(a) Left-hand endpoints
(b) Right-hand endpoints
(c) Midpoints

**Solution**

$$\Delta x = \frac{2-(-2)}{4} = 1$$
$$x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2$$

(a)

$$\begin{aligned}
\int_{-2}^{2}(-x^2+16)\ dx &\approx \Delta x(f(-2)+f(-1)+f(0)+f(1)) \\
&= 1\left[(-(-2)^2+16)+(-(-1)^2+16)+(-(0)^2+16)+(-(1)^2+16)\right] \\
&= (-4+16)+(-1+16)+(0+16)+(-1+16) \\
&= 12+15+16+15 \\
&= \boxed{58}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-2}^{2}(-x^2+16)\ dx &\approx \Delta x(f(-2)+f(-1)+f(0)+f(1)) \\
&= 1\left[((-(-1)^2+16)+(-(0)^2+16)+(-(1)^2+16)+(-(2)^2+16)\right] \\
&= (-1+16)+(0+16)+(-1+16)+(-4+16) \\
&= 15+16+15+12 \\
&= \boxed{58}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-2}^{2}(-x^2+16)\ dx &\approx \Delta x\left[f\left(\frac{-2-1}{2}\right)+f\left(\frac{-1+0}{2}\right)+f\left(\frac{0+1}{2}\right)+f\left(\frac{1+2}{2}\right)\right] \\
&= 1\left[f\left(\frac{-3}{2}\right)+f\left(\frac{-1}{2}\right)+f\left(\frac{1}{2}\right)+f\left(\frac{3}{2}\right)\right] \\
&= \left(-\left(\frac{-3}{2}\right)^2+16\right)+\left(-\left(\frac{-1}{2}\right)^2+16\right)+\left(-\left(\frac{1}{2}\right)^2+16\right)+\left(-\left(\frac{3}{2}\right)^2+16\right) \\
&= \left(-\frac{9}{4}+16\right)+\left(-\frac{1}{4}+16\right)+\left(-\frac{1}{4}+16\right)+\left(-\frac{9}{4}+16\right) \\
&= -\frac{20}{4}+64 \\
&= 64-5 \\
&= \boxed{59}
\end{aligned}$$

□

**Linear Approximations, Differentials, L'Hospital's Rule, and Newton's Method**

(1) Use linear approximation to estimate the following numbers (you do not need to simplify your answers):

(a) $(.95)^{10}$

(b) $\sqrt{10}$

(c) $\dfrac{1}{101}$ (using that $1/100 = 0.01$)

(d) $29^{1/3}$

**Solution**

(a) Let $f(x) = x^{10} \Rightarrow f'(x) = 10x^9$, $a = 1$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= (1)^{10} + 10(1)^9(x-1) \\ &= 1 + 10(x-1) \end{aligned}$$

$$\begin{aligned} (.95)^1 &\approx L(.95) \\ &= 1 + 10(.95-1) \\ &= 1 + 10(-.05) \\ &= 1 - 0.5 \\ &= \boxed{0.5} \end{aligned}$$

(b) Let $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$, $a = 9$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) \\ &= 3 + \frac{1}{6}(x-9) \end{aligned}$$

$$\begin{aligned} \sqrt{10} &\approx L(10) \\ &= 3 + \frac{1}{6}(10-9) \\ &= 3 + \frac{1}{6} \\ &= \boxed{\frac{19}{6}} \end{aligned}$$

(c) Let $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$, $a = 100$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \frac{1}{100} - \frac{1}{(100)^2}(x-100) \\ &= 0.01 - 0.0001(x-100) \end{aligned}$$

$$\begin{aligned} \frac{1}{101} &\approx L(101) \\ &= 0.01 - 0.0001(101-100) \\ &= 0.01 - 0.0001 \\ &= 0.0099 \\ &= \boxed{\frac{99}{10000}} \end{aligned}$$

(d) Let $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}}$, $a = 27$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 27^{1/3} + \frac{1}{3(27)^{2/3}}(x-27) \\ &= 3 + \frac{1}{3(9)}(x-27) \\ &= 3 + \frac{1}{27}(x-27) \end{aligned}$$

$$\begin{aligned} 29^{1/3} &\approx L(29) \\ &= 3 + \frac{1}{27}(29-27) \\ &= 3 + \frac{2}{27} \\ &= \boxed{\frac{83}{27}} \end{aligned}$$

□

(2) Use linear approximation to find the following approximations.

