Midterm 1 Practice Questions

Note: This document is split up based on major mathematical themes covered so far. A topic that is not on this practice exam may still show up on the actual exam if it was covered in class.

General Limits

(1) Evaluate $\lim_{x \to -1} \frac{x^2 - 5x - 6}{x + 1}$

Solution

$$\begin{aligned}
\lim_{x \to -1} \frac{x^2 - 5x - 6}{x + 1} &= \lim_{x \to -1} \frac{(x - 6)\cancel{(x + 1)}}{\cancel{x + 1}} \\
&= \lim_{x \to -1} (x - 6) \\
&= -1 - 6 \\
&= \boxed{-7}
\end{aligned}$$

□

(2) $\lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5}$

Solution

$$\begin{aligned}
\lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5} &= \lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5} \cdot \frac{\sqrt{x} + 5}{\sqrt{x} + 5} \\
&= \lim_{x \to 25} \frac{\cancel{(x - 25)}(\sqrt{x} + 5)}{\cancel{x - 25}} \\
&= \lim_{x \to 25} (\sqrt{x} + 5) \\
&= \sqrt{25} + 5 \\
&= \boxed{10}
\end{aligned}$$

□

(3) Evaluate $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$

Solution

$$\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} \to \frac{0}{0}$$

$$\begin{aligned}
\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} &= \lim_{t\to 1} \frac{(t+2)\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} \\
&= \lim_{t\to 1} \frac{(t+2)}{(t+1)} \\
&= \frac{1+2}{1+1} \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

□

(4) Evaluate $\lim_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$

Solution

$$\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} \to \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} &= \lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} \cdot \frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4} \\
&= \lim_{x\to 2} \frac{(x^2+12)-4^2}{(x-2)(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(\sqrt{x^2+12}+4)} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+12}+4} \\
&= \frac{2+2}{\sqrt{2^2+12}+4} \\
&= \frac{4}{\sqrt{16}+4} \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

□

(5) Evaluate $\lim_{x \to 0} \frac{1}{x^3 - 1} + 1$

Solution

$$\begin{aligned}\lim_{x \to 0} \frac{1}{x^3 - 1} + 1 &= \frac{1}{0 - 1} + 1 \\ &= -1 + 1 \\ &= \boxed{0}\end{aligned}$$

□

(6) Evaluate $\lim_{x \to 1} \frac{3x - 4}{x^2 + x + 1}$

Solution

$$\begin{aligned}\lim_{x \to 1} \frac{3x - 4}{x^2 + x + 1} &= \frac{3(1) - 4}{(1)^2 + 1 + 1} \\ &= \boxed{-\frac{1}{3}}\end{aligned}$$

□

(7) Evaluate $\lim_{x \to -1^-} \frac{|x + 1|}{x + 1}$

Solution

$$\begin{aligned}\lim_{x \to -1^-} \frac{|x + 1|}{x + 1} &= \lim_{x \to -1^-} \frac{-\cancel{(x + 1)}}{\cancel{x + 1}} \\ &= \lim_{x \to -1^-} -1 \\ &= \boxed{-1}\end{aligned}$$

□

(8) Evaluate $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Solution

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}} \\ &= \lim_{x \to 1} (x + 1) \\ &= 1 + 1 \\ &= \boxed{2}\end{aligned}$$

□

(9) Evaluate $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$

Solution

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \Rightarrow -x \le x\cos\left(\frac{1}{x}\right) \le x$$

$$\lim_{x \to 0}(-x) = 0 \text{ and } \lim_{x \to 0}(x) = 0$$

Thus by squeeze theorem $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = \boxed{0}$

□

(10) Evaluate $\lim_{x \to 3^-} \frac{|x-3|}{(6-2x)}$

Solution

$$\begin{aligned}\lim_{x \to 3^-} \frac{|x-3|}{(6-2x)} &= \lim_{x \to 3^-} \frac{-\cancel{(x-3)}}{-2\cancel{(x-3)}} \\ &= \lim_{x \to 3^-} \frac{-1}{-2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

□

(11) Evaluate $\lim_{h \to 1} \frac{\sqrt{h+8}-3}{1-h}$

Solution

$$\begin{aligned}\lim_{h \to 1} \frac{\sqrt{h+8}-3}{1-h} &= \lim_{h \to 1} \frac{\sqrt{h+8}-3}{1-h} \cdot \frac{\sqrt{h+8}+3}{\sqrt{h+8}+3} \\ &= \lim_{h \to 1} \frac{(h+8)-9}{(1-h)\sqrt{h+8}+3} \\ &= \lim_{h \to 1} \frac{\cancel{h-1}}{-\cancel{(h-1)}(\sqrt{h+8}+3)} \\ &= \lim_{h \to 1} \frac{1}{-(\sqrt{h+8}+3)} \\ &= -\frac{1}{\sqrt{9}+3} \\ &= \boxed{-\frac{1}{6}}\end{aligned}$$

□

(12) Evaluate $\lim_{x\to 0} x^2\cos\left(\frac{3}{x}\right) - 11$

Solution

$$-1 \le \cos\left(\frac{3}{x}\right) \le 1 \Rightarrow -x^2 \le x^2\cos\left(\frac{3}{x}\right) \le x^2$$
$$\Rightarrow -x^2 - 11 \le x^2\cos\left(\frac{3}{x}\right) - 11 \le x^2 - 11$$

$$\lim_{x\to 0}(-x^2 - 11) = -11 \text{ and } \lim_{x\to 0}(x^2 - 11) = -11$$

Thus by the squeeze theorem $\lim_{x\to 0} x^2\cos\left(\frac{3}{x}\right) - 11 = \boxed{-11}$

$\square$

(13) $\lim_{x\to 0^-} \frac{x^2}{2} - \frac{1}{x}$

Solution

$$\lim_{x\to 0^-} \frac{x^2}{2} - \frac{1}{x} \to \lim_{x\to 0^-} \frac{0^2}{2} - \frac{1}{0^-}$$
$$\to 0 - (-\infty)$$
$$\boxed{\infty}$$

$\square$

(14) $\lim_{x\to 3} 4x^3$

Solution

$$\lim_{x\to 3} 4x^3 = 4(3)^3$$
$$= 4(27)$$
$$= \boxed{108}$$

$\square$

(15) $\lim_{x\to -3} \sqrt{x-5}$

Solution

$$\lim_{x\to -3} \sqrt{x-5} = \sqrt{-3-5}$$
$$= \sqrt{-8}$$
$$\Rightarrow \boxed{\text{DNE}}$$

$\square$

(16) $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$

Solution

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x \leq x\cos\left(\frac{1}{x}\right) \leq x$$

$$\begin{aligned}\lim_{x\to 0} -x &= -0\\ &= 0\end{aligned}$$

$$\lim_{x\to 0} x = 0$$

Thus by the Squeeze Theorem, $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = \boxed{0}$

□

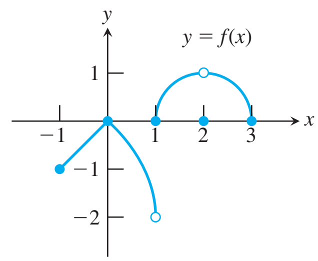
(17) $\lim_{x\to 2^+} \frac{8x-16}{|x-2|}$