(a) $\dfrac{1}{0.9}$ given that $\dfrac{1}{1} = 1$

(b) $\sqrt[3]{8.5}$ given that $\sqrt[3]{8} = 2$

(c) $\dfrac{1.3}{1+1.3}$ given that $\dfrac{1}{1+1} = \dfrac{1}{2}$.

**Solution**

(a) Let $f(x) = \dfrac{1}{x}$ then $f'(x) = -x^{-2}$. Choosing $a = 1$ we have

$$\begin{aligned} f(0.9) &\approx f(1) + f'(1)(0.9-1) \\ &= 1 + (-1)(-0.1) \\ &= 1 + 0.1 \\ &= 1.1 \end{aligned}$$

(b) Let $f(x) = \sqrt[3]{x}$ then $f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$. Choosing $a = 8$ we have

$$\begin{aligned} f(8.5) &\approx f(8) + f'(8)(8.5-8) \\ &= \sqrt[3]{8} + \frac{1}{3(\sqrt[3]{8})^2} \cdot \frac{1}{2} \\ &= 2 + \frac{1}{3(4)} \cdot \frac{1}{2} \\ &= 2 + \frac{1}{24} \\ &= \frac{49}{24} \end{aligned}$$

(c) Let $f(x) = \dfrac{x}{1+x}$ then

$$\begin{aligned} f'(x) &\approx \frac{(1+x)(1) - x(1)}{(1+x)^2} \\ &= \frac{1 + \cancel{x} - \cancel{x}}{(1+x)^2} \\ &= \frac{1}{(1+x)^2} \end{aligned}$$

Choosing $a = 1$ we have

$$\begin{aligned} f(1.3) &\approx f(1) + f'(1)(1.3 - 1) \\ &= \frac{1}{2} + \frac{1}{4}(0.3) \\ &= \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{10} \\ &= \frac{1}{2} + \frac{3}{40} \\ &= \frac{23}{40} \end{aligned}$$

□

(3) Approximate $\sqrt{37}$ using linear approximation. (Recall the formula for linear approximation is $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$)

**Solution** Let $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Choose $x_0 = 36$

$$\begin{aligned} f(x_0) &= \sqrt{36} \\ &= 6 \end{aligned}$$

$$\begin{aligned} f'(x_0) &= \frac{1}{2\sqrt{36}} \\ &= \frac{1}{2(6)} \\ &= \frac{1}{12} \end{aligned}$$

Thus we have

$$\begin{aligned} f(37) &= \sqrt{37} \\ &\approx f(36) + f'(36)(37 - 36) \\ &= 6 + \frac{1}{12} \end{aligned}$$

□

(4) Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find $x_2$.

**Solution**

$$f(x) = x^4 - 2 \Rightarrow f'(x) = 4x^3$$

$$\begin{aligned} x_1 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-1}{4} \\ &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{5}{4} - \frac{f(5/4)}{f'(5/4)} \\ &= \frac{5}{4} - \frac{(625/256) - 2}{(500/64)} \\ &= \frac{5}{4} - \frac{625 - 512}{2000} \\ &= \frac{5}{4} - \frac{113}{2000} \\ &= \boxed{\frac{2387}{2000}} \end{aligned}$$

□

(5) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$

(b) $\lim_{x\to 0} \frac{8x^2}{\cos x - 1}$

(c) $\lim_{x\to\infty} (\ln(2x) - \ln(x+1))$

**Solution**

(a)

$$\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1} \to \frac{\infty}{\infty} \text{ Indeterminate Form}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1} &\overset{L'H}{=} \lim_{x\to\infty} \frac{10x - 3}{14x} \left(\to \frac{\infty}{\infty}\right) \\ &\overset{L'H}{=} \lim_{x\to\infty} \frac{10}{14} \\ &= \frac{10}{14} \\ &= \boxed{\frac{5}{7}}\end{aligned}$$