Solution Coming from the right of 2 means that $x - 2 > 0$, thus $|x-2| = x - 2$. This give us:

$$\begin{aligned}\lim_{x\to 2^+} \frac{8x-16}{|x-2|} &= \lim_{x\to 2^+} \frac{8\cancel{(x-2)}}{\cancel{x-2}}\\ &= \lim_{x\to 2^+} 8\\ &= \boxed{8}\end{aligned}$$

□

(18) Find the following limits for



(a) $\lim_{x \to -1^+} f(x)$

(b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 2} f(x)$

Solution

(a) $\boxed{0}$

(b) $\boxed{-2}$

(c) $\boxed{1}$

□

(19) Evaluate the following limits for $f(x)$

$$f(x) = \begin{cases} x^2 - 3x + 4 & x \leq 1 \\ x + 1 & 1 < x \leq 3 \\ x^2 - 3x + 4 & x > 3 \end{cases}$$

(a) $\lim_{x \to 1^-} f(x)$

(b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 3^-} f(x)$

(e) $\lim_{x \to 3^+} f(x)$

(f) $\lim_{x \to 3} f(x)$

Solution

a)

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (x^2 - 3x + 4) \\ &= 1^2 - 3(1) + 4 = 1 - 3 + 4 \\ &= 2 \end{aligned}$$

b)

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (x + 1) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

c)

$$\lim_{x \to 1} f(x) = 2$$

d)

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} (x + 1) \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

e)

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3+} (x^2 - 3x + 4) \\ &= 3^2 - 3(3) + 4 \\ &= 9 - 9 + 4 \\ &= 4 \end{aligned}$$

f)

$$\lim_{x \to 3} f(x) = 4$$

□

Infinite Limits & Limits at Infinity

(1) Evaluate $\lim\limits_{x \to \infty} \dfrac{2x^3 - 4x^2 + 5x}{17x^2 + 1}$

Solution

$$\begin{aligned}
\lim_{x \to \infty} \frac{2x^3 - 4x^2 + 5x}{17x^2 + 1} &= \lim_{x \to \infty} \frac{\frac{2x^3}{x^2} - \frac{4x^2}{x^2} + \frac{5x}{x^2}}{\frac{17x^2}{x^2} + \frac{1}{x^2}} \\
&= \lim_{x \to \infty} \frac{2x - 4 + \frac{5}{x}}{17 + \frac{1}{x^2}} \\
&\to \frac{2(\infty) - 4 + 0}{17 + 0} \\
&\to \boxed{\infty}
\end{aligned}$$

$\square$

(2) Evaluate $\lim\limits_{x \to -1} \dfrac{x^2 + x - 2}{x^3 + 1}$

Solution

$$\begin{aligned}
\lim_{x \to -1} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{(-1)^2 - 1 - 2}{(-1)^3 + 1} \\
&\to \frac{1 - 1 - 2}{-1 + 1} \\
&\to -\frac{2}{0}
\end{aligned}$$

So we have to check the one-sided limits.

$$\begin{aligned}
\lim_{x \to -1^-} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{-2}{0^-} \\
&= \to \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to -1^+} \frac{x^2 + x - 2}{x^3 + 1} &\to \frac{-2}{0^+} \\
&\to -\infty
\end{aligned}$$

Thus the limit $\boxed{\text{DNE}}$

$\square$

(3) $\lim_{x\to\infty} \frac{x^3+x^2+2}{7x^3+x+1}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+x^2+2}{7x^3+x+1} &= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{7x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}} \\ &= \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{2}{x^3}}{7+\frac{1}{x^2}+\frac{1}{x^3}} \\ &= \frac{1+0+0}{7+0+0} \\ &= \boxed{\frac{1}{7}}\end{aligned}$$

□

(4) $\lim_{x\to 3^-} \frac{4}{(x-3)^2}$

Solution

$$\begin{aligned}\lim_{x\to 3^-} \frac{4}{(x-3)^2} &\to \frac{4}{(0^-)^2} \\ &\to \frac{4}{0^+} \\ &\to \boxed{\infty}\end{aligned}$$

□

(5) $\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} &= \lim_{x\to\infty} \frac{\frac{7x^3}{x^3}}{\frac{x^3}{x^3}-\frac{3x^2}{x^3}+\frac{6x}{x^3}} \\ &= \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \\ &= \frac{7}{1-0+0} \\ &= \boxed{7}\end{aligned}$$

□

(6) $\lim_{x\to-\infty} \dfrac{3x^2+4}{x+7}$

Solution

$$\begin{aligned}\lim_{x\to-\infty} \frac{3x^2+4}{x+7} &= \lim_{x\to-\infty} \frac{\frac{3x^2}{x}+\frac{4}{x}}{\frac{x}{x}+\frac{7}{x}} \\ &= \lim_{x\to-\infty} \frac{3x+\frac{4}{x}}{1+\frac{7}{x}} \\ &\to \frac{3(-\infty)+0}{1+0} \\ &\to \boxed{-\infty}\end{aligned}$$

□

(7) $\lim_{t\to2^-} \dfrac{t+2}{t-2}$

Solution

$$\begin{aligned}\lim_{t\to2^-} \frac{t+2}{t-2} &\to \frac{2+2}{0^-} \\ &\to \frac{4}{0^-} \\ &\to \boxed{-\infty}\end{aligned}$$

□

(8) $\lim_{x\to\infty} \dfrac{2x^2-11x+5}{3x-7}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2-11x+5}{3x-7} &= \lim_{x\to\infty} \frac{2x^2/x-11x/x+5/x}{3x/x-7/x} \\ &= \lim_{x\to\infty} \frac{2x-11+5/x}{3-7/x} \\ &= \infty\end{aligned}$$

□

(9) $\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

Solution

$$\begin{aligned}
\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} &= \lim_{x\to\infty} \frac{2x^{1/2} + x^{-1}}{3x - 7} \\
&= \lim_{x\to\infty} \frac{\frac{2x^{1/2}}{x} + \frac{x^{-1}}{x}}{\frac{3x}{x} - \frac{7}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{2}{x^{1/2}} + \frac{1}{x^2}}{3 - \frac{7}{x}} \\
&= \frac{0+0}{3-0} \\
&= \boxed{0}
\end{aligned}$$

□

(10) $\lim_{x\to\infty} \frac{\sqrt{x} + 3x^{-2}}{4 - 2x}$

Solution

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x} + 3x^{-2}}{4 - 2x} &= \lim_{x\to\infty} \frac{x^{1/2} + 3x^{-2}}{4 - 2x} \\
&= \lim_{x\to\infty} \frac{\frac{x^{1/2}}{x} + \frac{3x^{-2}}{x}}{\frac{4}{x} - \frac{2x}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^{1/2}} + \frac{3}{x^3}}{\frac{4}{x} - 2} \\
&= \frac{0+0}{0-2} \\
&= \boxed{0}
\end{aligned}$$

□

(11) $\lim_{x\to\infty} \frac{\sin x}{x}$

Solution

$$-1 \le \sin x \le 1 \Rightarrow \frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \text{ for } x > 0$$

$$\lim_{x\to\infty} \frac{-1}{x} = 0$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Thus by the Squeeze Theorem, $\lim_{x\to\infty} \frac{\sin x}{x} = \boxed{0}$

□

(12) $\lim_{t\to 2^-} \frac{t+2}{t-2}$

Solution

$$\begin{aligned}\lim_{t\to 2^-} \frac{t+2}{t-2} &\to \frac{2+2}{2^- - 2} \\ &\to \frac{4}{0^-} \\ &\to \boxed{-\infty}\end{aligned}$$

□

(13) $\lim_{x\to\infty} \frac{2x^3 - 4x^2 + 5x}{17x^2 + 1}$

Solution

$$\lim_{x\to\infty} \frac{2x^3 - 4x^2 + 5x}{17x^2 + 1} = \lim_{x\to\infty} \frac{\frac{2x^3}{x^2} - \frac{4x^2}{x^2} + \frac{5x}{x^2}}{\frac{17x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2x - 4 + \frac{5}{x}}{17 + \frac{1}{x^2}}$$

$$\to \boxed{\infty}$$

□

(14) $\lim_{x\to-\infty} \frac{3x^2 + 4}{x + 7}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2 + 4}{x + 7} = \lim_{x\to-\infty} \frac{\frac{3x^2}{x} + \frac{4}{x}}{\frac{x}{x} + \frac{7}{x}}$$

$$= \lim_{x\to-\infty} \frac{3x + \frac{4}{x}}{1 + \frac{7}{x}}$$

$$\to 3(-\infty)$$

$$\to \boxed{-\infty}$$

□

Continuity

(1) Describe on which interval(s) the following function is continuous: $y = \frac{\sin x}{x-2}$

Solution

$$x - 2 \neq 0 \Rightarrow x \neq 2 \Rightarrow \boxed{(-\infty, 2), (2, \infty)}$$

□

(2) Describe on which interval(s) the following function is continuous:

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & 2 \leq x < 4 \\ 3, & x \geq 4 \end{cases}$$

Solution Each piece is continuous. Check the breaks.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (3 - x) \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} \left(\frac{x}{2} + 1 \right) \\ &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(2) &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 4^-} f(x) &= \lim_{x \to 4^-} \left(\frac{x}{2} + 1 \right) \\ &= \frac{4}{2} + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 4^+} f(x) &= \lim_{x \to 4^+} 3 \\ &= 3 \end{aligned}$$

$$f(4) = 3$$

Thus f is not continuous at $x = 2$ (but is continuous from the right) and is continuous at $x = 4$. Accepted answers are either $\boxed{(-\infty, 2) \cup (2, \infty)}$ or $\boxed{(-\infty, 2), [2, \infty)}$

□

(3) Describe on which interval(s) the following function is continuous (show your work):

$$f(x) = \begin{cases} 1 - x^2 & x < -1 \\ 1 + x & -1 \leq x \leq 1 \\ -3 & x > 1 \end{cases}$$

Solution Each piece is continuous so we just have to check $x = -1$ and $x = 1$.

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (1 - x^2) \\ &= 1 - (-1)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (1 + x) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(=1) &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (1 + x) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (-3) \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(1) &= 1 + 1 \\ &= 2 \end{aligned}$$

So the function is continuous at $x = -1$ and not at $x = 1$ (continuous from the left). Accepted answers are either $\boxed{(-\infty, 1) \cup (1, \infty)}$ or $\boxed{(-\infty, 1], (1, \infty)}$

□

(4) Is the following function continuous:

$$f(x) = \begin{cases} x + 3 & x < 2 \\ 5 & x = 2 \\ x^4 - 11 & x > 2 \end{cases}$$

Solution Each piece is a polynomial so each piece is continuous. We must check $x = 2$

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x + 3) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^4 - 11) \\ &= 5 \end{aligned}$$

$$f(2) = 5$$

Since they are all equal, f is continuous everywhere.

□

(5) What values of m and b make the following function continuous:

$$f(x) = \begin{cases} x^2 - 7 & x < -2 \\ mx + b & -2 \leq x \leq 2 \\ 5 & x > 2 \end{cases}$$

Solution

We need $\lim_{x \to 2} f(x) = f(2)$ and $\lim_{x \to -2} f(x) = f(-2)$

$\underline{\lim_{x \to 2} f(x)}$:

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (mx + b) \\ &= 2m + b \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 5 \\ &= 5 \end{aligned}$$

So we want $f(2) = 5 = 2m + b$

$\underline{\lim_{x \to -2} f(x)}$:

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (x^2 - 7) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} (mx + b) \\ &= -2m + b \end{aligned}$$

So we want $f(-2) = -2 = -2m + b$

Putting these both together we have $b = 1$ and $m = 2$

□

(6) Show that the equation $x^3 - x^2 + 2x - 7 = 0$ has a solution in the interval $[1, 2]$. State any theorems you use to support your answer.

Solution Let $f(x) = x^3 - x^2 + 2x - 7$

$$\begin{aligned} f(1) &= 1 - 1 + 2 - 7 \\ &= -5 \end{aligned}$$

$$\begin{aligned} f(2) &= 8 - 4 + 4 - 7 \\ &= 1 \end{aligned}$$

f is continuous with $f(1) < 0$ and $f(2) > 0$ so by the Intermediate Value Theorem, there exists at least one solution in $[1, 2]$

$\square$

(7) Let $f(x) = 5 + x - x^4$. Use the intermediate value theorem to show that there is at least one point where $f(x) = 0$.

Solution

$$\begin{aligned} f(0) &= 5 + 0 - 0 \\ &= 5 \end{aligned}$$

$$\begin{aligned} f(2) &= 5 + 2 - 16 \\ &= -9 \end{aligned}$$

f is continuous with $f(0) > 0$ and $f(2) < 0$ so by the Intermediate Value Theorem there is at least one point where $f(x) = 0$.

$\square$

(8) Describe on which intervals the following functions are continuous (show your work):

(a) $y = \dfrac{\sin x}{x-2}$

(b) $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & 2 \le x < 4 \\ 3, & x \ge 4 \end{cases}$

(c) $f(x) = \begin{cases} 1 - x^2 & x < -1 \\ 1 + x & -1 \le x \le 1 \\ -3 & x > 1 \end{cases}$

Solution

(a) The function is undefined for $x - 2 = 0 \Rightarrow x = 2$. That gives us an answer of $\boxed{(-\infty, 2) \cup (2, \infty)}$

(b) Each piece is continuous, so we have to check the breaks $x = 2$ and $x = 4$

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (3 - x) \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \left(\frac{x}{2} + 1\right) \\ &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(2) &= \frac{2}{2} + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-} \left(\frac{x}{2} + 1\right) \\ &= \frac{4}{2} + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} 3 \\ &= 3 \end{aligned}$$

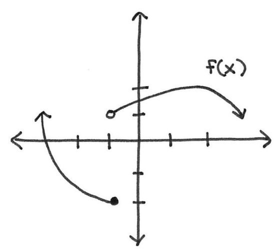
$$f(4) = 3$$

This shows us that the function is discontinuous at $x = 2$, but continuous from the right there. It also shows us that the function is continuous at $x = 4$. Putting this together we get $\boxed{(-\infty, 2), [2, \infty)}$

□

Reading/Using Graphs

(1) From the picture, decide whether the following limits exist. If they exist, find their value:



(a) $\lim_{x \to -1^-} f(x)$

(b) $\lim_{x \to -1^+} f(x)$

(c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 2} f(x)$

Solution

a) -2

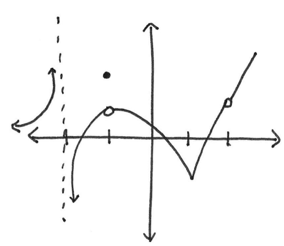
b) 1

c) DNE since $\lim_{x \to =1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

d) 2

□

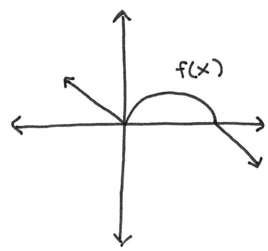
(2) Identify the x-values where $f(x)$ is discontinuous.