(b)

$$\lim_{x\to 0} \frac{8x^2}{\cos x - 1} \to \frac{0}{0} \text{ Indeterminate Form}$$

$$\begin{aligned}\lim_{x\to 0} \frac{8x^2}{\cos x - 1} &\overset{L'H}{=} \lim_{x\to 0} \frac{16x}{-\sin x} \left(\to \frac{0}{0}\right) \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{16}{-\cos x} \\ &= \frac{16}{-\cos 0} \\ &= \frac{16}{-1} \\ &= \boxed{-16}\end{aligned}$$

(c)

$$\lim_{x\to\infty}(\ln(2x) - \ln(x+1)) \to \infty - \infty \text{ Indeterminate Form}$$

$$\begin{aligned}
\lim_{x\to\infty}(\ln(2x) - \ln(x+1)) &= \lim_{x\to\infty} \ln \frac{2x}{x+1} \\
&= \ln \lim_{x\to\infty} \frac{2x}{x+1} \\
&\overset{L'H}{=} \ln \lim_{x\to\infty} \frac{2}{1} \\
&= \boxed{\ln 2}
\end{aligned}$$

□

(6) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{x^2+8}{6x^2-x}$

(b) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(c) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

**Solution**

(a)

$$\lim_{x\to\infty} \frac{x^2+8}{6x^2-x} \to \frac{\infty}{\infty} \text{ Indeterminate Form}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2+8}{6x^2-x} &\overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{12x-1} \left(\to \frac{\infty}{\infty}\right) \\ &\overset{L'H}{=} \lim_{x\to\infty} \frac{2}{12} \\ &= \frac{2}{12} \\ &= \boxed{\frac{1}{6}}\end{aligned}$$

(b)

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \to \frac{0}{0} \text{ Indeterminate Form}$$

$$\begin{aligned}\lim_{x\to 1} \frac{x^2-1}{x-1} &\overset{L'H}{=} \lim_{x\to 1} \frac{2x}{1} \\ &= \frac{2}{1} \\ &= \boxed{2}\end{aligned}$$

(c)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \to \frac{0}{0} \text{ Indeterminate Form}$$

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{x^2} &\overset{L'H}{=} \lim_{x\to 0} \frac{\sin x}{2x} \left(\to \frac{0}{0}\right) \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{2} \\ &= \frac{\cos 0}{2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

□

**Position, Velocity, Acceleration Problems**

(1) A particle's acceleration is given by $a(t) = 6t + 2$. Its velocity at 1 sec is $-1$ m/s. Its initial position is given by $s(0) = 5$. Find the position function $s(t)$.

**Solution**

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int (6t+2)\ dt \\ &= 6\left(\frac{t^2}{2}\right) + 2t + C \\ &= 3t^2 + 2t + C \end{aligned}$$

We are given that $v(1) = -1$

$$\begin{aligned} v(1) = -1 &\Leftrightarrow 3 + 2 + C = -1 \\ &\Leftrightarrow 5 + C = -1 \\ &\Leftrightarrow C = -6 \\ &\Rightarrow v(t) = 3t^2 + 2t - 6 \end{aligned}$$

$$\begin{aligned} s(t) &= \int v(t)\ dt \\ &= \int (3t^2 + 2t - 6)\ dt \\ &= 3\left(\frac{t^3}{3}\right) + 2\left(\frac{t^2}{2}\right) - 6t + D \\ &= t^3 + t^2 - 6t + D \end{aligned}$$

We are given that $s(0) = 5$

$$\begin{aligned} s(0) = 5 &\Leftrightarrow 0 + 0 - 0 + D = 5 \\ &\Leftrightarrow D = 5 \\ &\Rightarrow \boxed{s(t) = t^3 + t^2 - 6t + 5} \end{aligned}$$

□

(2) An object moves along the $x$-axis with velocity $v(t) = 2t - 2$. Its initial position was $x(0) = 5$. Find the position of the object at time $t = 3$.