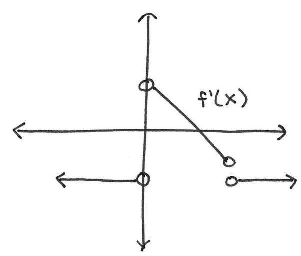
Solution Discontinuous at $x = -2$ because there is an asymptote so the function is not defined (also the limit DNE). Discontinuous at $x = -1$ because $\lim_{x \to -1} f(x) \neq f(-1)$. Discontinuous at $x = 2$ because $f(2)$ is undefined.

□

(3) Sketch the graph of the derivative of the function shown.



Solution



The graph is not exact; we only care that you know where the derivative is positive, negative, constant, and where it doesn't exist.

□

(4) Consider the function $f(x)$ given below. Find

(i) $\lim_{x \to k^-} f(x)$

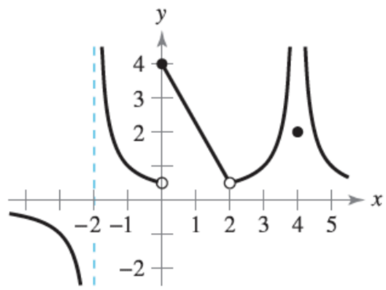
(ii) $\lim_{x \to k^+} f(x)$

(iii) $\lim_{x \to k} f(x)$

(iv) $f(k)$

(v) Is $f(x)$ continuous at k? (yes or no)

for each of the given values of k. If the given value does not exist, write "DNE", ∞, $-\infty$, or "undefined" as necessary:



(a) $k = -1$

(b) $k = 0$

(c) $k = 2$

(d) $k = 4$

Solution

(a) (i) $\boxed{1}$

(ii) $\boxed{1}$

(iii) $\boxed{1}$

(iv) $\boxed{1}$

(v) $\boxed{\text{yes}}$

(b) (i) $\boxed{\frac{1}{2}}$

(ii) $\boxed{4}$

(iii) $\boxed{\text{DNE}}$

(iv) $\boxed{4}$

(v) $\boxed{\text{no}}$

(c) (i) $\boxed{\frac{1}{2}}$

(ii) $\boxed{\frac{1}{2}}$

(iii) $\boxed{\frac{1}{2}}$

(iv) $\boxed{\text{undefined}}$

(v) $\boxed{\text{no}}$

(d) (i) $\boxed{\infty}$

(ii) $\boxed{\infty}$

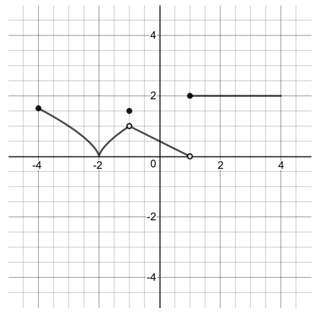
(iii) $\boxed{\infty}$

(iv) $\boxed{2}$

(v) $\boxed{\text{no}}$

□

(5) The function $f(x)$ is defined for $-4 \leq x \leq 4$ and is graphed below. Use the graph to answer the following questions:



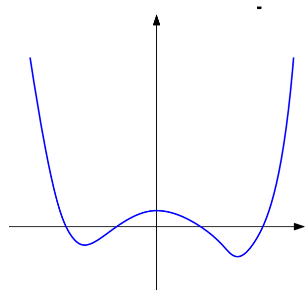
(a) What is $\lim_{x \to -1} f(x)$?

(b) What is $\lim_{x \to 1} f(x)$?

(c) Give the intervals where $f(x)$ is continuous, be careful to include the endpoints if necessary.

(d) Does the function appear to be differentiable at $x = -2$? Explain why or why not.

(e) Sketch the graph of f' given that the graph of f looks like the following:



Solution

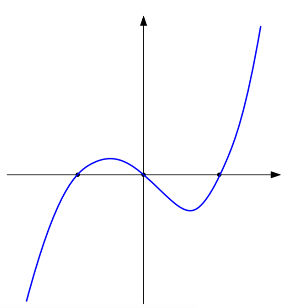
(a) $\boxed{1}$

(b) $\boxed{\text{DNE}}$

(c) $\boxed{[-4, -1), (-1, 1), [1, 4]}$

(d) $\boxed{\text{No, there is a corner}}$

(e)



□

Limit Definition of a Derivative

(1) (a) Using the limit definition of a derivative, differentiate the following:

$$f(x) = x^2 - 3x - 1$$

(b) Find the equation of the line tangent to $f(x)$ at $x = 1$

Solution

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - 3(x+h) - 1] - (x^2 - 3x - 1)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{1} - \cancel{x^2} + \cancel{3x} + \cancel{1}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x + h - 3) \\
&= \boxed{2x - 3}
\end{aligned}$$

(b)

$$f'(1) = -1 \text{ and } f(1) = -3$$

Using point-slope form we have

$$\boxed{y + 3 = -(x - 1)}$$

□

(2) (a) **Use the definition of derivative** to show that the derivative of $f(x) = x^2 - x$ at $x = -2$ is -5, i.e. $f'(-2) = -5$.

(b) Find an equation for the tangent line to $f(x) = x^2 - x$ at $x = -2$.

Solution

(a)

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{[(-2+h)^2 - (-2+h)] - [(-2)^2 - (-2)]}{h} \\
&= \lim_{h\to 0} \frac{[(4 - 4h + h^2) + 2 - h] - [4+2]}{h} \\
&= \lim_{h\to 0} \frac{\not{4} - 4h + h^2 + \not{2} - h - \not{4} - \not{2}}{h} \\
&= \lim_{h\to 0} \frac{\not{h}(-4 + h - 1)}{\not{h}} \\
&= \lim_{h\to 0} (-4 + h - 1) \\
&= -4 - 1 \\
&= -5 \checkmark
\end{aligned}
$$

(b)

$$f(-2) = (-2)^2 - (-2) = 6$$

Using the slope $m = -5$ and the point $(-2, 6)$ we have

$$\boxed{y - 6 = -5(x+2)}$$

□

(3) Use the definition of the derivative (limit definition) to find the derivative of $f(x) = \sqrt{x}$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \boxed{\frac{1}{2\sqrt{x}}}
\end{aligned}
$$

□

(4) Use the definition of the derivative (limit definition) to find the derivative of $f(x) = x^2 - x$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - (x+h)] - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x + h - 1) \\
&= \boxed{2x - 1}
\end{aligned}
$$

□

(5) Use the definition of the derivative (limit definition) to find the derivative of $f(x) = \frac{1}{x}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \boxed{-\frac{1}{x^2}}
\end{aligned}$$

□

(6) Consider the function $f(x) = 5 - x^2$.

(a) Find the equation for the secant line to the graph of $f(x)$ that passes through the points $(1, 4)$ and $(2, 1)$.

(b) Find $f'(x)$ using the definition of a derivative.

(c) Find the equation for the tangent line to the graph of $f(x)$ at the point $(1, 4)$.

(d) Find the equation for the tangent line to the graph of $f(x)$ at the point $(2, 1)$.

Solution

(a)

$$m = \frac{4-1}{1-2} = -3$$

Using point-slope form we have

$$\boxed{y - 4 = -3(x-1)}$$

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[5-(x+h)^2] - (5-x^2)}{h} \\ &= \lim_{h\to 0} \frac{5-(x^2+2xh+h^2) - 5 + x^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{5} - \cancel{x^2} - 2xh - h^2 - \cancel{5} + \cancel{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}} \\ &= \lim_{h\to 0} (-2x-h) \\ &= \boxed{-2x} \end{aligned}$$

(c)

$$m = f'(1) = -2$$

Using point-slope form we have

$$\boxed{y - 4 = -2(x-1)}$$

(d)

$$m = f'(2) = -4$$

Using point-slope form we have

$$\boxed{y - 1 = -4(x-2)}$$

□

(7) Use the definition of the derivative (limit definition) to find the derivatives of the following:

(a) $f(x) = \sqrt{x}$
(b) $f(x) = x^2 - x$
(c) $f(x) = \frac{1}{x}$

Solution

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + x)} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \boxed{\frac{1}{2\sqrt{x}}}
\end{aligned}$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x + h - 1) \\
&= \boxed{2x - 1}
\end{aligned}$$

(c)

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{(x+h)(x)}{(x+h)(x)} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)(x)} \\
&= \lim_{h \to 0} \frac{x - x - h}{xh(x+h)} \\
&= \lim_{h \to 0} \frac{-\cancel{h}}{x\cancel{h}(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x(x+0)} \\
&= \boxed{-\frac{1}{x^2}}
\end{aligned}
$$

□

(8) Consider the function $f(x) = 5 - x^2$.

(a) Find the equation for the secant line to the graph of $f(x)$ that passes through the points $(1,4)$ and $(2,1)$.

(b) Find $f'(x)$ using the definition of a derivative.

(c) Find the equation for the tangent line to the graph of $f(x)$ at the point $(1,4)$.

(d) Find the equation for the tangent line to the graph of $f(x)$ at the point $(2,1)$.

Solution

(a)

$$m = \frac{1-4}{2-1} = \frac{-3}{1} = -3$$

Using point-slope form we have

$$\boxed{y - 4 = -3(x-1) \text{ or } y - 1 = -3(x-2)}$$

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{5-(x+h)^2-(5-x^2)}{h} \\ &= \lim_{h\to 0} \frac{5-(x^2+2xh+h^2)-5+x^2}{h} \\ &= \lim_{h\to 0} \frac{5-x^2-2xh-h^2-5+x^2}{h} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}} \\ &= \lim_{h\to 0}(-2x-h) \\ &= \boxed{-2x} \end{aligned}$$

(c)

$$f'(1) = -2(1) = -2$$

Using point-slope form we have

$$\boxed{y - 4 = -2(x-1)}$$

(d)

$$f'(2) = -2(2) = -4$$

Using point-slope form we have

$$\boxed{y - 1 = -4(x-2)}$$

□

Basic Differentiation

(1) Differentiate. You do not need to simplify your answer: $y = 3x\sin x$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{d}{dx}(3x)\right)\sin x + 3x\frac{d}{dx}(\sin x)\\ &= \boxed{3\sin x + 3x\cos x}\end{aligned}$$

□

(2) Differentiate. You do not need to simplify your answer: $y = \frac{x+1}{x^2+2}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+2)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^2+2)}{(x^2+2)^2}\\ &= \boxed{\frac{(x^2+2)(1) - (x+1)(2x)}{(x^2+2)^2}}\end{aligned}$$

□

(3) Differentiate. You do not need to simplify your answer: $f(x) = 12x^2 - \frac{5}{\sqrt{x}} + 78$

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}(12x^2 - 5x^{-1/2} + 78)\\ &= 12(2x) - 5\left(-\frac{1}{2}x^{-3/2}\right)\\ &= \boxed{24x + \frac{5}{2}x^{-3/2}}\end{aligned}$$

□

(4) Differentiate. You do not need to simplify your answer: $y = \frac{x^3+1}{2-x}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2-x)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}(2-x)}{(2-x)^2}\\ &= \boxed{\frac{(2-x)(3x^2) - (x^3+1)(-1)}{(2-x)^2}}\end{aligned}$$

□

(5) Differentiate. You do not need to simplify your answer: $y = 6x^2 - 10x - 5x^{-2}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= 6(2)x^{2-1} - 10(1)x^{1-1} - 5(-2)x^{-2-1} \\ &= \boxed{12x - 10 + 10x^{-3}}\end{aligned}$$

□

(6) Differentiate. You do not need to simplify your answer: $y = x^2 \sin x + 2x \cos x - 2 \sin x$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{d}{dx}(x^2) \sin x\right) + x^2 \left(\frac{d}{dx}(\sin x)\right) + \left(\frac{d}{dx}(2x)\right) \cos x + 2x \left(\frac{d}{dx}(\cos x)\right) - 2 \cos x \\ &= \cancel{2x \sin x} + x^2 \cos x + \cancel{2 \cos x} - \cancel{2x \sin x} - \cancel{2 \cos x} \\ &= \boxed{x^2 \cos x}\end{aligned}$$

□

(7) Differentiate. You do not need to simplify your answer: $y = \frac{\cot x}{1+\cot x}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 + \cot x)(d/dx(\cot x)) - (\cot x)(d/dx(1 + \cot x))}{(1 + \cot x)^2} \\ &= \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2} \\ &= \frac{(1 + \cot x)(-\csc^2 x) + \cot x \csc^2 x}{(1 + \cot x)^2} \\ &= \frac{-\csc^2 x - \cancel{\cot x \csc^2 x} + \cancel{\cot x \csc^2 x}}{(1 + \cot x)^2} \\ &= -\frac{\csc^2 x}{(1 + \cot x)^2}\end{aligned}$$

□

(8) Differentiate. You do not need to simplify your answer: $f(x) = \sin x(x^2 + 3) + x^{4/3}$

Solution

$$\begin{aligned} f'(x) &= \sin(x)\frac{d}{dx}(x^2+3) + (x^2+3)\frac{d}{dx}(\sin(x)) + \frac{4}{3}x^{4/3-1} \\ &= \boxed{\sin(x)(2x) + (x^2+3)\cos(x) + \frac{4}{3}x^{1/3}} \end{aligned}$$