**Solution**

$$\begin{aligned} x(t) &= \int v(t)\ dt \\ &= \int (2t-2)\ dt \\ &= t^2 - 2t + C \end{aligned}$$

$$\begin{aligned} x(0) = 5 &\Leftrightarrow 5 = (0)^2 - 2(0) + C \\ &\Leftrightarrow C = 5 \\ &\Leftrightarrow x(t) = t^2 - 2t + 5 \end{aligned}$$

$$\begin{aligned} x(3) &= (3)^2 - 2(3) + 5 \\ &= 9 - 6 + 5 \\ &= 8 \end{aligned}$$

□

(3) If a particle's motion is given by the equation $s(t) = 4t^3 - 10t^2 + 5$, find its velocity and acceleration as functions of $t$. What is its speed at $t = 1$

**Solution**

$$v(t) = s'(t) = 12t^2 - 20t$$

$$a(t) = v'(t) = 24t - 20$$

$$\text{speed}|_{t=1} = |v(1)| = |12 - 20| = |-8| = 8$$

□

(4) The acceleration of an object is given by $\dfrac{3t}{8}$ find the position given that $v(4) = 3$ and $s(4) = 4$.

**Solution**

$$a(t) = \frac{3}{8}t \Rightarrow v(t) = \frac{3}{8}\left(\frac{t^2}{2}\right) + C = \frac{3}{16}t^2 + C$$

$$v(4) = \frac{3}{16}(16) + C = 3 + C = 3 \Rightarrow C = 0 \Rightarrow v(t) = \frac{3}{16}t^2$$

$$v(t) = \frac{3}{16}t^2 \Rightarrow s(t) = \frac{3}{16}\left(\frac{t^3}{3}\right) + D = \frac{1}{16}t^3 + D$$

$$s(4) = \frac{1}{16}(64) + D = 4 + D \Rightarrow D = 0 \Rightarrow \boxed{s(t) = \frac{1}{16}t^3}$$

□

(5) A ball is thrown from a cliff that is 6 feet from the ground ($s(0) = 6$) with initial velocity 100ft/sec ($v(0) = 100$). If the acceleration due to gravity is $-32$ ft/sec$^2$ ($a(t) = -32$), find the equation $s(t)$ for the position of the ball at time $t$.

**Solution** This is an initial value problem where we have

$$a(t) = -32, \quad v(0) = 100, \quad s(0) = 6$$

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int -32\ dt \\ &= -32t + C \end{aligned}$$

$$\begin{aligned} s(t) &- \int v(t)\ dt \\ &= \int (-32t + C)\ dt \\ &= -32\left(\frac{t^2}{2}\right) + Ct + D \\ &= -16t^2 + Ct + D \end{aligned}$$

Since $v(0) = 100$ we have

$$v(0) = C = 100 \Rightarrow v(t) = -32t + 100 \text{ and } s(t) = -16t^2 + 100t + D$$

And since $s(0) = 6$ we have

$$s(0) = D = 6 \Rightarrow \boxed{s(t) = -16t^2 + 100t + 6}$$

□

## Antiderivatives and Integrals

(1) Find the following (definite and indefinite) integrals.

(a) $\displaystyle\int_1^4 \frac{x+4}{\sqrt{x}}\ dx$

(b) $\displaystyle\int e^{5x}\ dx$

(c) $\displaystyle\int x\sqrt{4+x^2}\ dx$

(d) $\displaystyle\int_1^e \left(5x^4 - \frac{1}{x}\right)\ dx$

(e) $\displaystyle\int_0^3 \sqrt{9-x^2}\ dx$

**Solution**

(a)

$$\begin{aligned}\int_1^4 \frac{x+4}{\sqrt{x}}\ dx &= \int_1^4 (x^{1/2} + 4x^{-1/2})\ dx \\ &= \left(\frac{2}{3}x^{3/2} + 8x^{1/2}\right)\Big|_1^4 \\ &= \frac{16}{3} + 16 - \frac{2}{3} - 8 \\ &= \frac{38}{3}\end{aligned}$$