□

(9) Differentiate. You do not need to simplify your answer: $f(x) = 2x^3e^x + 1$

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^3)e^x + 2x^3\frac{d}{dx}(e^x) \\ &= 6x^2e^x + 2x^3e^x \leftarrow \textit{acceptable} \\ &= \boxed{2x^2e^x(3+x)} \end{aligned}$$

□

(10) Differentiate. You do not need to simplify your answer: $f(x) = x\sin^{-1} x$

Solution

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}(x)\right)\sin^{-1} x + x\frac{d}{dx}(\sin^{-1} x) \\ &= (1)\sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) \\ &= \boxed{\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}} \end{aligned}$$

□

(11) Differentiate. You do not need to simplify your answer: $h(x) = \frac{\log_2 x}{\cos^{-1} x}$

Solution

$$\begin{aligned} h'(x) &= \frac{\cos^{-1} x\left(\frac{d}{dx}(\log_2 x)\right) - \log_2 x\left(\frac{d}{dx}(\cos^{-1} x)\right)}{(\cos^{-1} x)^2} \\ &= \boxed{\frac{\cos^{-1} x\left(\frac{1}{x\ln 2}\right) - \log_2 x\left(-\frac{1}{\sqrt{1-x^2}}\right)}{(\cos^{-1} x)^2}} \end{aligned}$$

□

(12) Find an equation for the tangent line to the graph of $y = \dfrac{3}{x+1}$ for $x = 1$

Solution

$$x = 1 \Rightarrow y = \frac{3}{1+1} = \frac{3}{2}$$

so the point we have is $\left(1, \dfrac{3}{2}\right)$

$$\begin{aligned} y' &= \frac{d}{dx} 3(x+1)^{-1} \\ &= -3(x+1)^{-2} \\ &= \frac{-3}{(x+1)^2} \end{aligned}$$

$$\begin{aligned} y'(1) &= \frac{-3}{(1+1)^2} \\ &= \frac{-3}{2^2} \\ &= \frac{-3}{4} \end{aligned}$$

Using point-slope form we have the line is $\boxed{y - \frac{3}{2} = -\frac{3}{4}(x-1)}$

□

(13) Find an equation of the tangent line to the curve $y = 2 - x^3$ at $(1, 1)$

Solution

$$m = f'(1) = -3(1)^2 = -3$$

Using point-slope form we have

$$\boxed{y - 1 = -3(x-1) \text{ or } y = 4 - 3x}$$

□

Chain Rule

(1) Differentiate. You do not need to simplify your answer: $y = \frac{\cot x}{1+\cot(x^2+x)}$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1+\cot(x^2+x))\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot(x^2+x))}{(1+\cot(x^2+x))^2} \\
&= \frac{(1+\cot(x^2+x))(-\csc^2 x) - (\cot x)(-\csc^2(x^2+x)\frac{d}{dx}(x^2+x))}{(1+\cot(x^2+x))^2} \\
&= \boxed{\frac{(1+\cot(x^2+x))(-\csc^2 x) - (\cot x)(-\csc^2(x^2+x)(2x+1))}{(1+\cot(x^2+x))^2}}
\end{aligned}$$

□

(2) Differentiate. You do not need to simplify your answer: $h(x) = x\tan(2\sqrt{x}) + 7$

Solution

$$\begin{aligned}
h'(x) &= \left(\frac{d}{dx}(x)\right)\tan(2\sqrt{x}) + x\left(\frac{d}{dx}(\tan(2\sqrt{x}))\right) \\
&= \tan(2\sqrt{x}) + x\sec^2(2\sqrt{x})\frac{d}{dx}(2\sqrt{x}) \\
&= \tan(2\sqrt{x}) + x\sec^2(2\sqrt{x})(x^{-1/2}) \\
&= \boxed{\tan(2\sqrt{x}) + \sqrt{x}\sec^2(2\sqrt{x})}
\end{aligned}$$

□

(3) Differentiate. You do not need to simplify your answer: $h(t) = \cos^2(\pi t) + 3$

Solution

$$\begin{aligned}
h'(t) &= 2\cos(\pi t)\frac{d}{dt}(\cos(\pi t)) + 0 \\
&= 2\cos(\pi t)\cdot -\sin(\pi t)\frac{d}{dt}(\pi t) \\
&= \boxed{2\cos(\pi t)\cdot -\sin(\pi t)\cdot(\pi)}
\end{aligned}$$

□

(4) Differentiate. You do not need to simplify your answer: $y = \frac{\sin^2 x}{2x^3+4x^2+7}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x^3+4x^2+7)\frac{d}{dx}(\sin^2 x) - \sin^2 x \frac{d}{dx}(2x^3+4x^2+7)}{(2x^3+4x^2+7)^2} \\ &= \frac{(2x^3+4x^2+7)\left(2\sin x \frac{d}{dx}(\sin x)\right) - \sin^2 x(2(3x^2)+4(2x))}{(2x^3+4x^2+7)^2} \\ &= \boxed{\frac{(2x^3+4x^2+7)(2\sin x\cos x) - \sin^2 x(6x^2+8x)}{(2x^3+4x^2+7)^2}}\end{aligned}$$

□

(5) Differentiate. You do not need to simplify your answer: $y = \tan\left(x^2+1\right)$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(x^2+1)\cdot\frac{d}{dx}(x^2+1) \\ &= \boxed{\sec^2(x^2+1)(2x)}\end{aligned}$$

□

(6) Differentiate. You do not need to simplify your answer: $y = \left(1-\frac{x}{7}\right)^{-7}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= -7\left(1-\frac{x}{7}\right)^{-8}\cdot\frac{d}{dx}\left(1-\frac{x}{7}\right) \\ &= -7\left(1-\frac{x}{7}\right)^{-8}\left(-\frac{1}{7}\right) \\ &= \left(1-\frac{x}{7}\right)^{-8}\end{aligned}$$

□

(7) Differentiate. You do not need to simplify your answer: $y = 4x^2(3x-2)^5$

Solution By the product rule we have $y' = (4x^2)'(3x-2)^5 + 4x^2((3x-2)^5)'$

Using chain rule we have that $(3x-2)^5 = u(v(x))$ where
$v(x) = 3x - 2 \Rightarrow v'(x) = 3$
$u(x) = x^5 \Rightarrow u'(x) = 5x^4$
so $((3x-2)^5)' = u'(v(x))v'(x) = u'(3x-2)\cdot 3 = 5(3x-2)^4 \cdot 3 = 15(3x-2)^4$

Putting this all together we have $\boxed{y' = (8x)(3x-2)^5 + 4x^2(15(3x-2)^4)}$

□

(8) Differentiate. You do not need to simplify your answer: $\ln\left(x^4+1\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(x^4+1\right)\right) &= \frac{1}{x^4+1}\cdot\frac{d}{dx}(x^4+1)\\ &= \frac{1}{x^4+1}(4x^3)\\ &= \boxed{\frac{4x^3}{x^4+1}}\end{aligned}$$

□

(9) Differentiate. You do not need to simplify your answer: $2^{3+\sin x}$

Solution

$$\begin{aligned}\frac{d}{dx}(2^{3+\sin x}) &= 2^{3+\sin x}\ln 2\cdot\frac{d}{dx}(3+\sin x)\\ &= \boxed{2^{3+\sin x}\ln 2\cdot\cos x}\end{aligned}$$