(b) Let $u = 5x \Rightarrow du = 5dx$

$$\begin{aligned}\int e^{5x}\ dx &= \frac{1}{5}\int e^u\ du \\ &= \frac{1}{5}e^u + C \\ &= \frac{1}{5}e^{5x} + C\end{aligned}$$

(c) Let $u = 4 + x^2 \Rightarrow du = 2xdx$

$$\begin{aligned}\int x\sqrt{4+x^2}\ dx &= \frac{1}{2}\int u^{1/2}\ du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \\ &= \frac{1}{3}(4+x^2)^{3/2} + C\end{aligned}$$

(d)

$$\begin{aligned}\int_1^e \left(5x^4 - \frac{1}{x}\right)\ dx &= \left(x^5 - \ln|x|\right)\Big|_1^e \\ &= (e^5 - 1) - (1 - 0) \\ &= e^5 - 2\end{aligned}$$

(e) Notice that $y = \sqrt{9 - x^2}$ is the graph of a circle of radius 3. Since the integral goes from $x = 0$ to $x = 3$, it is a quarter of the circle. Thus

$$\int_0^3 \sqrt{9 - x^2}\, dx = \frac{1}{4}(9\pi)$$

□

(2) Find the following (definite and indefinite) integrals.

(a) $\int_1^4 \frac{2-5x^2-3x}{\sqrt{x}}dx$

(b) $\int xe^{x^2+2}dx$

(c) $\int \frac{\sin(\ln x)}{x}dx$

**Solution**

(a)

$$\int_1^4 \frac{2-5x^2-3x}{\sqrt{x}}dx = \int_1^4 (2x^{-1/2}-5x^{3/2}-3x^{1/2})dx$$
$$= (4x^{1/2}-2x^{5/2}-2x^{3/2})\Big|_1^4$$
$$= (4\cdot 4^{1/2}-2\cdot 4^{5/2}-2\cdot 4^{3/2})-(4-2-2)$$
$$= 4\cdot 2-2\cdot 2^5-2\cdot 2^3$$
$$= -72$$

(b) Let $u = x^2+2 \Rightarrow du = 2xdx$

$$\int xe^{x^2+2}dx = \frac{1}{2}\int e^u du$$
$$= \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{x^2+2} + C$$

(c) Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\; du$$
$$= -\cos u + C$$
$$= -\cos(\ln x) + C$$

□

(3) Find the following integrals:

(a) $\displaystyle\int \frac{1+2t^3}{4t}\ dt$

(b) $\displaystyle\int \tan^4 x \sec^2 x\ dx$

(c) $\displaystyle\int_{1/2}^{e/2} \frac{\ln(2x)}{x}\ dx$

**Solution**

(a)

$$\begin{aligned}\int \frac{1+2t^3}{4t}\ dt &= \int \left(\frac{1}{4t} + \frac{2t^3}{4t}\right)\ dt \\ &= \frac{1}{4}\int \frac{1}{t}\ dt + \frac{1}{2}\int t^2\ dt \\ &= \frac{1}{4}\ln|t| + \frac{1}{6}t^3 + C\end{aligned}$$

(b) Let $u = \tan x \Rightarrow du = \sec^2 x dx$

$$\begin{aligned}\int \tan^4 x \sec^2 x\ dx & \int u^4\ du \\ &= \frac{u^5}{5} + C \\ &= \frac{\tan^5 x}{5} + C\end{aligned}$$

(c) Let $u = \ln(2x) \Rightarrow du = \dfrac{2}{2x}dx = \dfrac{1}{x}dx$

$$\begin{aligned}\int_{1/2}^{e/2} \frac{\ln(2x)}{x}\ dx &= \int_0^1 \int u\ du \\ &= \left.\frac{u^2}{2}\right|_0^1 \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2}\end{aligned}$$

□

(4) Find the most general antiderivative for the following. Check your answer by differentiation.

(a) $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$ (b) $\int 2x(1 - x^{-3})\ dx$

**Solution**

(a)

$$\begin{aligned}\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right)\ dx &= \int \left(x^{-2} - x^2 - \frac{1}{3}\right)\ dx \\ &= \frac{x^{-2+1}}{-2+1} - \frac{x^{2+1}}{2+1} - \frac{1}{3}\cdot\frac{x^{0+1}}{0+1} + C \\ &= \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{x}{3} + C \\ &= -\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C\end{aligned}$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C\right) = \frac{1}{x^2} - x^2 - \frac{1}{3}$$

(b)

$$\begin{aligned}\int 2x(1 - x^{-3})\ dx &= \int (2x - 2x^{-2})\ dx \\ &= 2\cdot\frac{x^{1+1}}{1+1} - 2\cdot\frac{x^{-2+1}}{-2+1} + C \\ &= 2\cdot\frac{x^2}{2} - 2\cdot\frac{x^{-1}}{-1} + C \\ &= x^2 + \frac{2}{x} + C\end{aligned}$$

Check:

$$\frac{d}{dx}\left(x^2 + \frac{2}{x} + C\right) = 2x - \frac{2}{x^2} = 2x(1 - x^{-3})$$

□

(5) Evaluate the following integrals

(a) $\displaystyle\int_1^4 \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right)\, dx$

(b) $\displaystyle\int_1^2 x^{-3}(x+1)\, dx$

(c) $\displaystyle\int_0^{\pi/3} 2\sec^2 x\, dx$

**Solution**

(a)

$$\begin{aligned}
\int_1^4 \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx &= \int_1^4 \left(\frac{1}{2}x^{1/2} + 2x^{-1/2}\right) dx \\
&= \left[\frac{1}{2}\left(\frac{x^{3/2}}{3/2}\right) + 2\left(\frac{x^{1/2}}{1/2}\right)\right]\Bigg|_1^4 \\
&= \left(\frac{1}{2}\cdot\frac{2}{3}x^{3/2} + 2\cdot(2x^{1/2})\right)\Bigg|_1^4 \\
&= \left(\frac{1}{3}x^{3/2} + 4x^{1/2}\right)\Bigg|_1^4 \\
&= \left(\frac{1}{3}(4)^{3/2} + 4(4)^{1/2}\right) - \left(\frac{1}{3}(1)^{3/2} + 4(1)^{1/2}\right) \\
&= \left(\frac{8}{3} + 8\right) - \left(\frac{1}{3} + 4\right) \\
&= \frac{8}{3} + 8 - \frac{1}{3} - 4 \\
&= \frac{7}{3} + 4 \\
&= \boxed{\frac{19}{3}}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_1^2 x^{-3}(x+1)\ dx &= \int_1^2 (x^{-2}+x^{-3})\ dx \\
&= \left(\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}\right)\Bigg|_1^2 \\
&= \left(-\frac{1}{x}-\frac{1}{2x^2}\right)\Bigg|_1^2 \\
&= \left(-\frac{1}{2}-\frac{1}{8}\right)-\left(-\frac{1}{1}-\frac{1}{2}\right) \\
&= -\frac{1}{2}-\frac{1}{8}+1+\frac{1}{2} \\
&= \boxed{\frac{7}{8}}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^{\pi/3} 2\sec^2 x\ dx &= 2\tan x\big|_0^{\pi/3} \\
&= 2\tan(\pi/3) - 2\tan 0 \\
&= \boxed{2\sqrt{3}}
\end{aligned}$$

□

(6) Calculate the following integrals.