□

(10) Differentiate. You do not need to simplify your answer: $\log_2 \frac{8}{\sqrt{2x+1}}$

Solution

$$\begin{aligned}
\frac{d}{dx} \log_2 \frac{8}{\sqrt{2x+1}} &= \frac{1}{\frac{8}{\sqrt{2x+1}} \ln 2} \cdot \frac{d}{dx}\left(\frac{8}{\sqrt{2x+1}}\right) \\
&= \frac{\sqrt{2x+1}}{8\ln 2} \cdot \frac{d}{dx}(8(2x+1)^{-1/2}) \\
&= \frac{\sqrt{2x+1}}{8\ln 2} \cdot \left(8 \cdot -\frac{1}{2}(2x+1)^{-3/2} \cdot \frac{d}{dx}(2x+1)\right) \\
&= \boxed{\frac{\sqrt{2x+1}}{8\ln 2}\left(-4(2x+1)^{-3/2}(2)\right)}
\end{aligned}$$

□

(11) Differentiate. You do not need to simplify your answer: $\tan^{-1}(e^{4x})$

Solution

$$\begin{aligned}
\frac{d}{dx} \tan^{-1}(e^{4x}) &= \frac{1}{1+(e^{4x})^2} \cdot \frac{d}{dx}(e^{4x}) \\
&= \frac{1}{1+e^{8x}} \cdot e^{4x} \cdot \frac{d}{dx}(4x) \\
&= \frac{1}{1+e^{8x}} \cdot e^{4x}(4) \\
&= \boxed{\frac{4e^{4x}}{1+e^{8x}}}
\end{aligned}$$

□

Implicit Differentiation

(1) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $2\sqrt{y} = x - y$

Solution

$$\begin{aligned}
2\sqrt{y} = x - y &\Rightarrow y^{-1/2}\frac{dy}{dx} = 1 - \frac{dy}{dx} \\
&\Rightarrow (y^{-1/2} + 1)\frac{dy}{dx} = 1 \\
&\Rightarrow \frac{dy}{dx} = \frac{1}{y^{-1/2} + 1} \text{ or } \frac{\sqrt{y}}{1 + \sqrt{y}}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} = \frac{\sqrt{y}}{1+\sqrt{y}} &\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\sqrt{y}}{1+\sqrt{y}}\right) \\
&\Rightarrow \frac{d^2y}{dx^2} = \frac{(1+\sqrt{y})((1/2)y^{-1/2}(dy/dx)) - \sqrt{y}(1/2)(y^{-1/2})(dy/dx)}{(1+\sqrt{y})^2} \\
&\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} \cdot \frac{(1+\sqrt{y}) - \sqrt{y}}{2\sqrt{y}(1+\sqrt{y})^2} \\
&\Rightarrow \frac{d^2y}{dx^2} = \frac{\cancel{\sqrt{y}}}{1+\sqrt{y}} \cdot \frac{1}{2\cancel{\sqrt{y}}(1+\sqrt{y})^2} \\
&\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+\sqrt{y})^3}
\end{aligned}$$

□

(2) Find an equation of the tangent line to $x^2 + xy + y^2 = 3$ at $(1,1)$

Solution Taking the derivative of both sides with respect to x we have

$$2x + \left[\left(\frac{d}{dx}(x)\right)y + x\frac{d}{dx}(y)\right] + 2y\frac{dy}{dx} = 0 \Leftrightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Plugging in $x = 1$ and $y = 1$ we have

$$\begin{aligned} 2(1) + (1) + (1)\frac{dy}{dx} + 2(1)\frac{dy}{dx} = 0 &\Leftrightarrow 3 + 3\frac{dy}{dx} = 0 \\ &\Leftrightarrow 3\frac{dy}{dx} = -3 \\ &\Leftrightarrow \frac{dy}{dx} = -1 \end{aligned}$$

Using point-slope form we have

$$\boxed{y - 1 = -(x-1)}$$

□

(3) Find $\frac{dy}{dx}$ if $y \sin\left(\frac{1}{y}\right) = 1 - xy$.

Solution

$$\begin{aligned} y\sin\left(\frac{1}{y}\right) = 1 - xy &\Rightarrow \frac{dy}{dx}\sin\left(\frac{1}{y}\right) + y\cos\left(\frac{1}{y}\right)\frac{d}{dx}\left(\frac{1}{y}\right) = -y - x\frac{dy}{dx} \\ &\Rightarrow \frac{dy}{dx}\sin\left(\frac{1}{y}\right) + y\cos\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)\frac{dy}{dx} = -y - x\frac{dy}{dx} \\ &\Rightarrow \frac{dy}{dx}\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)\frac{1}{y}\frac{dy}{dx} = -y - x\frac{dy}{dx} \\ &\Rightarrow \frac{dy}{dx}\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)\frac{1}{y}\frac{dy}{dx} + x\frac{dy}{dx} = -y \\ &\Rightarrow \frac{dy}{dx}\left(\sin\left(\frac{1}{y}\right) - \frac{1}{y}\cos\left(\frac{1}{y}\right) + x\right) = -y \\ &\Rightarrow \frac{dy}{dx} = -\frac{y}{\sin(1/y) - (1/y)\cos(1/y) + x} \\ &\Rightarrow \frac{dy}{dx} = -\frac{y^2}{y\sin(1/y) - \cos(1/y) + xy} \end{aligned}$$

□

(4) Find an equation for the tangent line to $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1, 0)$.

Solution

$$6x^2 + 3xy + 2y^2 + 17y - 6 = 0 \Rightarrow 12x + 3x\frac{dy}{dx} + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

Plugging in $x = -1$ and $y = 0$ gives:

$$-12 - 3\frac{dy}{dx} + 17\frac{dy}{dx} = 0 \Rightarrow 14\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{7}$$

Thus the slope is $6/7$ for the tangent line and the point is $(-1, 0)$. Using point-slope form we get:

$$y = \frac{6}{7}(x+1) \Leftrightarrow y = \frac{6}{7}x + \frac{6}{7}$$

The slope for the normal line is $-7/6$ so using point slope form we get:

$$y = -\frac{7}{6}(x+1) \Leftrightarrow y = -\frac{7}{6}x - \frac{7}{6}$$

□

(5) Find an equation of the tangent line to $x^2 + xy + y^2 = 3$ at $(1, 1)$

Solution Taking the derivative of both sides with respect to x we have

$$2x + \left[\left(\frac{d}{dx}(x)\right)y + x\frac{d}{dx}(y)\right] + 2y\frac{dy}{dx} = 0 \Leftrightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Plugging in $x = 1$ and $y = 1$ we have

$$\begin{aligned} 2(1) + (1) + (1)\frac{dy}{dx} + 2(1)\frac{dy}{dx} = 0 &\Leftrightarrow 3 + 3\frac{dy}{dx} = 0 \\ &\Leftrightarrow 3\frac{dy}{dx} = -3 \\ &\Leftrightarrow \frac{dy}{dx} = -1 \end{aligned}$$