(a) $\int \left( \dfrac{x^2 + 7x^5 + 5}{x^2} \right) dx$

(b) $\int_1^2 (x^2 + 3x - 1)\ dx$

(c) $\int_0^4 \dfrac{x}{\sqrt{x^2+9}}\ dx$

**Solution**

(a)

$$\begin{aligned}\int \left( \frac{x^2 + 7x^5 + 5}{x^2} \right) dx &= \int \left( \frac{x^2}{x^2} + \frac{7x^5}{x^2} + \frac{5}{x^2} \right) dx \\ &= \int (1 + 7x^3 + 5x^{-2})\ dx \\ &= x + 7\left(\frac{x^4}{4}\right) + 5\left(\frac{x^{-1}}{-1}\right) + C \\ &= \boxed{x + \frac{7x^4}{4} - \frac{5}{x} + C}\end{aligned}$$

(b)

$$\begin{aligned}\int_1^2 (x^2 + 3x - 1)\ dx &= \left( \frac{x^3}{3} + 3\left(\frac{x^2}{2}\right) - x \right)\Bigg|_1^2 \\ &= \left( \frac{x^3}{3} + \frac{3x^2}{2} - x \right)\Bigg|_1^2 \\ &= \left( \frac{8}{3} + \frac{12}{2} - 2 \right) - \left( \frac{1}{3} + \frac{3}{2} - 1 \right) \\ &= \frac{8}{3} + 6 - 2 - \frac{1}{3} - \frac{3}{2} + 1 \\ &= \frac{7}{3} + 5 - \frac{3}{2} \\ &= \frac{14}{6} + \frac{30}{6} - \frac{9}{6} \\ &= \boxed{\frac{35}{6}}\end{aligned}$$

(c)

$$u = x^2 + 9 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

Changing the bounds we have

$$x = 0 \Rightarrow u = 0^2 + 9 = 9 \text{ and } u = 4 \Rightarrow u = 4^2 + 9 = 25$$

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int_9^{25} u^{-1/2}\,du$$

$$\cancel{\frac{1}{2}}\left(\frac{u^{1/2}}{\cancel{1/2}}\right)\Bigg|_9^{25}$$

$$= \sqrt{u}\big|_9^{25}$$

$$= \sqrt{25} - \sqrt{9}$$

$$= 5 - 3$$

$$= \boxed{2}$$

□

(7) Find the following integrals:

(a) $\int_0^4 2(\sqrt{t} - t)\ dt$

(b) $\int \frac{1 + 2t^3}{t^3}\ dt$

(c) $\int_0^\pi 2 \sin x \cos^2 x$

(d) $\int \frac{x}{(x^2+2)^3}$

**Solution**

(a)

$$\begin{aligned}
\int_0^4 2(\sqrt{t} - t)\ dt &= \int_0^4 (2\sqrt{t} - 2t)\ dt \\
&= \int_0^4 (2t^{1/2} - 2t)\ dt \\
&= \left[2\left(\frac{t^{3/2}}{3/2}\right) - 2\left(\frac{t^2}{2}\right)\right]_0^4 \\
&= \left[2 \cdot \frac{2}{3} t^{3/2} - t^2\right]_0^4 \\
&= \left(\frac{4}{3} t^{3/2} - t^2\right)\Big|_9^4 \\
&= \left(\frac{4}{3}(4)^{3/2} - (4)^2\right) - \left(\frac{4}{3}(0) - (0)^2\right) \\
&= \frac{4}{3}(2)^3 - 16 \\
&= \frac{4}{3}(8) - 16 \\
&= \frac{32}{3} - \frac{48}{3} \\
&= \boxed{-\frac{16}{3}}
\end{aligned}$$

(b)

$$\int \frac{1+2t^3}{t^3}\ dt = \int \left(\frac{1}{t^3} + \frac{2t^3}{t^3}\right)\ dt$$

$$= \int (t^{-3} + 2)\ dt$$

$$= \boxed{-\frac{t^{-2}}{2} + 2t + C}$$

(c)

$$u = \cos x \Rightarrow du = -\sin x\ dx \Rightarrow dx = \frac{-\sin x}{du}$$

Since this is a definite integral we have to change the bounds.

$$x = 0 \Rightarrow u = \cos(0) = 1 \text{ and } x = \pi \Rightarrow u = \cos(\pi) = -1$$

$$\int_0^{\pi} 2\sin x \cos^2 x = \int_1^{-1} 2\cancel{\sin x} u^2 \cdot \frac{du}{-\cancel{\sin x}}$$

$$= -2\int_1^{-1} u^2\ du$$

$$= -2\left(\frac{u^3}{3}\right)\Bigg|_1^{-1}$$

$$= -2\left(\frac{(-1)^3}{3} - \frac{(1)^3}{3}\right)$$

$$= -2\left(-\frac{1}{3} - \frac{1}{3}\right)$$

$$= -2\left(\frac{-2}{3}\right)$$

$$= \boxed{\frac{4}{3}}$$

(d)

$$u = x^2 + 2 \Rightarrow du = 2x\ dx \Rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned}\int \frac{x}{(x^2+2)^3} &= \int \frac{\cancel{x}}{u^3} \cdot \frac{du}{2\cancel{x}} \\ &= \frac{1}{2}\int u^{-3}\ du \\ &= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C \\ &= -\frac{1}{4}u^{-2} + C \\ &= \boxed{-\frac{1}{4}(x^2+2)^{-2} + C}\end{aligned}$$

□

(8) Solve the initial value problem

(a) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, $y(4) = 0$

(b) $\dfrac{ds}{dt} = 12t(3t^2 - 1)^2$, $s(1) = 3$

**Solution**

(a)

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

$$y = \frac{1}{2} \cdot \frac{x^{1/2}}{1/2} + C = x^{1/2} + C$$

$$y(4) = 2 + C = 0 \Rightarrow C = -2 \Rightarrow \boxed{y = x^{1/2} - 2}$$

(b)

$$\frac{ds}{dt} = 12t(9t^4 - 6t^2 + 1) = 108t^5 - 72t^3 + 12t$$

$$s = 108\left(\frac{t^6}{6}\right) - 72\left(\frac{t^4}{4}\right) + 12\left(\frac{t^2}{2}\right) + C = 18t^6 - 18t^4 + 6t^2 + C$$

$$s(1) = 18 - 18 + 6 + C = 3 \Rightarrow C = -3 \Rightarrow \boxed{s = 18t^6 - 18t^4 + 6t^2 - 3}$$

□

(9) Solve the initial value problem $\dfrac{dy}{dx} = 9x^2 - 4x + 5,\ y(-1) = 0$

**Solution**

$$\begin{aligned} y &= \int (9x^2 - 4x + 5)\ dx \\ &= 9\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5x + C \\ &= 3x^3 - 2x^2 + 5x + C \end{aligned}$$

Using the initial value we were given we have

$$\begin{aligned} y(-1) = 0 &\Leftrightarrow 3(-1)^3 - 2(-1)^2 + 5(-1) + C = 0 \\ &\Leftrightarrow -3 - 2 - 5 + C = 0 \\ &\Leftrightarrow -10 + C = 0 \\ &\Leftrightarrow C = 10 \\ &\Leftrightarrow y = \boxed{3x^3 - 2x^2 + 5x + 10} \end{aligned}$$

□

(10) Solve the following initial value problems.

(a) $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \; r(0) = 0$

(b) $\dfrac{d^3y}{dx^3} = 6; \; y''(0) = -8, \; y'(0) = 0, \; y(0) = 5$

**Solution**

(a)

$$r = \int (-\pi \sin(\pi\theta)) \; d\theta = \cos(\pi\theta) + C$$

$$0 = \cos(0) + C \Leftrightarrow C = -1 \Rightarrow \boxed{r = \cos(\pi\theta) - 1}$$

(b)

$$y''(x) = \int 6 \; dx = 6x + C$$

$$-8 = 0 + C \Leftrightarrow C = -8 \Rightarrow y''(x) = 6x - 8$$

$$y'(x) = \int (6x - 8) \; dx = 6 \cdot \frac{x^2}{2} - 8 \cdot \frac{x^{0+1}}{0+1} = 3x^2 - 8x + C$$

$$0 = 0 - 0 + C \Leftrightarrow C = 0 \Rightarrow y'(x) = 3x^2 - 8x$$

$$y = \int (3x^2 - 8x) \; dx = 3 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + C = x^3 - 4x^2 + C$$

$$5 = 0 - 0 + C \Leftrightarrow C = 5 \Rightarrow \boxed{y = x^3 - 4x^2 + 5}$$

□