Using point-slope form we have

$$\boxed{y - 1 = -(x - 1)}$$

□

(6) Find y' at $(1,1)$ for $y^4 - \sqrt{x} + 2y = 2$

Solution

$$\frac{d}{dx}(y^4 - \sqrt{x} + 2y) = \frac{d}{dx}(2)$$

$$4y^3y' - \frac{d}{dx}(x^{1/2}) + 2y' = 0$$

$$4y^3y' - \frac{1}{2}x^{-1/2} + 2y' = 0$$

Plugging in $x = 1$ and $y = 1$, we get

$$4 \cdot 1^3y' - \frac{1}{2} \cdot 1^{-1/2} + 2y' = 0$$

$$6y' - \frac{1}{2} = 0$$

$$6y' = \frac{1}{2}$$

$$y' = \boxed{\frac{1}{12}}$$

□

Related Rates

(1) The length of a rectangle is decreasing at the rate of 2 cm/sec while the width is increasing at the rate of 2 cm/sec. When the length is 12cm and the width is 5cm, find the rates of change of **a)** the area, **b)** the perimeter, and **c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

Solution

(a)

$$A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt}$$

Plugging in $l = 12$, $w = 5$, $\frac{dl}{dt} = -2$, and $\frac{dw}{dt} = 2$ we have

$$\frac{dA}{dt} = -2(5) + (12)(2) = \boxed{14 \text{ cm}^2\text{/sec, which is increasing.}}$$

(b)

$$P = 2l + 2w \Rightarrow \frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

Plugging in $\frac{dl}{dt} = -2$ and $\frac{dw}{dt} = 2$ we have

$$\frac{dP}{dt} = 2(-2) + 2(2) = \boxed{0 \text{ cm/sec, which is neither decreasing nor increasing}}$$

(c) The diagonal is related to the sides by the pythagorean theorem:

$$D^2 = l^2 + w^2 \Rightarrow 2d\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt}$$

When $l = 12$ and $w = 5$ we have

$$D^2 = 144 + 25 = 169 \Rightarrow D = 13$$

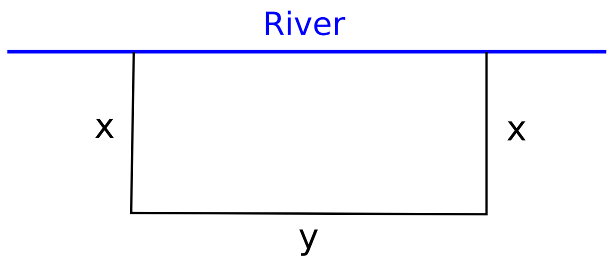
Plugging in $l = 12$, $w = 5$, $D = 13$, $\frac{dl}{dt} = -2$, and $\frac{dw}{dt} = 2$ we have

$$26\frac{dD}{dt} = 24(-2) + 10(2) \Leftrightarrow \frac{dD}{dt} = \boxed{-\frac{14}{13} \text{ cm/sec, which is decreasing}}$$

□

(2) A rectangular plot of land will be bounded on one side by a river and on the other three sides by some sort of fence. With 800 m of fencing at your disposal, what is the largest area you can enclose, and what are its dimensions?

Solution The picture is



We are given

$$2x + y = 800 \Rightarrow y = 800 - 2x$$

Thus

$$A = xy = x(800 - 2x) = 800x - 2x^2$$

Differentiating with respect to x we have

$$\frac{dA}{dx} = 800 - 4x$$

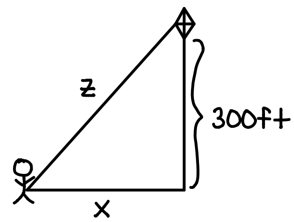
This gives one critical point of $x = 200$. Testing the intervals we have that there is a relative and thus absolute max at $x = 200 \Rightarrow y = 800 - 400 = 400$. So the dimensions are $\boxed{200 \text{ m by } 400 \text{ m}}$

$$A = 200(400) = \boxed{80000\text{m}^2}$$

□

(3) A child flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from them at a rate of 25 ft/sec. How fast must they let out the string when the kite is 500 ft away from them?

Solution The picture is the following:



Using the pythagorean theorem we have

$$x^2 + 300^2 = z^2 \Rightarrow 2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

When $z = 500$ we have $x^2 + 300^2 = 500^2 \Rightarrow x = 400$. We are also given $\frac{dx}{dt} = 25$ and we want to find $\frac{dz}{dt}$. Plugging in our known information we have

$$2(400)(25) = 2(500)\frac{dz}{dt} \Leftrightarrow \frac{dz}{dt} = \boxed{20 \text{ ft/sec}}$$

□

(4) The volume V of a circular cylinder of height h and radius r is given by $V = \pi r^2 h$. Assume that V is kept constant ($V = 8\pi$) as r and h are changing. Calculate the rate of change of h with respect to r when $h = 2$ and $r = 2$.

Solution Differentiating with respect to r we have:

$$0 = \pi r^2 \frac{dh}{dr} + 2\pi r h$$

Plugging in $h = 2$ and $r = 2$, we have

$$0 = \pi(2)^2\frac{dh}{dr} + 8\pi \Leftrightarrow 4\pi\frac{dh}{dr} = -8\pi$$

$$\Leftrightarrow \frac{dh}{dr} = -2$$

□

(5) A spherical snowball is placed in the sun. The sun melts the snowball so that its radius **decreases** 1/4 in. per hour. Find the rate of change of the volume with respect to time at the instant the radius is 4 in. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Solution

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

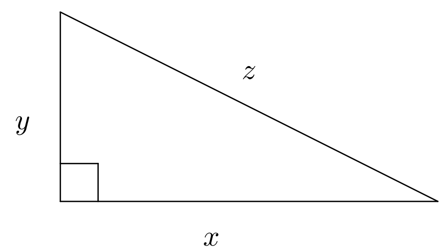
We are given that $\frac{dr}{dt} = \frac{-1}{4}$ and we want to find $\frac{dV}{dt}$ when $r = 4$. Plugging these in we have

$$\frac{dV}{dt} = 4\pi(4)^2\left(-\frac{1}{4}\right) = \boxed{-16\pi \text{ in}^3/\text{hr}}$$

□

(6) A person leaves a given point and travels north at 3 mph. Another person leaves the same point at the same time and travels east at 4 mph. At what rate is the distance between the two people changing at the instant when they have traveled 2 hours?

Solution The following is what the problem sketch will look like:



We are given

$$\frac{dy}{dt} = 3 \text{ and } \frac{dx}{dt} = 4$$

We want to find $\frac{dz}{dt}$ when $x = 8$ and $y = 6$ (as that is how far the cars with have traveled in 2 hours)

When $x = 8$ and $y = 6$ we have

$$z^2 = (8)^2 + (6)^2 = 100 \Rightarrow z = 10$$

Taking the derivative with respect to time and plugging in our given information we have

$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow 2(8)(4) + 2(6)(3) = 2(10)\frac{dz}{dt}$$

$$\Leftrightarrow 64 + 36 = 20\frac{dz}{dt}$$

$$\Leftrightarrow 100 = 20\frac{dz}{dt}$$

$$\Leftrightarrow \frac{dz}{dt} = \boxed{5 \text{ mph}}$$

□